Solutions - Midterm in number theory - Fall 2006

1) The algorithm is

$$\begin{aligned} 221 &= 2 \cdot 91 + 39 \\ 91 &= 2 \cdot 39 + 13 \\ 39 &= 3 \cdot 13. \end{aligned}$$

Hence $13 = \gcd(221, 91)$. Now

$$\begin{aligned} 13 &= 91 - 2 \cdot 39 = 91 - 2(221 - 2 \cdot 91) \\ &= -2 \cdot 221 + 5 \cdot 91. \end{aligned}$$

Since $52 = 4 \cdot 13$, it follows that

$$52 = -8 \cdot 221 + 20 \cdot 91,$$

yielding the solutions $x = -8$, $y = 20$. The solutions are

$$\left\{ \begin{array}{l} x = -8 + \frac{91}{13}t = -8 + 7t \\ y = 20 - \frac{221}{13}t = 20 - 17t \end{array} \right. , \quad t = 0, \pm 1, \pm 2, \ldots.$$

(A convenient one is $x = -1$, $y = 3$.) The right hand member of the equation

$$221x + 91y = 50$$

is divisible by 13, but the right-hand member 50 is not. Hence this equation does not have solutions.

2) Proof by induction that $21 | 4^{n+1} + 5^{2n-1}$.

1°) When $n = 1$, we have $4^{1+1} + 5^{2 \cdot 1 - 1} = 21$. Thus the statement is true for $n = 1$.
2°) Induction hypothesis: $4^{k+1} + 5^{2k-1} = 21N$.
3°) $4^{k+2} + 5^{2k+1} = 4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1}$
$= 4(4^{k+1} + 5^{2k-1}) + 21 \cdot 5^{2k-1} = 21(4N + 5^{2k-1})$,
i.e., $21 | 4^{k+2} + 5^{2k+1}$

By the principle of induction, the statement is true for each $n = 1, 2, 3, \ldots$.

3) To find the last digit of 7^{2007}, we calculate modulo 10.

$$\begin{aligned} 7^2 &\equiv 49 \equiv -1 \pmod{10}, \\ 7^4 &\equiv (-1)(-1) = 1 \pmod{10}, \\ 7^{2007} &= 7^{4 \cdot 501 + 3} = (7^4)^{501} 7^3 \equiv 1^{501}(-1) \cdot 7 \equiv -7 \equiv 3 \pmod{10}. \end{aligned}$$

Thus the last digit is 3.

4) Recall Euclid's proof about the infinitude of primes. The cruical ingredient is the number

$$p_1 p_2 \cdots p_n + 1.$$

Please, consult the book.