A CHARACTERISATION OF ± 1-FRAMED UNKNOTS

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Abstract. We give a new, diagrammatic, proof, involving virtual knots, that if the quotient of the fundamental group of a blackboard-framed, oriented knot by the normal closure of its longitude is trivial, then it is a ± 1-framed unknot. The arguments make use only of Reidemeister moves (no Kirby moves), providing new proofs of several important theorems concerning the 3-sphere.

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1. Introduction

Questions about 3-manifolds can sometimes be reformulated entirely in terms of diagrammatic knot theory, by means of the bridge between the two fields provided by Dehn surgery and Kirby's theorem. In this article, we shall give a new proof of the following (the definition of longitude with which we shall work will be recalled at the beginning of §2; technically, it is defined with respect to a chosen point of K). We shall always speak of the fundamental group or rack of an oriented (classical or virtual) knot diagram, as opposed to of a knot itself, thinking of this group or rack as defined directly from the diagram using Wirtinger relations.

Theorem 1. *Let K be an oriented (classical) knot diagram, corresponding to a knot equipped with its blackboard framing. If $\pi_1(K)/\langle l \rangle$ is trivial, where $\langle l \rangle$ is the normal closure in $\pi_1(K)$ of the longitude l of K, then K is equivalent under R2 and R3 moves to a ± 1-framed unknot (see Figure 1).*

Though we will not go into the details of this, Theorem 1 immediately implies, via the afore-mentioned bridge, a version of the Property P conjecture: if a 3-manifold obtained by non-trivial integral Dehn surgery on a knot has trivial fundamental group, then the knot is the unknot. This can also be formulated as follows, sometimes referred to as the Poincaré conjecture for knots: if a 3-manifold obtained from a knot by integral Dehn surgery has trivial fundamental group, then it is the 3-sphere (it is only the ± 1-framing of the unknot that corresponds to the 3-sphere under Dehn surgery). In addition, Theorem 1 implies the Gordon–Luecke

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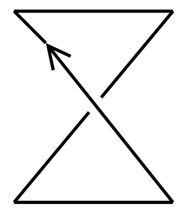

Figure 1. A -1-framed unknot. In a $+1$-framed unknot, the over-arc and under-arcs of the crossing are switched.

theorem that if an integer Dehn surgery on a knot yields the 3-sphere, then the knot is equivalent to the unknot. All of this was already known ([7, 8, 10, 9, 4]), but the proofs are very different to that we shall give of Theorem 1.

Let w be any word belonging to the free group on the arcs of K whose equivalence class in $\pi_1(K)/\langle l \rangle$ is the empty word. Our approach to Theorem 1 involves showing that if w satisfies a certain condition, then one can construct an oriented virtual knot diagram V, into which the over-arcs of K inject, whose longitude is exactly w. This is Lemma 4.

Diagrammatic approaches to Theorem 1 have been attempted before, but are notoriously thorny. There is no simple way to achieve a construction as just described using classical knot diagrams, because it is not necessarily local; a sequence of Reidemeister moves to adjust part of the longitude might have consequences for other parts of it. Passing to virtual knots allows us to entirely avoid intricacies of this kind.

A specific ingredient in our proof of Lemma 4 is what appears to be a new notion of n-parallelisation to a virtual knot of a classical knot. In the most paradigmatic case, it has the property that its longitude is the product of the longitude of the classical knot with itself n times.

We demonstrate that if an oriented virtual knot diagram has a longitude which cancels to a single arc or its formal inverse in the free group on the arcs of the diagram, then this virtual knot is equivalent as a welded knot to a (classical) ± 1-framed unknot. This is Lemma 5. To conclude the proof of Theorem 1, we combine Lemma 4 and Lemma 5 in a straightforward way using fundamental racks.

In §3, we indicate directions in which the techniques we introduce in this article might be further explored.

2. Proof

Let K be as in Theorem 1. Fix a point on an arc of K. Recall that when working with blackboard framings, the longitude of K can be obtained by starting with the empty word at our chosen point, then walking around K in accordance with its orientation, appending a (resp a^{-1}) to the word whenever one passes under a crossing whose over-arc is a, and where the orientation of the arcs is as on the left (resp right) in Figure 2. The opposite sign convention is sometimes used. The choice of point makes no difference to $\pi_1(K)/\langle l \rangle$.

The longitude of an oriented virtual knot diagram is defined [6] similarly: one simply walks through any virtual crossings one encounters without appending anything, ie only classical crossings contribute.

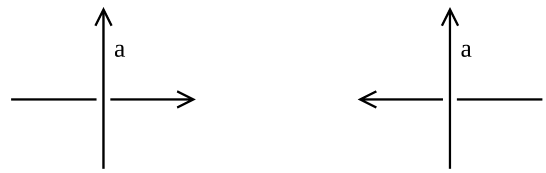

Figure 2. Longitude orientations

By a *welded Reidemeister move* on a virtual knot diagram, we shall mean a replacement of the kind depicted in Figure 3. This was first considered, as far as we know, in [1].

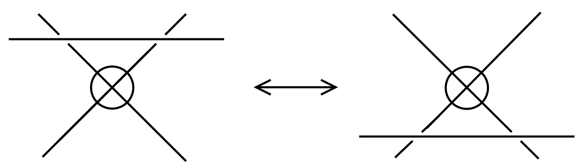

Figure 3. Welded Reidemeister move

We shall say that a virtual knot diagram is *welded-equivalent* [11] to another if there is a finite sequence of classical R2 and R3 moves, virtual Reidemeister moves [6] (allowing virtual R1 moves too), and welded Reidemeister moves taking the one diagram to the other. More properly, this notion would be referred to as framed welded equivalence, but we omit the first adjective since we work throughout in a (blackboard) framed context.

We shall work with labellings of the arcs of oriented virtual knot diagrams. Whenever we apply a Reidemeister move to a certain diagram V, we intend that the labelling of its arcs be adjusted in the standard way. In the case of an R2 move of a certain arc a under another arc, for instance, it is split into three arcs, the first and last of which (when we travel in accordance with the orientation of V) retain the label a, whilst the second is given a new label that was not previously used.

The importance of this is that the labelling of the post-move diagram defines a presentation of the fundamental group and fundamental rack of it. The set of labels of the diagram defines the generators of the presentation, whilst the relations are the Wirtinger relations coming from each crossing (with respect to the labelling).

The following notion will play a central role in this article. That we ask that the condition defining realisability holds for any point q might at first glance seem strong, but in fact will be seen in the proof of Lemma 2 to be natural.

We shall not make use of welded Reidemeister moves in the course of the proof of Lemma 4, so that it would be enough for the purposes of this article to ask only for ordinary virtual equivalence in the definition of realisability and complete realisability; but welded Reidemeister moves will occur in the proof of Lemma 5, and for simplicity we prefer to use the same notion of equivalence throughout.

Definition. We shall say that a word $w = a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}$ in the arcs of an oriented virtual knot diagram V is *realisable* if, for every point q on an arc of V, there is an oriented virtual knot diagram V', welded-equivalent to V, such that if we start at

q (following q through the welded equivalence), the longitude of V' is exactly (on the nose) wl, where l is the longitude of V, defined starting at q.

Given a point q on an arc of V, we shall say that w is *completely realisable from* q if there is an oriented virtual knot diagram V', welded-equivalent to V, such that if we start at q (following q through the welded equivalence), the longitude of V' is exactly w.

We shall also make use of the following construction. The author is not aware that it has previously been considered; a somewhat akin notion of parallelisation was introduced in a recent article [5], but it is not exactly the same. Figure 4 gives an example of the construction, for a trefoil.

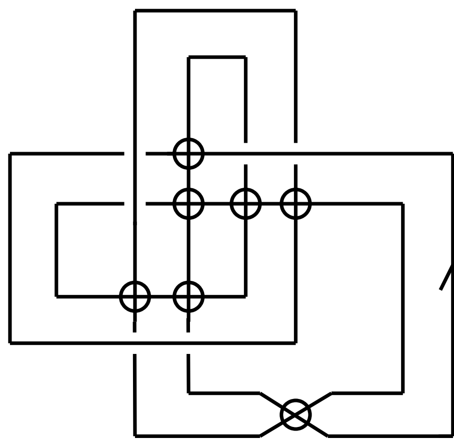

FIGURE 4. Doubling/2-parallelisation of a trefoil

Definition. Let K be an oriented classical knot diagram, and let p be a point on an arc of it. An *n-parallelisation* of K at p, where $n \geq 1$, is an oriented virtual knot diagram obtained as follows.

1. Starting at p, walk around K in accordance with its orientation, adding a parallel path as one goes, slightly offset from the original path. Wherever the original path crosses classically under an arc of K, ensure that the parallel path does the same. Wherever other crossings are necessary, make these virtual, including where the original path crosses classically over another arc. See Figure 5 for the various possibilities. Do not join the parallel path up at its ends; leave a small gap.
2. Repeat this procedure until one has n parallel paths in total (including the original one). Remove a small gap around p from the original path too.
3. Reverse the orientations of any number of the n parallel paths.
4. For $1 \leq i < n$, join the end of the i-th parallel path to the start of the $(i+1)$-th, without introducing any crossings; up to planar isotopy, a diagonal line will suffice if the paths have the same orientation, and a 'cap' if not. Join the end of the n-th parallel path to the start of the first; where crossings are necessary for this, make these virtual. Some examples are given in Figure 6.

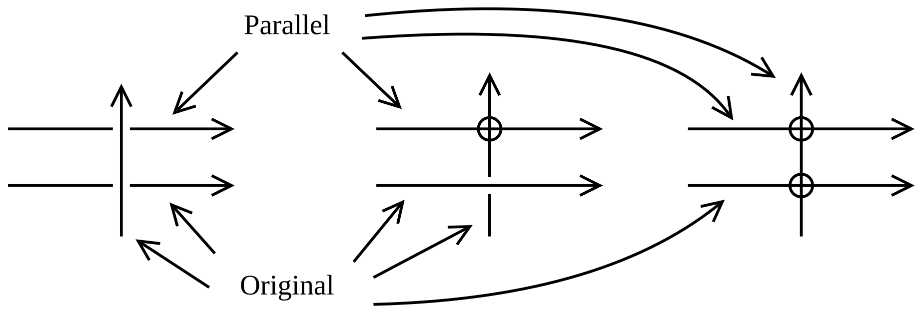

Figure 5. Crossings involving a parallel track

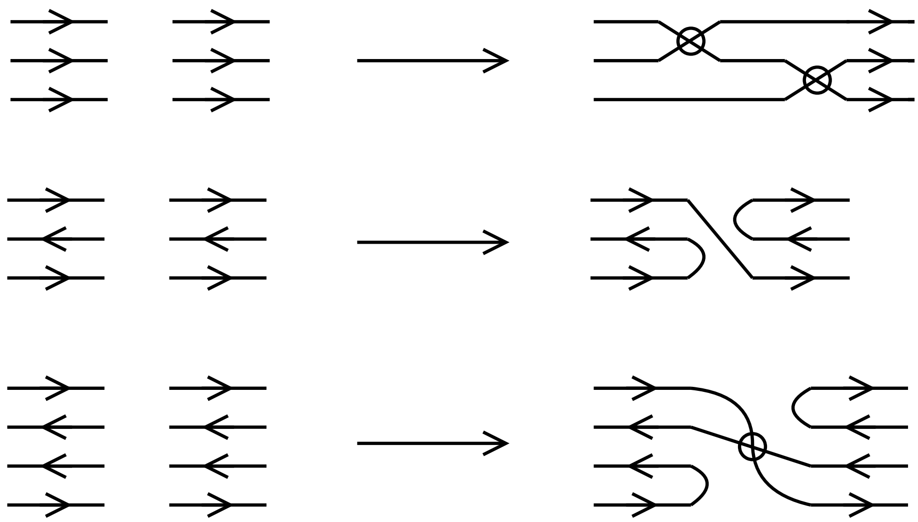

Figure 6. Joining parallel tracks

We shall refer to the i-th parallel path introduced in 1. and 2. as the *i-th track* of an n-parallelisation. We regard the arc upon which we emerge having walked under the final under-crossing of the i-th track for $i < n$ (resp the n-th track) as belonging to the $(i+1)$-th track (resp first track), ie as being the first arc of the latter, as well as being the final arc of the i-th track (resp n-th track).

When $n = 2$ (resp $n = 3$), we might also refer to an n-parallelisation of K as a *doubling* (resp *trebling*) of K.

We shall, throughout this article, view arcs and crossings of an oriented classical knot diagram K as belonging to an n-parallelisation V of it. By this we shall mean that we identify the arcs and crossings of K with those of the first track of V.

If we do not reverse the orientations of any tracks, the longitude of the n-parallelisation of K is exactly l^n, where l is the longitude of K. Lemma 4, which we shall now build up to, relies (up to reversing of orientations) on this observation.

Throughout the remainder of this article, given a classical crossing C of an oriented virtual knot diagram V, we shall denote by w_C the word in the free group on the arcs of V which is $c^{-1}b^{-1}ab$ if C has the orientation on the left in Figure 7, and $c^{-1}bab^{-1}$ if it has the orientation on the right in Figure 7. By K, we shall always mean an oriented classical knot diagram as in the statement of Theorem 1.

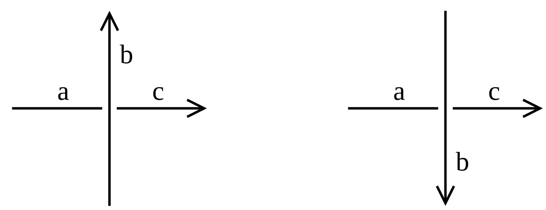

FIGURE 7. Crossing orientations

Lemma 2. *Let V be an m-parallelisation of K for some $m \geq 1$. Every word*

$$g^{-1}w_C^{\pm 1}g$$

in the free group on the arcs of V is realisable, where C is a classical crossing of K viewed as belonging to V, and there are arcs $a_1, \ldots, a_n$ of K, viewed as belonging to V, such that $g = a_1^{\pm 1} \cdots a_n^{\pm 1}$.

Proof. Suppose first that C is oriented as on the left in Figure 7, and that the exponent of both w_C and a_1 is $+1$. Using only virtual R2 moves and, if necessary, a single virtual R1 move, drag a small piece of a_1 so that it lies as shown in Figure 8 in relation to C.

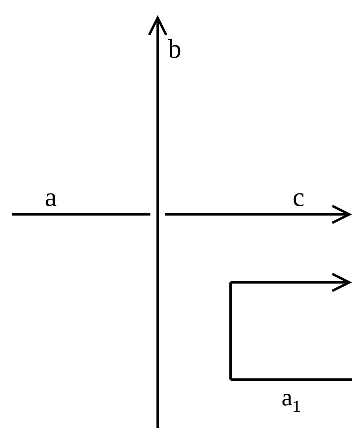

FIGURE 8. Dragging a_1.

Let q be a point on any arc of V. Take a small piece of the arc on which q lies, just after q. Using only virtual R2 moves and, if necessary, a single virtual R1 move, drag it so that it is positioned as shown in Figure 9 in relation to C.

Using three classical R2 moves followed by a classical R3 move, drag this piece of arc further under C, to obtain what is shown in Figure 10. The oriented virtual knot

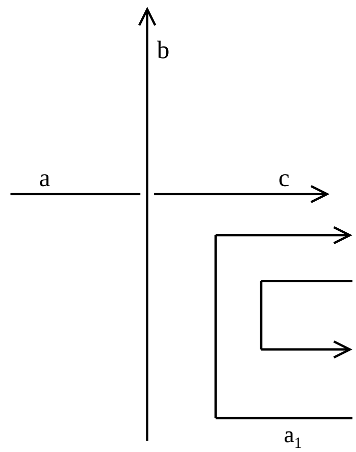

FIGURE 9. Dragging a piece of arc located just after q

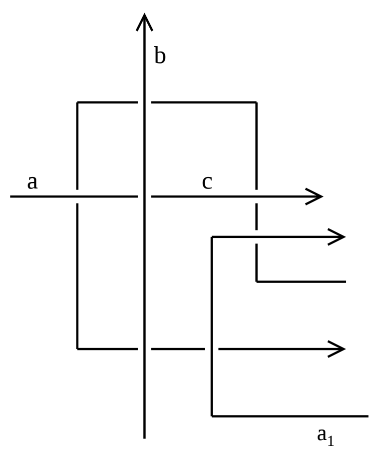

FIGURE 10. Dragging under C

diagram we have arrived at exhibits $a_1^{-1}c^{-1}b^{-1}aba_1$ to be realisable with respect to q.

If the exponent of a_1 is -1, we proceed in exactly the same way except that the dragged piece of a_1 should have the opposite orientation, as in Figure 11. A virtual R1 move can be used if necessary to achieve this.

Suppose now that the exponent of a_2 is $+1$. To exhibit the realisability of

$$a_2^{-1}a_1^{-1}c^{-1}b^{-1}aba_1a_2,$$

we proceed in the same way as above, but drag a piece of the arc a_2 in addition so that it nestles inside the dragged piece of a_1, to end up with the configuration shown in Figure 12 when the exponent of a_1 is $+1$, and a similar one with the a_2 (resp a_1) arcs having the opposite orientation if the exponent of a_2 (resp a_1) is -1.

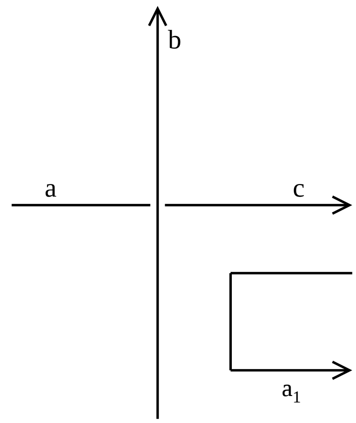

FIGURE 11. Dragging a_1 so that we have the opposite orientation

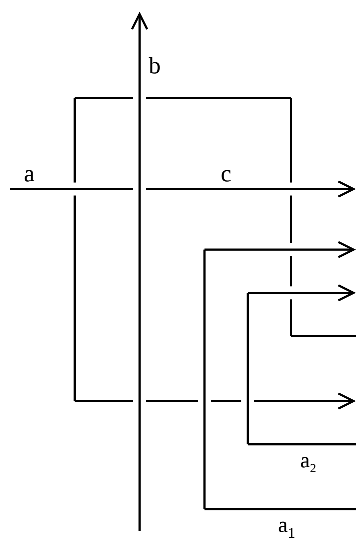

FIGURE 12. With both a_1 and a_2

Proceeding entirely analogously for all of $a_3, \ldots, a_n$ exhibits the realisability of

$$g^{-1} w_C^{\pm 1} g$$

for this type of C when the exponent of w_C is $+1$.

The proof when the arc b has the opposite orientation in C, as on the right in Figure 7, is identical. For example, Figure 13 is the analogue of Figure 10 when it comes to exhibiting that $a_1^{-1} c^{-1} b a b^{-1} a_1$ is realisable.

The proof when b has either orientation and when w_C has exponent -1 is entirely analogous to that when the exponent of w_C is $+1$, except that the piece of arc just after q which is dragged is given the opposite orientation, using a virtual R1 move

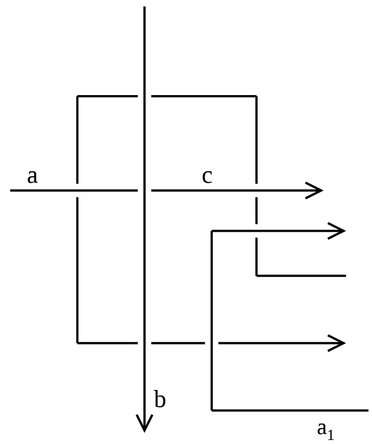

FIGURE 13. Dragging under C when b has the opposite orientation

if necessary. For example, Figure 14 is the analogue of Figure 10 when it comes to exhibiting that

$$a_1^{-1}b^{-1}a^{-1}bca_1$$

is realisable.

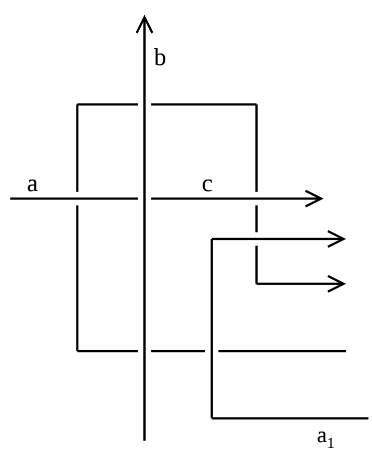

FIGURE 14. Dragging under C when w_C has exponent -1

□

We shall also require the following.

Lemma 3. *Let $m \geq 1$, and let p be the point of K at which we start when defining the longitude l of K. Let w be a word in the free group on the arcs of K which is completely realisable from p when viewed as a word in the free group on the arcs of* *an $(m-1)$-parallelisation of K, taking w to be the empty word when $m = 1$. Then there is an m-parallelisation V of K such that every word*

$$wg^{-1}l^{\pm 1}g$$

in the free group on the arcs of K is completely realisable from p when viewed in the free group on the arcs of V, where there are arcs $a_1, \ldots, a_n$ of K such that $g = a_1^{\pm 1} \cdots a_n^{\pm 1}$.

Proof. Take V to be an m-parallelisation of K at p in which the orientation of the m-th track is chosen so that we obtain $l^{\pm 1}$ by walking along it, constructing a word in the same manner as in the definition of longitude. The i-th track of V is taken to be that of the $(m-1)$-parallelisation of K in the statement of the lemma.

Suppose first that the exponent of a_1 is $+1$. Using only virtual R2 moves, drag a piece of the arc a_1 so that it is near the point p' corresponding to p on the m-th track of V, using a virtual R1 move if necessary so that the orientation of it is as in Figure 15. Apply a classical R2 move to obtain Figure 16.

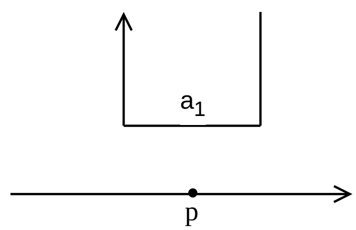

FIGURE 15. Dragging a_1

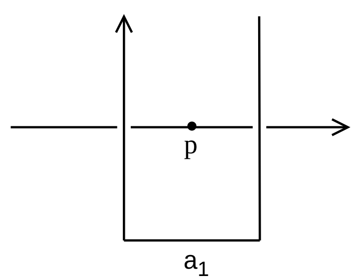

FIGURE 16. After an R2 move

If instead the exponent of a_1 is -1, proceed identically except for ensuring, using a virtual R1 move if necessary, that the arc a_1 has the opposite orientation. We end up with Figure 17 in this case.

Similarly, drag the arc a_2 near p' using only virtual R2 moves, and if necessary a virtual R1 move. Follow this by a classical R2 move, so that we end up with Figure 18 in the case that the exponent of a_2 is $+1$, and the same but with a_2 having the opposite orientation if the exponent of a_2 is -1.

Do the same for the remaining arcs $a_3, \ldots, a_n$. The longitude of the oriented virtual knot diagram we obtain in the end is exactly $wg^{-1}lg$. □

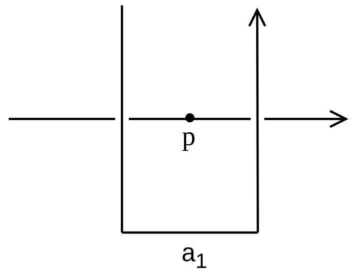

FIGURE 17. With the opposite orientation of a_1

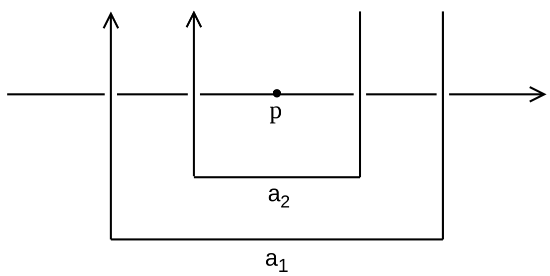

FIGURE 18. With a_2 as well

We deduce the following.

Lemma 4. *Let p be the point on an arc of K at which we start when defining the longitude l of K. Let w be a word in the free group on the arcs of K of the form*

$$g_1^{-1}w_1g_1 \cdots g_n^{-1}w_ng_n,$$

where each g_i is as in the statement of Lemma 3; where each w_i is either $w_C^{\pm 1}$ for a classical crossing C of K or is $l^{\pm 1}$; and where there are exactly m indices i for which w_i is $l^{\pm 1}$. Then there is an m-parallelisation V of K such that w is completely realisable from p when viewed as a word in free group on the arcs of V.

Proof. Take V to be an m-parallelisation of K at p in which the orientation of the j-th track of V is the sign of the exponent of l in the j-th (reading from the left) term w_i of w which is equal to $l^{\pm 1}$.

If w_1 is $w_C^{\pm 1}$ for a crossing C of K, viewed as a crossing of V, then by Lemma 2, we have that $g_1^{-1}w_1g_1$ is realisable with respect to p. If instead w_1 is $l^{\pm 1}$, then by Lemma 3 (when $m = 1$) we have that $g_1^{-1}w_1g_1$ is completely realisable from p.

Similarly, if w_2 is $w_C^{\pm 1}$ for a crossing C of K, and w_1 is $l^{\pm 1}$, then by Lemma 2, with respect to the point p' which corresponds to p on the second track of V, we have that

$$g_1^{-1}w_1g_1g_2^{-1}w_2g_2$$

is completely realisable from p. If, instead, w_1 too is $w_{C'}$ for a crossing C' of K, then by Lemma 2 with respect to a point p' on the arc upon which one emerges after passing under the final crossing involved in exhibiting the realisability of $g_1^{-1}w_1g_1$, we obtain that

$$g_1^{-1} w_1 g_1 g_2^{-1} w_2 g_2$$

is realisable with respect to p.

If both w_1 and w_2 are $l^{\pm 1}$ (their exponents not necessarily having the same sign), then by Lemma 3 (when $m = 2$), it follows from the fact that $g_1^{-1} w_1 g_1$ is completely realisable with respect to p that so is

$$g_1^{-1} w_1 g_1 g_2^{-1} w_2 g_2.$$

Proceeding inductively in this way for all i, appealing to Lemma 2 or to Lemma 3 as required, we obtain that

$$g_1^{-1} w_1 g_1 \cdots g_n^{-1} w_n g_n$$

is completely realisable from p.

□

One can prove an analogue of Lemma 4 using a connected sum of m copies of K with itself, rather than an m-parallelisation of K. Where we shall later work with a quotient of the fundamental rack of an m-parallelisation of K, one could then instead work with the quotient of the fundamental rack of the connected sum by the kernel of the fold morphism to the fundamental rack of K (this quotient rack is isomorphic to the fundamental rack of K).

This would allow one to identify the label of an arc on a copy of K with the label of the corresponding arc of K, and thus label all of the arcs of the connected sum using only the labels of the arcs of K. However, Lemma 5, to which we shall now turn, does not immediately adapt to such a setting.

It might, though, be possible to prove a version of Lemma 5 algebraically, working directly with the fundamental rack. Such a proof would go through when working with connected sums too, allowing one to carry out an analogue of the proof we shall give of Theorem 1 in that setting.

Lemma 5. *Let V be an oriented virtual knot diagram whose longitude is equal in the free group on the arcs of V to $b^{\pm 1}$, for a single arc b of V. Then V is welded-equivalent to the (classical) ± 1-framed unknot.*

Proof. If the longitude l of V is equal to $b^{\pm 1}$ on the nose, ie V has a single classical crossing, the lemma holds, since any virtual crossings can be removed using virtual Reidemeister moves. Otherwise, since l cancels to b using free group relations, there must be an arc a of V such that there is a sub-word of l of the form $a^{\pm 1} w a^{\mp 1}$ in which the two outer terms involving a cancel one another, ie w cancels to the empty word. In fact there must be such a sub-word in which w is of the form $a^{\pm 1} a^{\mp 1}$, ie in which w is empty, but the more general situation will be more convenient as an assumption.

Without loss of generality, we can assume that $a \neq b$. For if there is no such a, every term of the longitude of V involves b, ie all crossings of V have b as their over-arc. A straightforward argument, using welded Reidemeister moves as we defined them where needed, in addition to classical R2 moves, proves the lemma in this case; one can argue as below, for instance, with the simplification that in all cases, there are no classical crossings of the arc c, and no classical crossings in the interior of R.

By definition of l, to obtain a sub-word of it of the form $a^{\pm1}wa^{\mp1}$ in which the two outer terms involving a cancel one another, we must pass under the arc a (at least) twice when walking around V, in accordance with the latter's orientation. Thus V is locally as in Figure 19, except that the arc a might have the opposite orientation. There might be any number of classical crossings under or over the horizontal strand; any number of classical crossings under, but not over, the part of the arc a that is depicted; and any number of virtual crossings of either arc. Let us denote any of the arcs into which the horizontal strand is broken by c.

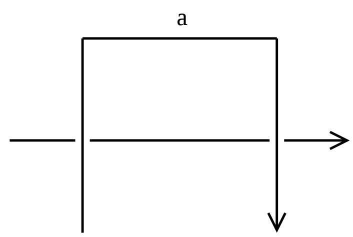

FIGURE 19. Part of V

Let R be the region defined by the horizontal strand and a. Without loss of generality, we can assume that $c \neq b$. For if $c = b$, there is a sub-word of l of the form $a^{\pm1}wb^{\pm1}$ or $b^{\pm1}wa^{\pm1}$, where the exponents of a and w may be the same or differ, where w does not contain a or a^{-1}, and where the term $b^{\pm1}$ is that which survives when cancelling l. In either case, the term involving $a^{\pm1}$ must be cancelled by a second such term. Together with a, the horizontal strand between the crossings under a giving rise to this cancellation defines a second region as in Figure 19, but this strand cannot have b as one of its arcs: we walk along the entirety of it either before or after we walk along b and respectively enter or leave R.

Suppose that it is possible to walk (in accordance with the orientation of V) under c via a classical crossing into the interior of R, and later under c again to exit R, without passing under a in the meantime. There is then a region of the plane which is exactly as in Figure 19, but whose interior is entirely strictly within R.

By induction on region size, we deduce that we can assume that if we walk into the interior of R under a crossing of c, we must exit R via a crossing under a. We might encounter classical and virtual crossings along the way, but no classical crossings under c.

Applying classical R2 and R3 moves and welded Reidemeister moves in the sense we defined them, slide the arc a downwards in the plane and over c to replace Figure 19 by Figure 20 (where again there may be virtual and classical crossings which are not depicted).

Suppose that an arc passes (classically) under the part of the arc a that is above c via a crossing C. After having walked (in accordance with the orientation of V) under the arc a via C, we must (though we might encounter classical or virtual crossings along the way) eventually walk under the arc a for a second time.

Indeed, even if we first walk under c, because such a crossing contributes to the longitude of V and must therefore have a term of the longitude which cancels it, we must later re-enter the region R by walking under c; this entering and re-entering may occur any number of times for any of arcs of the horizontal strand,

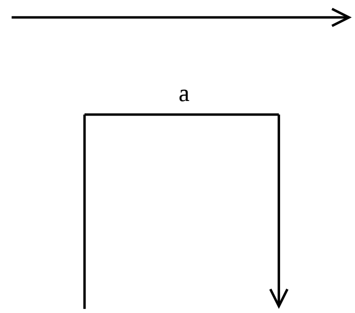

FIGURE 20. After the slide of a

encountering any number of classical or virtual crossings along the way (including possibly of the arc a), but eventually there must be a point at which we exit R by passing under a, and after which there are no further crossings under the horizontal strand. Let us denote this crossing of a by C'.

The longitude of V then has a sub-word x which either i) cancels to the empty word, and is of the form $a^{\pm 1} w a^{\mp 1}$, with the term $a^{\pm 1}$ coming from C, and the term $a^{\mp 1}$ coming from C'; or ii) cancels either to the empty word or to $b^{\pm 1}$, and is of the form

$$\left(a^{\mp 1} v_1 a^{\pm 1}\right) w \left(a^{\mp 1} v_2 a^{\pm 1}\right),$$

with the first $a^{\pm 1}$ term coming from C, the second $a^{\mp 1}$ term coming from C', and where the two parentheses each cancel to the empty word.

If i) holds, the effect of the slide of the arc a resulting in Figure 20 is to replace x by w, which is cancellable to the empty word since x is. If ii) holds, the effect of the slide is to replace x by $a^{\mp 1} v_1 w v_2 a^{\pm 1}$, which is cancellable to the empty word (resp $b^{\pm 1}$) if x is, since, under the assumptions of ii), v_1, v_2 both individually cancel to the empty word, and w cancels to the empty word (resp $b^{\pm 1}$) if x does.

Thus, in all cases, the slide from Figure 19 to Figure 20 results in an oriented virtual knot diagram whose longitude is still cancellable to $b^{\pm 1}$. Moreover, this diagram has strictly fewer classical crossings than V, since at least the two crossings in Figure 19 are removed, whilst our assumptions on R, specifically that if an arc enters R under c then it must exit under a, ensure that every crossing under the arc c in Figure 20 has a corresponding crossing of a above c in Figure 19.

The lemma follows immediately by induction on the number of classical crossings of V.

□

We shall also require the following.

Lemma 6. *Let V be an m-parallelisation of K at a point p of K, for any $m \geq 1$. The quotient of the fundamental rack of V defined by identifying the first arc of the first track of V with the final arc of this track is isomorphic to the fundamental rack of K.*

Proof. Suppose that $m > 1$ (the lemma is tautological when $m = 1$). Let us denote the final arc of the first track of V by a. The first classical crossing C of the second track of V gives rise, depending on the orientation of the over-arc b of C, to a relation $a \triangleright b = x$ or $a \triangleleft b = x$ in the fundamental rack of V, where x is the label of the arc of the second track of V along which we travel after passing under C.

Suppose that it is the relation $a \triangleright b = x$ that we have; the argument in the case that $a \triangleleft b = x$ is entirely analogous. But, since we identify a with the first arc of the first track of V, we have that $a \triangleright b = c$ in the quotient rack, where c is the arc of K, viewed as an arc of V, along which we travel having passed under b in K. We deduce that $x = c$ in the quotient rack.

Continuing to argue in this way, walking around the entirety of V, we obtain that every arc introduced in the construction of V becomes equal in the quotient rack to an arc of K (viewed as an arc of V), in such a way that the Wirtinger relations for all of the classical crossings introduced in the construction of V are identical to the ones coming from crossings of K.

□

We now bring everything together. We shall make use of the fact that the fundamental rack for virtual knots agrees with the usual definition for classical knots, and that it is unchanged up to isomorphism by virtual Reidemeister moves and welded Reidemeister moves [1].

Proof of Theorem 1. Since $\pi_1(K)/\langle l \rangle$ is trivial, every element of the free group on the arcs of K lies within the normal closure of the set

$$\{w_C \mid C \text{ is a crossing of } K\} \cup \{l\}.$$

There must be at least one arc b of K which is not equal in the free group on the arcs of K to a product of conjugates of words of the form w_C for C a classical crossing of K, since otherwise $\pi_1(K)$ would be trivial, which is impossible for a knot.

We then have that

$$b = g_1^{-1} w_1 g_1 \cdots g_n^{-1} w_n g_n,$$

in the free group on the arcs of K, where $g_1, \ldots, g_n$ and $w_1, \ldots, w_n$ are as in the hypotheses of Lemma 4, and w_i is equal to l for at least one i. Appealing to Lemma 4, we deduce that there is an m-parallelisation V of K, for some $m \geq 1$, whose longitude is equal (on the nose) to the right-hand side of the equality above.

By Lemma 5, V is welded-equivalent to a classical ± 1-framed unknot. Hence the fundamental rack of V is isomorphic to the fundamental rack of the ± 1-framed unknot, which is the unique rack with a single element.

It follows from Lemma 6 that the fundamental rack of K is the unique rack with a single element. Since the fundamental rack is a complete invariant of framed classical knots [2], we deduce that K is equivalent under classical R2 and R3 moves to a ± 1-framed unknot, as required.

□

3. Further directions

The author believes that the analogue of Theorem 1 for links should be able to be proven as in this article, though certain Kirby moves will be necessary, and aspects of our arguments must be adjusted.

One might also try to replace the trivial group by other groups in the statement of Theorem 1. For the integers, one would aim to prove that K must be equivalent to the 0-framed unknot. This is the Property R conjecture, proven in a very different way in [3, 4]. For finite cyclic groups, the picture is more complicated, but in certain cases one would aim to prove that K is equivalent to a $\pm n$-framed unknot for some $n > 1$.

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