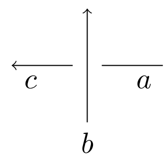
All framed knots that I will consider have the blackboard framing, and will all be equipped with an orientation (any will do).

I'll begin with the case of framed knots (i.e. framed links with only one component). The Poincaré conjecture in this case says that if $\pi_1(K)/\langle l \rangle$ is trivial, for a knot K and l its longitude, then K is equivalent under the framed Reidemeister moves and Kirby moves to the empty knot (or equivalently, and this is the form I will actually use, to the unknot with ± 1 framing, i.e. a figure of eight). I will actually prove something stronger: this is true without using Kirby moves. That is to say: if $\pi_1(K)/\langle l \rangle$ is trivial, then K is isotopic as a framed knot to the unknot with ± 1 framing. This stronger statement, in addition to proving the Property P conjecture (the Poincaré conjecture for knots), also proves the Gordon-Luecke theorem, that a Dehn surgery on a knot can give S^3 if and only if the knot is trivial (and then the only possibility is ± 1-framing).

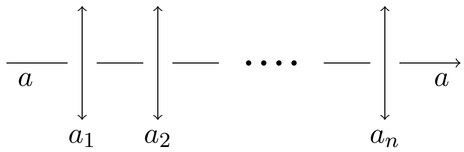
I am going to make fundamental use of virtual knot theory. In particular, I am going to rely on a theorem which says that a pair of framed classical knots are isotopic if and only if they are isotopic as framed virtual knots (i.e. using the classical framed Reidemeister moves and all the virtual Reidemeister moves). This is almost trivial to prove, using the fact that the fundamental rack is a complete invariant of framed knots. But it is a deep result: there is no known way to prove this result combinatorially (i.e. how to replace a sequence involving virtual frames Reidemeister moves with one involving only classical framed Reidemeister moves).

The crucial fact that I will rely on is the following. Let K be a classical (blackboard) framed knot. Let l be the longitude of K. By a word in the arcs of K, I shall mean a monomial of the form $a_1^{\pm 1} \cdots a_n^{\pm 1}$, where $a_1, \ldots, a_n$ are arcs of K. Given a crossing C of K which looks as follows



we denote by w_C the word $c^{-1}b^{-1}ab$ in the arcs of K.

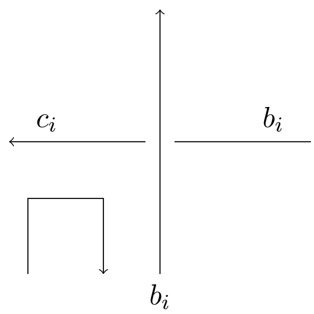
Let a be the arc of K at which we started when calculating the longitude of K. Pick a point p on this arc. Let us say that a word $w = a_1^{\pm 1} \cdots a_n^{\pm 1}$ in the arcs of K is *realisable* if there is a connected sum $\underbrace{K\#K\#\cdots\#K}_{n}$ of copies of K with itself, for some $n \geq 0$, and a virtual knot V which is equivalent to this connected sum under the classical $R2$ and $R3$ moves and the virtual $R2$ and $R3$ moves, such that we can walk from p around part or all of V and back to some other point of a in such a way that the classical arcs we pass under are amongst the following arcs. I do not depict any arcs that pass under the horizontal arcs, or cross these virtually. The arrow for the arc a_i points up if the power of a_i is -1, and down if the power of a_i is 1. By 'amongst the following arcs', I mean that not all of the a_i's need appear; but those do appear must do so in this order, from left to right, and if not all appear, then at least two must not appear.



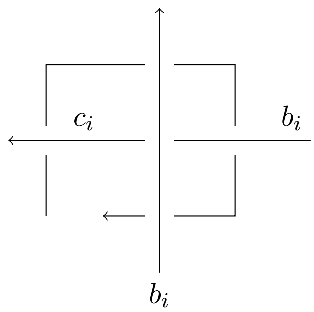
Let $C_1, \ldots C_n$ be the crossings of K. Let N be the normal subgroup of $F(K)$ generated by l, the longitude of K, and the words $w_{C_1}, \ldots, w_{C_n}$. I claim that every element of N is realisable.

Let us prove this. We shall do this in stages.

1) We first show that w_{C_i} is realisable, for every i. To see this, take a small piece of the arc a just after p. Drag it, using virtual $R2$ moves, so that it is near the crossing C_i, so that we have the following local picture.

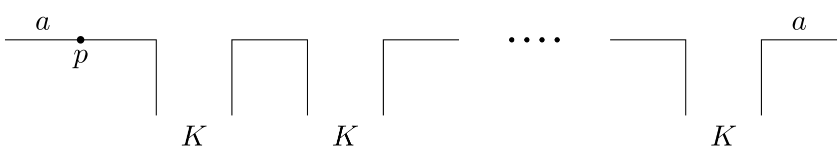


We then apply a pair of $R2$ moves and an $R3$ move to obtain the following local picture.



Taking our termination point to be the end of the small piece of a which we began by dragging, we are done.

2) We now show that any concatenation of the words l, $w_{C_1}, \ldots, w_{C_n}$ is realisable. To see this, let n be the number of times that l appears in the concatenation. Take n further copies of K, and take the connected sum of these with the original copy of K in such a way that we have the following picture.



Each copy of K in this picture is connected by removing a small arc around the point p in that copy.

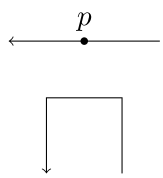
When l does not appear at all in the concatenation, we do nothing, just keeping our original copy of K.

Suppose that the words $w_{C_{i_1}} \cdots w_{C_{i_{j_1}}}$ appear in the concatenation before the first occurrence of l. Then we successively apply the construction of 1), for each of these crossings, between p and the first copy of K in the concatenation. We then travel all the way around the first copy of K in the concatenation. It is immediate from the definition of l that this will contribute l to the concatenation.

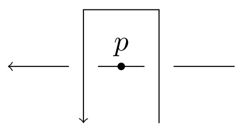
Suppose that the words $w_{C_{i_1}} \cdots w_{C_{i_{j_2}}}$ appear in the concatenation between the first and the second occurrence of l. Then we successively apply the construction of 1), for each of these crossings, between the first copy of K and the second copy of K in the concatenation. After that, we travel all the way around the second copy of K in the concatenation.

We do exactly the same kind of thing for the appearance of the words $w_{C_1}, \ldots, w_{C_n}$ between any of the occurrences of l, and after the final occurrence.

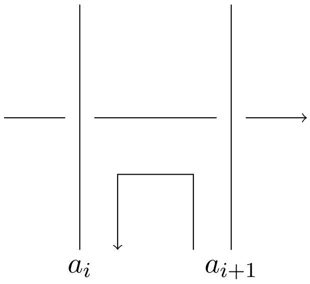
3) We now show that if w is any concatenation of the words l, $w_{C_1}, \ldots, w_{C_n}$, then $b^{-1}wb$ is realisable, for any arc b of K. To see this, drag, using virtual R2 moves, a small piece of b through the virtual knot which we obtain after 2) such that we have the following local picture.



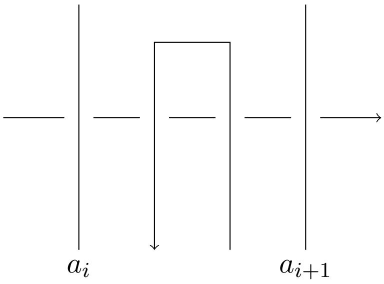
Now simply apply a classical R2 move to obtain the following local picture.



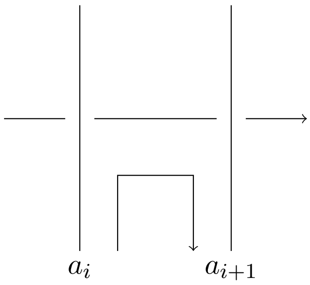
4) We now show that, if c is any arc of K, then any word obtained by adding or removing a consecutive pair cc^{-1} or $c^{-1}c$ to a realisable word w is realisable. To add a pair cc^{-1} between a_i and a_{i+1}, say, we drag, using virtual R2 moves, a small piece of c through the virtual knot which realises w in such a way that we have the following local picture.



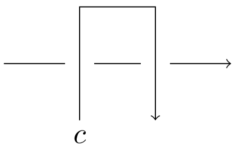
We now simply apply a classical R2 move to obtain the following picture.



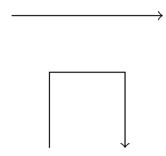
Similarly for adding a pair $c^{-1}c$, but using the virtual R2 moves in such a way as to obtain a local picture as follows.



Suppose now that we have a pair cc^{-1}. Then, ignoring virtual crossings and other under crossings, we encounter the following local picture as we travel along the virtual knot which realises w.



We can then apply R2 and R3 moves to 'reel in' the arc c to obtain the following corresponding local picture.

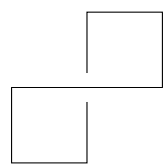


In the course of this, we may remove more than the two crossings of V depicted in the figure before last. But this is fine: this is the reason that, in our definition of realisability, we allowed for the arcs to be 'amongst' those depicted, as opposed to exactly these.

The argument for removing a pair $c^{-1}c$ is exactly the same, with the opposite orientation of the arc c.

The claim is now proven: N consists exactly of the words $b^{-1}wb$ in the arcs of K, up to the equivalence relation generated by adding or removing consecutive pairs.

We can now complete our proof of our stronger version of the Poincaré conjecture for knots very easily. Suppose that $\pi_1(K)/\langle l \rangle$ is trivial. We have that this group is exactly the quotient of $F(K)$ by N. Hence, in this case, N is all of $F(K)$. Then, since N is all of $F(K)$, we have that the arc a is realisable. This means that there is a connected sum K' of some number of copies of K such that K' is isotopic as a framed virtual knot to a virtual knot with the property that there is an arc a such that we start at a, cross under a, and return to a, without crossing under any other classical crossings. Up to applying virtual Reidemeister moves, the only possibility is that we have the following classical knot, with one of the two possible choices of orientation, as required.



Thus, we have demonstrated that if $\pi_1(K)/\langle l \rangle$ is trivial, then a connected sum K' of copies of K is isotopic as a framed virtual knot to the ± 1-framed unknot. As I mentioned at the beginning, this implies that K' is isotopic as a framed *classical* knot to the ± 1-framed unknot. Now, all knots in a connected sum which gives the unknot are unknots. Thus we in fact have that K is the unknot; and the ± 1 framed unknot is the only framed unknot such that $\pi_1(K)/\langle l \rangle$ is trivial, so K must be ± 1-framed.

The argument for links with more than one component is slightly more intricate, and here we do need the Kirby moves. It follows the same pattern, though. I'll explain that argument in due course.