Cylindrical Model Structures

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Dedicated to Kari Chard

Abstract

We build a model structure from the simple point of departure of a structured interval in a monoidal category — more generally, a structured cylinder and a structured co-cylinder in a category.

Contents

I. Introduction

Model structures are cherished — powerful, but hard to construct. In this work we build a model structure from the simple point of departure of a structured interval in a monoidal category — more generally, a structured cylinder and a structured co-cylinder in a category.

Abstract homotopy theory

The first steps towards an abstract homotopy theory were, for us, taken in the early 1950s by Kan — though there is a considerable prehistory, for example in the work of Whitehead. Kan isolated in [24] the notion of a cylinder in a category and of homotopy with respect to a cylinder.

Kan's immediate interest lay in the categories of cubical and simplicial sets. Whilst both categories admit a cylinder, the corresponding homotopies cannot be composed or reversed. What are now known as Kan complexes were introduced as a remedy.

Abstract homotopy theory has subsequently evolved in two branches, which have been explored rather independently. The first is the theory of model categories and its many variants and weakenings. The second is much less known. Its origins lie in the observation that the cylinder in topological spaces admits a much richer structure than the cylinder of the categories of simplicial or cubical sets. Two directions have been followed in capturing this richer structure of the topological cylinder in an abstract setting.

The first, begun by Kamps in the late 1960s in works such as [21], explores the homotopy theory with respect to a cylinder whose associated cubical set satisfies properties similar to those of a Kan complex. This is a global approach, with the axioms requiring consideration of all arrows of a category.

The second emerges out of works of Brown, Higgins, and others on cubical sets with connections, for example the paper [6]. It is of a structural and categorical nature, involving a rainbow of natural transformations intertwining a cylinder and its corresponding double cylinder. This approach has been explored by Grandis in works such as [13].

The book [23] of Kamps and Porter gives a nice overview of both directions, with an emphasis on the first.

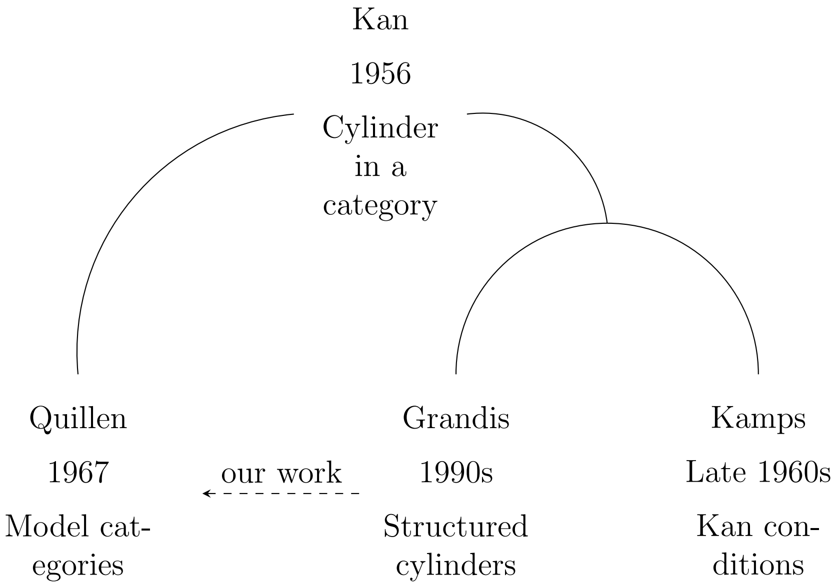

Figure: Approaches to abstract homotopy theory

Outline

The present work builds a bridge to model categories from the categorical approach to homotopy theory via structured intervals, cylinders, and co-cylinders. The theorems towards which all of our work leads are to be found in XV. Given a richly structured cylinder and co-cylinder in a category satisfying a certain strictness hypothesis, we prove that homotopy equivalences, cofibrations, and fibrations — defined from an abstract point of view in the same way as for topological spaces — equip this category with a model structure.

More precisely, our theory typically gives rise to not one, but two model structures. We introduce in VIII a notion of a normally cloven cofibration, and of a normally cloven fibration. One of our model structures is defined by homotopy equivalences, cofibrations, and normally cloven fibrations, whilst the other is defined by homotopy equivalences, normally cloven cofibrations, and fibrations.

The structures on a cylinder and a co-cylinder with which we work are defined in III, along with our strictness hypothesis. Often, in practise, we construct a structured cylinder and a structured co-cylinder by means of a structured interval in a monoidal category. This is discussed in VI.

We work in a 2-categorical setting, introduced in II, which allows us to express a duality between homotopy theory with respect to a cylinder on the one hand, and homotopy theory with respect to a co-cylinder on the other. This duality manifests itself throughout.

In IV, we discuss a notion of adjunction between a cylinder and a co-cylinder, in the presence of which the corresponding homotopy theories coincide. For the remainder of this outline, we shall indicate neither the particular structures involved at different points, nor whether we are working with a cylinder, a co-cylinder, or both. These matters are carefully treated in the remainder of the work.

In VII, we define homotopies and relative homotopies with respect to a cylinder or co-cylinder. We demonstrate that we can compose and reverse them.

If our strictness hypothesis holds, we prove in IX that the mapping cylinder of a map gives rise to a factorisation of it into a normally cloven cofibration followed by a strong deformation retraction, and that the mapping co-cylinder of a map gives rise to a factorisation of it into a section of strong deformation retraction followed by a normally cloven fibration.

In XI, we characterise trivial fibrations as strong deformation retractions, and characterise trivial cofibrations as sections of strong deformation retractions. This is by means of an abstraction of Dold's theorem for topological spaces, on homotopy equivalences under or over an object.

Assuming once more that our strictness hypothesis holds, we prove in XIII that a trivial cofibration is exactly a section of a strong deformation retraction, and dually that a trivial fibration is exactly a strong deformation retraction. With our mapping cylinder and mapping co-cylinder factorisations to hand, we deduce that the factorisation axioms for a model structure hold.

In XII, we prove that the canonical map from the mapping cylinder of a map to the cylinder at its target admits a strong deformation retraction. We prove that normally cloven fibrations have the right lifting property with respect to sections of strong deformation retractions. We deduce that normally cloven fibrations have the covering homotopy extension property — introduced in X — with respect to cofibrations.

This allows us to prove that cofibrations have the left lifting property with respect to trivial normally cloven fibrations. We conclude that the lifting axioms hold for one of our model structures. The lifting axioms for the other model structure follow by duality.

Folk model structure

This work was conceived as a step towards the construction of a model category of n-groupoids satisfying, in a strong sense, the homotopy hypothesis. The aim was to construct the folk model structure on categories and groupoids in a way which we could generalise to n-groupoids.

We demonstrate in XVI that our work indeed gives a new construction of the folk model structure on categories and groupoids. The construction of a model category of n-groupoids by means of the present work is the point of departure of joint work with Marius Thaule which is in preparation.

Further examples

Our work gives rise to many other model structures. We discuss three.

(1) Let $\mathsf{Ch}(\mathcal{A})$ denote the category of chain complexes in an additive category $\mathcal{A}$ with finite limits and colimits. The cylinder and co-cylinder functors

$$\mathsf{Ch}(\mathcal{A}) \longrightarrow \mathsf{Ch}(\mathcal{A})$$

of homological algebra can be equipped with all the structures of III. We refer the reader to §4.4.2 of the book [15] of Grandis, for example. Our strictness hypothesis is satisfied.

Our work thus gives a model structure on $\mathsf{Ch}(\mathcal{A})$ whose weak equivalences are chain homotopy equivalences. This model structure was constructed in a quite different way by Golasiński and Gromadzki in [11], appealing to a characterisation due to Kamps in [22] of the fibrations and cofibrations.

(2) Let Kan_{Δ} denote the category of algebraic Kan complexes introduced by Nikolaus in [27]. The objects of Kan_{Δ} are Kan complexes with a chosen filling for every horn. The arrows of Kan_{Δ} are morphisms of simplicial sets which respect the chosen horn fillings.

Our work gives a model structure on Kan_{Δ}, which we think of as akin to the model structure on topological spaces constructed by Strøm in [34]. A different model structure on Kan_{Δ} was constructed by Nikolaus in [27], which we think of as akin to the Serre model structure on topological spaces.

The author conjectures that the identity functor defines a Quillen equivalence between these two model structures on Kan_{Δ}.

(3) Let Top denote the category of all topological spaces. The unit interval is exponentiable with respect to the cartesian monoidal structure on Top and can be equipped with all of the structures of VI.

Our strictness hypothesis does not however hold. Thus our work does not immediately give rise to a model structure on Top.

Nevertheless, homotopy equivalences in Top can be understood as homotopy equivalences with respect to the Moore co-cylinder, which to a topological space X associates the set of pairs (t, f) of a real number $t \in [0, \infty)$ and a map

$$[0, \infty) \xrightarrow{f} X$$

such that $f(x) = t$ for all $x \geq t$, where this set is equipped with the subspace topology with respect to $[0, \infty) \times X^{[0,\infty)}$. Our strictness hypothesis does hold for the Moore co-cylinder.

The Moore co-cylinder does not however admit connection structures. There are two ways to get around this. Firstly, it is possible to generalise our work slightly to double cylinders and double co-cylinders which are not necessarily obtained by applying the cylinder or co-cylinder functor twice. We can then take our Moore double co-cylinder to consist of Moore rectangles as considered in the paper [5] of Brown. Secondly, it should be possible to replace the connection structures in our work by the 'strengths' of the paper [36] of van den Berg and Garner.

Following either route, we can obtain the model structure on Top constructed by Strøm in [34] by working with respect to both the Moore co-cylinder and the usual cylinder and co-cylinder in Top.

Around Easter 2012, Tobias Barthel and Bill Richter suggested to the author a construction of the Moore co-cylinder which could be carried out in a quite general setting. Thus our side-stepping of the failure of the strictness hypothesis to hold in Top may be able to be carried out more widely.

The significance of the Moore co-cylinder for the construction of the Strøm model structure on Top is explored in the paper [2] of Barthel and Riehl.

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II. Formal category theory preliminaries

In III, we introduce various structures with which a cylinder or co-cylinder may be able to be equipped. The structures upon a co-cylinder are formally dual to those upon a cylinder. This duality, which is of a 2-categorical rather than a 1-categorical nature, as was already observed by Gray in [17], will manifest itself throughout this work.

In order to express the duality we shall work throughout in a strict 2-category equipped with a strict final object. We think of the objects of this 2-category as formal categories. We now introduce these ideas as far as we shall need.

Assumption II.1. Let $\mathcal{C}$ be a strict 2-category, and let 1 be a final object of $\mathcal{C}$.

Definition II.2. Let $\mathcal{A}$ be an object of $\mathcal{C}$. An *object* of $\mathcal{A}$ is a 1-arrow

$$1 \longrightarrow \mathcal{A}$$

of $\mathcal{C}$.

Definition II.3. Let $\mathcal{A}$ be an object of $\mathcal{C}$. Given objects a_0 and a_1 of $\mathcal{A}$, an *arrow* of $\mathcal{A}$ from a_0 to a_1 is a 2-arrow

$$a_0 \longrightarrow a_1$$

of $\mathcal{C}$.

Remark II.4. Let $\underline{\mathsf{CAT}}$ denote the strict 2-category of large categories. The definitions above are motivated by the observation that, for any category $\mathcal{A}$, the category $\underline{\mathsf{Hom}}_{\underline{\mathsf{CAT}}}(1, \mathcal{A})$ is isomorphic to $\mathcal{A}$.

Definition II.5. Let $\mathcal{A}$ be an object of $\mathcal{C}$, and let

$$a_0 \xrightarrow{f_0} a_1$$

and

$$a_1 \xrightarrow{f_1} a_2$$

be arrows of $\mathcal{A}$. The *composition* $f_1 \circ f_0$ of f_1 and f_0 in $\mathcal{A}$ is their composition in $\underline{\mathsf{Hom}}_{\mathcal{C}}(1, \mathcal{A})$.

Definition II.6. A *diagram* in $\mathcal{A}$ is a diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(1, \mathcal{A})$. We define a *commutative diagram* in $\mathcal{A}$, a *co-cartesian square* in $\mathcal{A}$, and a *cartesian square* in $\mathcal{A}$ in the same way.

Notation II.7. Given 1-arrows

$$\begin{tikzcd} \mathcal{A}_0 \arrow[r, shift left, "F_0"] \arrow[r, shift right, "F_1"'] & \mathcal{A}_1 \arrow[r, "G"] & \mathcal{A}_2 \end{tikzcd}$$

of $\mathcal{C}$, and a 2-arrow

$$F_0 \xrightarrow{\eta} F_1$$

of $\mathcal{C}$, we denote by $G \cdot \eta$ the 2-arrow

$$GF_0 \longrightarrow GF_1$$

of $\mathcal{C}$ obtained by horizontal compositition of $id(G)$ and η. It is sometimes referred to as the *whiskering* of G and η.

Similarly, given 1-arrows

$$\begin{tikzcd} \mathcal{A}_0 \arrow[r, "G"] & \mathcal{A}_1 \arrow[r, shift left, "F_0"] \arrow[r, shift right, "F_1"'] & \mathcal{A}_2 \end{tikzcd}$$

of $\mathcal{C}$, and a 2-arrow

$$F_0 \xrightarrow{\eta} F_1$$

of $\mathcal{C}$, we denote by $\eta \cdot G$ the 2-arrow

$$F_0G \longrightarrow F_1G$$

of $\mathcal{C}$ obtained by horizontal composition of η and $id(G)$. It is also sometimes referred to as the *whiskering* of η and G.

Notation II.8. Let

$$\mathcal{A}_0 \xrightarrow{F} \mathcal{A}_1$$

be a 1-arrow of $\mathcal{C}$. If a is an object of $\mathcal{A}_0$, we denote by $F(a)$ the object of $\mathcal{A}_1$ defined by the 1-arrow $F \circ a$ of $\mathcal{C}$.

If

$$a_0 \xrightarrow{f} a_1$$

is an arrow of $\mathcal{A}_0$, we denote by $F(f)$ the arrow of $\mathcal{A}_1$ from $F(a_0)$ to $F(a_1)$ defined by the whiskered 2-arrow $F \cdot f$ of $\mathcal{C}$.

Remark II.9. In this way, we think of a 1-arrow

$$\mathcal{A}_0 \xrightarrow{F} \mathcal{A}_1$$

of $\mathcal{C}$ as a functor from $\mathcal{A}_0$ to $\mathcal{A}_1$, corresponding to the functor

$$\underline{\mathsf{Hom}}_{\mathcal{C}}(1, \mathcal{A}_0) \xrightarrow{\underline{\mathsf{Hom}}_{\mathcal{C}}(1, F)} \underline{\mathsf{Hom}}_{\mathcal{C}}(1, \mathcal{A}_1).$$

Notation II.10. Let

$$\begin{tikzcd} \mathcal{A}_0 \arrow[r, shift left, "F_0"] \arrow[r, shift right, "F_1"'] & \mathcal{A}_1 \end{tikzcd}$$

be 1-arrows of $\mathcal{C}$, and let

$$F_0 \xrightarrow{\eta} F_1$$

be a 2-arrow of $\mathcal{C}$. If a is an object of $\mathcal{A}_0$, we denote by $\eta(a)$ the arrow of $\mathcal{A}_1$ from $F_0(a)$ to $F_1(a)$ defined by the whiskered 2-arrow $\eta \cdot a$ of $\mathcal{C}$.

Remark II.11. Let

$$\begin{tikzcd} \mathcal{A}_0 \arrow[r, shift left, "F_0"] \arrow[r, shift right, "F_1"'] & \mathcal{A}_1 \end{tikzcd}$$

be 1-arrows of $\mathcal{C}$, thought of as functors from $\mathcal{A}_0$ to $\mathcal{A}_1$, as in Remark II.9. We think of a 2-arrow

$$F_0 \xrightarrow{\eta} F_1$$

of $\mathcal{C}$ as a natural transformation from F_0 to F_1, corresponding to the natural transformation

$$\underline{\mathsf{Hom}}_{\mathcal{C}}(1, F_0) \xrightarrow{\underline{\mathsf{Hom}}_{\mathcal{C}}(1, \eta)} \underline{\mathsf{Hom}}_{\mathcal{C}}(1, F_1).$$

Notation II.12. We denote by $\mathcal{C}^{op}$ the 2-category obtained from $\mathcal{C}$ by reversing all 2-arrows. If f is a 2-arrow of $\mathcal{C}$, we denote by f^{op} the corresponding 2-arrow of $\mathcal{C}^{op}$. When viewing an object $\mathcal{A}$ of $\mathcal{C}$ as an object of $\mathcal{C}^{op}$, we denote it by $\mathcal{A}^{op}$. Thus $\underline{\mathsf{Hom}}_{\mathcal{C}^{op}}(1, \mathcal{A}^{op})$ is the opposite category of $\underline{\mathsf{Hom}}_{\mathcal{C}}(1, \mathcal{A})$. In particular, if

$$a_0 \xrightarrow{f} a_1$$

defines an arrow of $\mathcal{A}$, then the 2-arrow f^{op} of $\mathcal{C}^{op}$ defines an arrow of $\mathcal{A}^{op}$ from a_1 to a_0.

Recollection II.13. An *adjunction* between a pair (F, G) of 1-arrows

$$\begin{tikzcd} \mathcal{A}_0 \arrow[r, shift left, "F"] & \mathcal{A}_1 \arrow[l, shift left, "G"] \end{tikzcd}$$

of $\mathcal{C}$ is a 2-arrow

$$id(\mathcal{A}_0) \xrightarrow{\eta} GF$$

of $\mathcal{C}$, and a 2-arrow

$$FG \xrightarrow{\zeta} id(\mathcal{A}_1)$$

of $\mathcal{C}$, such that the diagram

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "id"'] & FGF \arrow[d, "\zeta F"] \\ & F \end{tikzcd}$$

in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}_0, \mathcal{A}_1)$ commutes, and such that the diagram

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "id"'] & GFG \arrow[d, "G\zeta"] \\ & G \end{tikzcd}$$

in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}_1, \mathcal{A}_0)$ commutes. We refer to F as a *left adjoint* of G, and to G as a *right adjoint* of F.

If we have an adjunction between F and G, then for any object $\mathcal{A}$ of $\mathcal{C}$, the natural transformations $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \eta)$ and $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \zeta)$ define an adjunction between the following pair of functors.

$$\begin{tikzcd} \underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A}_0) \arrow[r, shift left, "\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, F)"] & \underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A}_1) \arrow[l, shift left, "\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, G)"] \end{tikzcd}$$

In particular, we have a natural isomorphism

$$\mathsf{Hom}_{\underline{\mathsf{Hom}}_{\mathcal{C}}(1, \mathcal{A}_1)}\Big(\underline{\mathsf{Hom}}_{\mathcal{C}}(1, F(-)), -\Big) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\underline{\mathsf{Hom}}_{\mathcal{C}}(1, \mathcal{A}_0)}\Big(-, \underline{\mathsf{Hom}}_{\mathcal{C}}(1, G(-))\Big)$$

of functors

$$(\mathsf{Hom}_{\mathcal{C}}(1, \mathcal{A}_0))^{op} \times \mathsf{Hom}_{\mathcal{C}}(1, \mathcal{A}_1) \longrightarrow \mathsf{Set}.$$

Adopting the shorthand

$$\mathcal{A} \longleftrightarrow \underline{\mathsf{Hom}}_{\mathcal{C}}(1, \mathcal{A}),$$

$$F \longleftrightarrow \underline{\mathsf{Hom}}_{\mathcal{C}}(1, F),$$

$$G \longleftrightarrow \underline{\mathsf{Hom}}_{\mathcal{C}}(1, G),$$

we shall write the above natural isomorphism as

$$\mathsf{Hom}_{\mathcal{A}_1}(F(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}_0}(-, G(-)).$$

Definition II.14. If for any objects $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{C}$, any object a of $\mathcal{A}_0$, and any co-cartesian (respectively cartesian) square

$$\begin{tikzcd} F_0 \arrow[r, "\eta_0"] \arrow[d, "\eta_2"'] & F_1 \arrow[d, "\eta_1"] \\ F_2 \arrow[r, "\eta_3"'] & F_3 \end{tikzcd}$$

in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}_0, \mathcal{A}_1)$, the square

$$\begin{tikzcd} F_0(a) \arrow[r, "\eta_0(a)"] \arrow[d, "\eta_2(a)"'] & F_1(a) \arrow[d, "\eta_1(a)"] \\ F_2(a) \arrow[r, "\eta_3(a)"'] & F_3(a) \end{tikzcd}$$

in $\mathcal{A}_1$ is co-cartesian (respectively cartesian), we write that *pushouts (respectively pullbacks) of 2-arrows of $\mathcal{C}$ give rise to pushouts (respectively pullbacks) in formal categories.*

Remark II.15. In both $\underline{\mathsf{CAT}}$ and $\underline{\mathsf{CAT}}^{op}$, pushouts and pullbacks of 2-arrows give rise to pushouts and pullbacks in formal categories, more or less by definition. The author expects that colimits and limits of 2-arrows equally give rise to colimits and limits in formal categories for quite general 2-categories, for instance 2-topoi.

Remark II.16. It would certainly be possible for us to work with weak rather than strict 2-categories. However, we are motivated by ordinary categories. In addition to

$\underline{\mathsf{CAT}}$, the only 2-category of importance to us is $\underline{\mathsf{CAT}}^{op}$, exactly in order to capture duality. Thus it is sufficient for us to work with strict 2-categories, and we do so in order to avoid the distraction of coherency.

III. Structures upon a cylinder or a co-cylinder

We introduce the notion of a cylinder or a co-cylinder in a formal category. We define the structures — contraction, involution, subdivision, and three flavours of connection — upon a cylinder and a co-cylinder which play a role in our work, and introduce axioms expressing their compatibility. We refer the reader to XVI for an example of these structures in the category of categories.

These structures and axioms have previously appeared in the literature. However, the precise definitions and terminology vary from author to author and paper to paper. Thus we collect in one place and from a single point of view all that we shall need. Most of our structures and axioms can be found in §4 of Chapter I and at the end of §3 of Chapter II of the book [23] of Kamps and Porter, or in §2.1 and §2.3 of the paper [13] of Grandis. Compatibility of right connections with subdivision appears implicitly, and in a slightly different context, in §6.4 of [7].

We also introduce strictness hypotheses which our structures upon a cylinder or a co-cylinder may satisfy. We shall come in VII to define the notion of a homotopy with respect to a cylinder or a co-cylinder. The first of our strictness hypotheses, which we shall refer to as strictness of left or right identities, ensures that the left or right composition of an identity homotopy with a homotopy h is exactly h.

Strictness of identities will also allow us to prove in XIII that the mapping cylinder (respectively the mapping co-cylinder) of any arrow f yields a factorisation of f into a normally cloven cofibration followed by a trivial fibration (respectively into a trivial cofibration followed by a normally cloven fibration). It will moreover be crucial in establishing that the lifting axioms for a model category hold, in XII.

The significance of strictness of identities with regard to lifting has to the author's knowledge not previously been observed. Its importance with regard to factorisation was independently identified by van den Berg and Garner in [36]. We particularly draw the reader's attention to Remark 4.3.3 in [36]. Our strictness of left (respectively right) identities condition corresponds to the left (respectively right) unitality condition of van den Berg and Garner.

The second of our hypotheses, which we shall refer to as strictness of left inverses, ensures that the composition of a homotopy with its inverse is exactly an identity homotopy. It also allows us, given a lower right connection structure Γ_{lr}, to construct an upper right connection structure Γ_{ur} such that Γ_{lr} and Γ_{ur} are compatible with subdivision. The compatibility of right connections with subdivision will be vital for us when in X we investigate the covering homotopy extension property.

Assumption III.1. Let $\mathcal{C}$ be a 2-category, and let $\mathcal{A}$ be an object of $\mathcal{C}$.

Definition III.2. A *cylinder* in $\mathcal{A}$ is a 1-arrow

$$\mathcal{A} \xrightarrow{\mathsf{Cyl}} \mathcal{A}$$

of $\mathcal{C}$, together with a pair of 2-arrows

$$\begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & \mathsf{Cyl} \end{tikzcd}$$

of $\mathcal{C}$.

Definition III.3. A *co-cylinder* in $\mathcal{A}$ is a 1-arrow

$$\mathcal{A} \xrightarrow{\mathsf{co\text{-}Cyl}} \mathcal{A}$$

of $\mathcal{C}$, together with a pair of 2-arrows

$$\begin{tikzcd} \mathsf{co\text{-}Cyl} \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & \mathsf{id}_{\mathcal{A}} \end{tikzcd}$$

of $\mathcal{C}$.

Remark III.4. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Then $(\mathsf{co\text{-}Cyl}, e_0^{op}, e_1^{op})$ defines a cylinder in $\mathcal{A}^{op}$, which we denote by $\underline{\mathsf{co\text{-}Cyl}}^{op}$.

Definition III.5. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. A *contraction structure* with respect to $\underline{\mathsf{Cyl}}$ is a 2-arrow

$$\mathsf{Cyl} \xrightarrow{p} \mathsf{id}_{\mathcal{A}}$$

of $\mathcal{C}$, such that the following diagrams in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commute.

$$\begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"] \arrow[dr, "id"'] & \mathsf{Cyl} \arrow[d, "p"] \\ & \mathsf{id}_{\mathcal{A}} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, "i_1"] \arrow[dr, "id"'] & \mathsf{Cyl} \arrow[d, "p"] \\ & \mathsf{id}_{\mathcal{A}} \end{tikzcd}$$

Definition III.6. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. A *contraction structure* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is a 2-arrow

$$\mathsf{id}_{\mathcal{A}} \xrightarrow{c} \mathsf{co\text{-}Cyl}$$

of $\mathcal{C}$, such that c^{op} equips the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ with a contraction structure.

Definition III.7. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. An *involution structure* with respect to $\underline{\mathsf{Cyl}}$ is a 2-arrow

$$\mathsf{Cyl} \xrightarrow{v} \mathsf{Cyl}$$

of $\mathcal{C}$, such that the following diagrams in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commute.

$$\begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"] \arrow[dr, "i_1"'] & \mathsf{Cyl} \arrow[d, "v"] \\ & \mathsf{Cyl} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, "i_1"] \arrow[dr, "i_0"'] & \mathsf{Cyl} \arrow[d, "v"] \\ & \mathsf{Cyl} \end{tikzcd}$$

Definition III.8. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. An *involution structure* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is a 2-arrow

$$\mathsf{co\text{-}Cyl} \xrightarrow{v} \mathsf{co\text{-}Cyl}$$

of $\mathcal{C}$, such that v^{op} defines an involution structure with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Definition III.9. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. An involution structure v with respect to $\underline{\mathsf{Cyl}}$ is *compatible with* p if the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, "v"] \arrow[dr, "p"'] & \mathsf{Cyl} \arrow[d, "p"] \\ & \mathsf{id}_{\mathcal{A}} \end{tikzcd}$$

Definition III.10. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. An involution structure v with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is *compatible with* c if the involution structure v^{op} with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ is compatible with the contraction structure defined by c^{op}.

Definition III.11. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. A *subdivision structure* with respect to $\underline{\mathsf{Cyl}}$ is a 1-arrow

$$\mathcal{A} \xrightarrow{\mathsf{S}} \mathcal{A}$$

of $\mathcal{C}$, together with a pair of 2-arrows

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & \mathsf{S} \end{tikzcd}$$

of $\mathcal{C}$, such that the diagram

$$\begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"] \arrow[d, "i_1"'] & \mathsf{Cyl} \arrow[d, "r_0"] \\ \mathsf{Cyl} \arrow[r, "r_1"'] & \mathsf{S} \end{tikzcd}$$

in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ is co-cartesian, and a 2-arrow

$$\mathsf{Cyl} \xrightarrow{s} \mathsf{S}$$

of $\mathcal{C}$, such that the following diagrams in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commute.

$$\begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"] \arrow[d, "i_0"'] & \mathsf{Cyl} \arrow[d, "s"] \\ \mathsf{Cyl} \arrow[r, "r_1"'] & \mathsf{S} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, "i_1"] \arrow[d, "i_1"'] & \mathsf{Cyl} \arrow[d, "s"] \\ \mathsf{Cyl} \arrow[r, "r_0"'] & \mathsf{S} \end{tikzcd}$$

Definition III.12. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. A *subdivision structure* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is a 1-arrow

$$\mathcal{A} \xrightarrow{\mathsf{S}} \mathcal{A}$$

of $\mathcal{C}$, together with a pair of 2-arrows

$$\begin{tikzcd} \mathsf{S} \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & \mathsf{co\text{-}Cyl} \end{tikzcd}$$

of $\mathcal{C}$ and a 2-arrow

$$\mathsf{S} \xrightarrow{s} \mathsf{co\text{-}Cyl}$$

of $\mathcal{C}$, such that $(\mathsf{S}, r_0^{op}, r_1^{op}, s^{op})$ defines a subdivision structure with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Definition III.13. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Let

$$\mathsf{S} \xrightarrow{\overline{p}} \mathsf{id}_{\mathcal{A}}$$

denote the canonical 2-arrow of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"] \arrow[d, "i_1"'] & \mathsf{Cyl} \arrow[d, "r_0"] \arrow[ddr, bend left, "p"] & \\
\mathsf{Cyl} \arrow[r, "r_1"'] \arrow[drr, bend right, "p"'] & \mathsf{S} \arrow[dr, "\overline{p}"] & \\
& & \mathsf{id}_{\mathcal{A}}
\end{tikzcd}$$

in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes. The subdivision structure $(\mathsf{S}, r_0, r_1, s)$ is *compatible with* p if the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "s"] \arrow[dr, "p"'] & \mathsf{S} \arrow[d, "\overline{p}"] \\
& \mathsf{id}_{\mathcal{A}}
\end{tikzcd}$$

Definition III.14. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. A subdivision structure $(\mathsf{S}, r_0, r_1, s)$ with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is *compatible with* c if the subdivision structure $(\mathsf{S}, r_0^{op}, r_1^{op}, s^{op})$ with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ is compatible with the contraction structure defined by c^{op}.

Definition III.15. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Then Cyl *preserves subdivision* with respect to $\underline{\mathsf{Cyl}}$ if the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ is co-cartesian.

$$\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_0"] \arrow[d, "\mathsf{Cyl} \cdot i_1"'] & \mathsf{Cyl}^2 \arrow[d, "\mathsf{Cyl} \cdot r_0"] \\
\mathsf{Cyl}^2 \arrow[r, "\mathsf{Cyl} \cdot r_1"'] & \mathsf{Cyl} \circ \mathsf{S}
\end{tikzcd}$$

Definition III.16. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, \mathsf{S}, r_0, r_1, s)$ be a co-cylinder in $\mathcal{A}$ equipped with a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Then $\mathsf{co\text{-}Cyl}$ *preserves subdivision* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if $\mathsf{co\text{-}Cyl}$ preserves subdivision with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the subdivision structure $(\mathsf{S}, r_0^{op}, r_1^{op}, s^{op})$.

Definition III.17. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. An *upper left connection structure* with respect to $\underline{\mathsf{Cyl}}$ is a 2-arrow

$$\mathsf{Cyl}^2 \xrightarrow{\Gamma_{ul}} \mathsf{Cyl}$$

of $\mathcal{C}$, such that the following diagrams in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A},\mathcal{A})$ commute.

$$\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "i_0 \cdot \mathsf{Cyl}"] \arrow[dr, "id"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{ul}"] \\
& \mathsf{Cyl}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_0"] \arrow[dr, "id"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{ul}"] \\
& \mathsf{Cyl}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "i_1 \cdot \mathsf{Cyl}"] \arrow[d, "p"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{ul}"] \\
\mathsf{id}_{\mathcal{A}} \arrow[r, "i_1"'] & \mathsf{Cyl}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_1"] \arrow[d, "p"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{ul}"] \\
\mathsf{id}_{\mathcal{A}} \arrow[r, "i_1"'] & \mathsf{Cyl}
\end{tikzcd}$$

Definition III.18. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. An *upper left connection structure* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is a 2-arrow

$$\mathsf{Cyl} \xrightarrow{\Gamma_{ul}} \mathsf{Cyl}^2$$

of $\mathcal{C}$, such that $(\Gamma_{ul})^{op}$ defines an upper left connection structure with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op}.

Definition III.19. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. A *lower right connection structure* with respect to $\underline{\mathsf{Cyl}}$ is a 2-arrow

$$\mathsf{Cyl}^2 \xrightarrow{\Gamma_{lr}} \mathsf{Cyl}$$

of $\mathcal{C}$, such that the following diagrams in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A},\mathcal{A})$ commute.

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, "i_1 \cdot \mathsf{Cyl}"] \arrow[dr, "id"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{lr}"] \\ & \mathsf{Cyl} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_1"] \arrow[dr, "id"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{lr}"] \\ & \mathsf{Cyl} \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, "i_0 \cdot \mathsf{Cyl}"] \arrow[d, "p"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{lr}"] \\ \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"'] & \mathsf{Cyl} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_0"] \arrow[d, "p"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{lr}"] \\ \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"'] & \mathsf{Cyl} \end{tikzcd}$$

Definition III.20. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. A *lower right connection structure* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is a 2-arrow

$$\mathsf{Cyl} \xrightarrow{\Gamma_{lr}} \mathsf{Cyl}^2$$

of $\mathcal{C}$, such that $(\Gamma_{lr})^{op}$ defines a lower right connection structure with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op}.

Definition III.21. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. A lower right connection structure Γ_{lr} with respect to $\underline{\mathsf{Cyl}}$ is *compatible with* p if the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}^2 \arrow[r, "\Gamma_{lr}"] \arrow[d, "p \cdot \mathsf{Cyl}"'] & \mathsf{Cyl} \arrow[d, "p"] \\ \mathsf{Cyl} \arrow[r, "p"'] & \mathsf{id}_{\mathcal{A}} \end{tikzcd}$$

Definition III.22. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. A lower right connection structure Γ_{lr} with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is *compatible with* c if the lower right connection structure $(\Gamma_{lr})^{op}$ with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ is compatible with the contraction structure c^{op}.

Remark III.23. We shall not need to consider compatibility of an upper left connection structure with a contraction structure, or compatibility of an upper right connection structure, which we shall define next, with a contraction structure.

Definition III.24. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p and an involution structure v. An *upper right connection structure* with respect to $\underline{\mathsf{Cyl}}$ is a 2-arrow

$$\mathsf{Cyl}^2 \xrightarrow{\Gamma_{ur}} \mathsf{Cyl}$$

of $\mathcal{C}$, such that the following diagrams in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commute.

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, "i_0 \cdot \mathsf{Cyl}"] \arrow[dr, "id"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{ur}"] \\ & \mathsf{Cyl} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_1"] \arrow[dr, "v"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{ur}"] \\ & \mathsf{Cyl} \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_0"] \arrow[d, "p"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{ur}"] \\ \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"'] & \mathsf{Cyl} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cyl} \arrow[r, "i_1 \cdot \mathsf{Cyl}"] \arrow[d, "p"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{ur}"] \\ \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"'] & \mathsf{Cyl} \end{tikzcd}$$

Definition III.25. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c and an involution structure v. An *upper right connection structure* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is a 2-arrow

$$\mathsf{Cyl} \xrightarrow{\Gamma_{ur}} \mathsf{Cyl}^2$$

of $\mathcal{C}$, such that $(\Gamma_{ur})^{op}$ defines an upper right connection structure with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op} and the involution structure v^{op}.

Remark III.26. Analogously, one can define a *lower left connection structure* with respect to a cylinder or a co-cylinder. Everything concerning upper and lower right connections below can equally be carried out for upper and lower left connections.

Definition III.27. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr}, \Gamma_{ur})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, an involution structure v, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, an upper right connection structure Γ_{ur}, and a lower right connection structure Γ_{lr}. Let

$$\mathsf{S} \circ \mathsf{Cyl} \xrightarrow{x} \mathsf{Cyl}$$

denote the canonical 2-arrow of $\mathcal{C}$ such that the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "i_0 \cdot \mathsf{Cyl}"] \arrow[d, "i_1 \cdot \mathsf{Cyl}"'] & \mathsf{Cyl}^2 \arrow[d, "r_0 \cdot \mathsf{Cyl}"] \arrow[ddr, bend left, "\Gamma_{lr}"] & \\
\mathsf{Cyl}^2 \arrow[r, "r_1 \cdot \mathsf{Cyl}"'] \arrow[drr, bend right, "\Gamma_{ur}"'] & \mathsf{S} \circ \mathsf{Cyl} \arrow[dr, "x"] & \\
 & & \mathsf{Cyl}
\end{tikzcd}$$

Then Γ_{lr} and Γ_{ur} are *compatible with* $(\mathsf{S}, r_0, r_1, s)$ if the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2 \arrow[r, "s \cdot \mathsf{Cyl}"] \arrow[dr, "p \cdot \mathsf{Cyl}"'] & \mathsf{S} \circ \mathsf{Cyl} \arrow[d, "x"] \\
 & \mathsf{Cyl}
\end{tikzcd}$$

Definition III.28. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr}, \Gamma_{ur})$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, an involution structure v, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, an upper right connection structure Γ_{ur}, and a lower right connection structure Γ_{lr}.

Then Γ_{lr} and Γ_{ur} are *compatible with* $(\mathsf{S}, r_0, r_1, s)$ if the right connections Γ_{lr}^{op} and Γ_{ur}^{op} with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op} and the involution structure v^{op} are compatible with the subdivision structure $(\mathsf{S}, r_0^{op}, r_1^{op}, s^{op})$.

Proposition III.29. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \Gamma_{lr})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, an involution structure v compatible with p, and a lower right connection structure Γ_{lr}. Then the 2-arrow*

$$\begin{tikzcd}
\mathsf{Cyl}^2 \arrow[r, "\Gamma_{lr} \circ (v \cdot \mathsf{Cyl})"] & \mathsf{Cyl}
\end{tikzcd}$$

of $\mathcal{C}$ defines an upper right connection structure with respect to $\underline{\mathsf{Cyl}}$.

Proof. Firstly, the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "i_0 \cdot \mathsf{Cyl}"] \arrow[d, "id"'] \arrow[dr, "i_1 \cdot \mathsf{Cyl}"] & \mathsf{Cyl}^2 \arrow[d, "v \cdot \mathsf{Cyl}"] \\
\mathsf{Cyl} & \mathsf{Cyl}^2 \arrow[l, "\Gamma_{lr}"]
\end{tikzcd}
$$

Secondly, the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes, appealing to the compatibility of v with p.

$$
\begin{tikzcd}
\mathsf{Cyl} \arrow[rr, "\mathsf{Cyl} \cdot i_0"] \arrow[dr, "v"] \arrow[dd, "p"'] & & \mathsf{Cyl}^2 \arrow[d, "v \cdot \mathsf{Cyl}"] \\
& \mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_0"] \arrow[dl, "p"] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{lr}"] \\
\mathsf{id}_{\mathcal{A}} \arrow[rr, "i_0"'] & & \mathsf{Cyl}
\end{tikzcd}
$$

Thirdly, the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "i_1 \cdot \mathsf{Cyl}"] \arrow[d, "p"'] \arrow[dr, "i_0 \cdot \mathsf{Cyl}"] & \mathsf{Cyl}^2 \arrow[d, "v \cdot \mathsf{Cyl}"] \\
\mathsf{id}_{\mathcal{A}} \arrow[dr, "i_0"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{lr}"] \\
& \mathsf{Cyl}
\end{tikzcd}
$$

Fourthly, the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_1"] \arrow[d, "v"'] & \mathsf{Cyl}^2 \arrow[d, "v \cdot \mathsf{Cyl}"] \\
\mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_1"] \arrow[dr, "id"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{lr}"] \\
& \mathsf{Cyl}
\end{tikzcd}
$$

$\square$

Corollary III.30. *Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \Gamma_{lr})$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, an involution structure v compatible with c, and a lower right connection structure Γ_{lr}. Then the 2-arrow*

$$
\mathsf{co\text{-}Cyl} \xrightarrow{(v \cdot \mathsf{co\text{-}Cyl}) \circ \Gamma_{lr}} \mathsf{co\text{-}Cyl}^2
$$

of $\mathcal{C}$ defines an upper right connection structure with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition III.29 by duality. $\square$

Definition III.31. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$, equipped with a contraction structure p, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Let

$$
\mathsf{S} \xrightarrow{q_l} \mathsf{Cyl}
$$

denote the canonical 2-arrow of $\mathcal{C}$ such that the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$
\begin{tikzcd}
\mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"] \arrow[d, "i_1"'] & \mathsf{Cyl} \arrow[d, "r_0"] \arrow[ddr, bend left, "id"] & \\
\mathsf{Cyl} \arrow[r, "r_1"'] \arrow[drr, bend right, "i_0 \circ p"'] & \mathsf{S} \arrow[dr, "q_l"'] & \\
& & \mathsf{Cyl}
\end{tikzcd}
$$

Then $\underline{\mathsf{Cyl}}$ has *strictness of left identities* if the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, "s"] \arrow[dr, "id"'] & \mathsf{S} \arrow[d, "q_l"] \\ & \mathsf{Cyl} \end{tikzcd}$$

Definition III.32. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, \mathsf{S}, r_0, r_1, s)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Then $\underline{\mathsf{co\text{-}Cyl}}$ has *strictness of left identities* if the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op} and the subdivision structure $(\mathsf{S}^{op}, r_0^{op}, r_1^{op}, s^{op})$ has strictness of left identities.

Definition III.33. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Let

$$\mathsf{S} \xrightarrow{q_r} \mathsf{Cyl}$$

denote the canonical 2-arrow of $\mathcal{C}$ such that the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"] \arrow[d, "i_1"'] & \mathsf{Cyl} \arrow[d, "r_0"] \arrow[ddr, bend left, "i_1 \circ p"] & \\ \mathsf{Cyl} \arrow[r, "r_1"'] \arrow[drr, bend right, "id"'] & \mathsf{S} \arrow[dr, "q_r"] & \\ & & \mathsf{Cyl} \end{tikzcd}$$

Then $\underline{\mathsf{Cyl}}$ has *strictness of right identities* if the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, "s"] \arrow[dr, "id"'] & \mathsf{S} \arrow[d, "q_r"] \\ & \mathsf{Cyl} \end{tikzcd}$$

Definition III.34. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, \mathsf{S}, r_0, r_1, s)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Then $\underline{\mathsf{co\text{-}Cyl}}$ has *strictness of right identities* if the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op} and the involution structure v^{op} has strictness of right identities.

Definition III.35. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Then $\underline{\mathsf{Cyl}}$ has *strictness of identities* if it has both strictness of left identities and strictness of right identities.

Definition III.36. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, \mathsf{S}, r_0, r_1, s)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Then $\underline{\mathsf{co\text{-}Cyl}}$ has *strictness of identities* if it has both strictness of left identities and strictness of right identities.

Definition III.37. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, v, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with an involution structure v and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Let

$$\mathsf{S} \xrightarrow{w} \mathsf{Cyl}$$

denote the canonical 2-arrow of $\mathcal{C}$ such that the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd}
\mathsf{id}_{\mathcal{A}} \arrow[r, "i_0"] \arrow[d, "i_1"'] & \mathsf{Cyl} \arrow[d, "r_0"] \arrow[ddr, bend left, "id"] & \\
\mathsf{Cyl} \arrow[r, "r_1"'] \arrow[drr, bend right, "v"'] & \mathsf{S} \arrow[dr, "w"] & \\
& & \mathsf{Cyl}
\end{tikzcd}$$

Then $\underline{\mathsf{Cyl}}$ has *strictness of left inverses* if the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "s"] \arrow[d, "p"'] & \mathsf{S} \arrow[d, "w"] \\
\mathsf{id}_{\mathcal{A}} \arrow[r, "i_1"'] & \mathsf{Cyl}
\end{tikzcd}$$

Definition III.38. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, v, \mathsf{S}, r_0, r_1, s)$ be a co-cylinder in $\mathcal{A}$ equipped with an involution structure v, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Then $\underline{\mathsf{co\text{-}Cyl}}$ has *strictness of left inverses* if the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ has strictness of left inverses.

Remark III.39. We shall not need to consider strictness of right inverses, for which there is an analogue of Proposition III.40.

Proposition III.40. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v *compatible with* p*, a subdivision* *structure* $(\mathsf{S}, r_0, r_1, s)$*, and a lower right connection structure* Γ_{lr}*. Suppose that* $\underline{\mathsf{Cyl}}$ *has strictness of left inverses. Let* Γ_{ur} *denote the upper right connection structure with respect to* $\underline{\mathsf{Cyl}}$ *constructed in Proposition III.29. Then* Γ_{lr} *and* Γ_{ur} *are compatible with* $\underline{(\mathsf{S}, r_0, r_1, s)}$*.*

Proof. Let

$$\mathsf{S} \xrightarrow{w} \mathsf{Cyl}$$

denote the canonical 2-arrow of $\mathcal{C}$ of Definition III.37. The following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}^2 \arrow[r, "r_0 \cdot \mathsf{Cyl}"] \arrow[dr, "id"'] & \mathsf{S} \circ \mathsf{Cyl} \arrow[d, "w \cdot \mathsf{Cyl}"] \\ & \mathsf{Cyl}^2 \end{tikzcd}$$

Hence the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}^2 \arrow[r, "r_0 \cdot \mathsf{Cyl}"] \arrow[dr, "\Gamma_{lr}"'] & \mathsf{S} \circ \mathsf{Cyl} \arrow[d, "\Gamma_{lr} \circ (w \cdot \mathsf{Cyl})"] \\ & \mathsf{Cyl} \end{tikzcd}$$

The following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ also commutes.

$$\begin{tikzcd} \mathsf{Cyl}^2 \arrow[r, "r_1 \mathsf{Cyl}"] \arrow[d, "\Gamma_{ur}"'] \arrow[dr, "v \cdot \mathsf{Cyl}"'] & \mathsf{SCyl} \arrow[d, "w \cdot \mathsf{Cyl}"] \\ \mathsf{Cyl} & \mathsf{Cyl}^2 \arrow[l, "\Gamma_{lr}"] \end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl} \arrow[r, "i_0 \cdot \mathsf{Cyl}"] \arrow[d, "i_1 \cdot \mathsf{Cyl}"'] & \mathsf{Cyl}^2 \arrow[d, "r_0 \cdot \mathsf{Cyl}"] \arrow[rdd, bend left, "\Gamma_{lr}"] & \\
\mathsf{Cyl}^2 \arrow[r, "r_1 \cdot \mathsf{Cyl}"'] \arrow[rrd, bend right, "\Gamma_{ur}"'] & \mathsf{S} \circ \mathsf{Cyl} \arrow[rd, "\Gamma_{lr} \circ (w \cdot \mathsf{Cyl})"'] & \\
 & & \mathsf{Cyl}
\end{tikzcd}
$$

By the universal property of $\mathsf{S} \circ \mathsf{Cyl}$, we deduce that $\Gamma_{lr} \circ (w \cdot \mathsf{Cyl}) = x$, where

$$
S \circ \mathsf{Cyl} \xrightarrow{x} \mathsf{Cyl}
$$

is the canonical 2-arrow of $\mathcal{C}$ of Definition III.27.

The following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2 \arrow[r, "s \cdot \mathsf{Cyl}"] \arrow[d, "p \cdot \mathsf{Cyl}"'] & \mathsf{S} \circ \mathsf{Cyl} \arrow[d, "w \cdot \mathsf{Cyl}"] \\
\mathsf{Cyl} \arrow[r, "i_1 \cdot \mathsf{Cyl}"] \arrow[rd, "id"'] & \mathsf{Cyl}^2 \arrow[d, "\Gamma_{lr}"] \\
 & \mathsf{Cyl}
\end{tikzcd}
$$

We conclude that the following diagram in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commutes, as required.

$$
\begin{tikzcd}
\mathsf{Cyl}^2 \arrow[r, "s \cdot \mathsf{Cyl}"] \arrow[rd, "p \cdot \mathsf{Cyl}"'] & \mathsf{S} \circ \mathsf{Cyl} \arrow[d, "x"] \\
 & \mathsf{Cyl}
\end{tikzcd}
$$

□

Corollary III.41. *Let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*, an involution structure* v *compatible with* c*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and a lower right connection structure* Γ_{lr}*. Suppose that* $\underline{\mathsf{co\text{-}Cyl}}$ *has strictness of left inverses. Let* Γ_{ur} *denote the upper right connection with respect to* $\underline{\mathsf{co\text{-}Cyl}}$ *of Corollary III.30. Then* Γ_{lr} *and* Γ_{ur} *are compatible with* $(\mathsf{S}, r_0, r_1, s)$*.*

Proof. Follows immediately from Proposition III.40 by duality. □

IV. Cylindrical adjunctions

We introduce a notion of adjunction between a cylinder and a co-cylinder, which is discussed for example in §3 of [23]. In VI we shall explain that an interval in a category gives rise to a cylinder and co-cylinder which are adjoint.

In VII we shall define a notion of homotopy with respect to a cylinder or a co-cylinder. Given both a cylinder and a co-cylinder, it will be vital for us to know that the corresponding notions of homotopy equivalence coincide. If the cylinder and co-cylinder are adjoint, we shall see that this is the case.

Furthermore, we shall in VIII define cofibrations with respect to a cylinder, and, dually, define fibrations with respect to a co-cylinder. If we have both a cylinder $\underline{\mathsf{Cyl}}$ and a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}$, with $\underline{\mathsf{Cyl}}$ left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, we shall be able to characterise fibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$ via a homotopy lifting property with respect to $\underline{\mathsf{Cyl}}$, and shall be able to characterise cofibrations with respect to $\underline{\mathsf{Cyl}}$ via a homotopy lifting property with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

We refer the reader to Recollection II.13 for the notion of an adjunction between 1-arrows of $\mathcal{C}$.

Assumption IV.1. Let $\mathcal{C}$ be a 2-category with a final object, and let $\mathcal{A}$ be an object of $\mathcal{C}$.

Definition IV.2. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Then $\underline{\mathsf{Cyl}}$ is *left adjoint* to $\underline{\mathsf{co\text{-}Cyl}}$ if the following conditions are satisfied.

(i) Cyl is left adjoint to $\mathsf{co\text{-}Cyl}$.

(ii) Suppose that (i) holds. Let

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the corresponding natural isomorphism, adopting the shorthand of Recollection II.13. We require that for every arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

of $\mathcal{A}$, the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "\mathsf{adj}(h)"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\ \mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "i_1(a_0)"] \arrow[d, "\mathsf{adj}(h)"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\ \mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_1(a_1)"'] & a_1 \end{tikzcd}$$

Definition IV.3. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$.

Let

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the corresponding natural isomorphism, adopting the shorthand of Recollection II.13. The adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is *compatible with p and c* if, for every arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "f"] \arrow[dr, "\mathsf{adj}(f \circ p(a_0))"'] & a_1 \arrow[d, "c(a_1)"] \\ & \mathsf{co\text{-}Cyl}(a_1) \end{tikzcd}$$

Remark IV.4. Given a cylinder $\underline{\mathsf{Cyl}}$ in $\mathcal{A}$, and a 1-arrow

$$\mathcal{A} \xrightarrow{\mathsf{co\text{-}Cyl}} \mathcal{A}$$

of $\mathcal{C}$ which is left adjoint to Cyl, one can always equip $\mathsf{co\text{-}Cyl}$ with the structure of a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}$ in $\mathcal{A}$ via the adjunction.

Moreover, one can transfer structures upon $\underline{\mathsf{Cyl}}$ across the adjunction to structures upon $\underline{\mathsf{co\text{-}Cyl}}$. This is explained for example in §1.8 of the paper [16] of Grandis and MacDonald. It goes back at least to §7 of the paper [21] of Kamps.

V. Monoidal category theory preliminaries

In VI we shall introduce the notion of an interval in a monoidal category $\mathcal{A}$. Under natural hypotheses, a structured interval in $\mathcal{A}$ will give rise to a structured cylinder and a structured co-cylinder in $\mathcal{A}$.

We shall need the notion of an exponentiable object of $\mathcal{A}$. There are two possible definitions, as we shall not assume that our monoidal structures are symmetric. We make the following choice.

Assumption V.1. Let $(\mathcal{A}, \otimes)$ be a monoidal category. Let 1 denote its unit object, and let λ denote its natural isomorphism

$$- \otimes 1 \longrightarrow -.$$

Definition V.2. An object a of $\mathcal{A}$ is *exponentiable* with respect to $\otimes$ if the functor

$$\mathcal{A} \xrightarrow{- \otimes a} \mathcal{A}$$

admits a right adjoint, which we shall denote by

$$\mathcal{A} \xrightarrow{(-)^a} \mathcal{A}.$$

Remark V.3. We have that 1 is exponentiable with respect to $\otimes$, since the natural isomorphism

$$\mathsf{Hom}_{\mathcal{A}}(- \otimes 1, -) \xrightarrow{\mathsf{Hom}_{\mathcal{A}}(\lambda^{-1}, -)} \mathsf{Hom}_{\mathcal{A}}(-, -)$$

exhibits the identity functor as a right adjoint of $- \otimes 1$.

Notation V.4. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ such that both a_0 and a_1 are exponentiable with respect to $\otimes$. Then, for any object a of $\mathcal{A}$, we have a natural isomorphism

$$\mathsf{Hom}_{\mathcal{A}}(- \times a_0, a) \xrightarrow{\mathsf{adj}(a_0)} \mathsf{Hom}_{\mathcal{A}}(-, a^{a_0})$$

and a natural isomorphism

$$\mathsf{Hom}_{\mathcal{A}}(- \times a_1, a) \xrightarrow{\mathsf{adj}(a_1)} \mathsf{Hom}_{\mathcal{A}}(-, a^{a_1}).$$

Let

$$a^{a_1} \xrightarrow{a^f} a^{a_0}$$

denote the arrow of $\mathcal{A}$ corresponding via the Yoneda lemma to the following natural transformation.

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{A}}(-, a^{a_1}) \arrow[r, "\mathsf{adj}(a_1)^{-1}"] \arrow[d, dashed] & \mathsf{Hom}_{\mathcal{A}}(- \otimes a_1, a) \arrow[d, "\mathsf{Hom}_{\mathcal{A}}(- \otimes f, a)"] \\
\mathsf{Hom}_{\mathcal{A}}(-, a^{a_0}) & \mathsf{Hom}_{\mathcal{A}}(- \otimes a_0, a) \arrow[l, "\mathsf{adj}(a_0)"]
\end{tikzcd}$$

Remark V.5. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ such that both a_0 and a_1 are exponentiable with respect to $\otimes$. Associating to an object a of $\mathcal{A}$ the arrow

$$a^{a_1} \xrightarrow{a^f} a^{a_0}$$

of $\mathcal{A}$ defines a natural transformation

$$(-)^{a_1} \xrightarrow{(-)^f} (-)^{a_0}.$$

Definition V.6. The monoidal structure upon $\mathcal{A}$ defined by $\otimes$ is *closed* if, for every object a of $\mathcal{A}$, the functor

$$\mathcal{A} \xrightarrow{a \otimes -} \mathcal{A}$$

admits a right adjoint.

Remark V.7. Suppose that the monoidal structure upon $\mathcal{A}$ defined by $\otimes$ is symmetric. This monoidal structure is closed if and only if every object of $\mathcal{A}$ is exponentiable with respect to $\otimes$.

Remark V.8. Let

$$\begin{tikzcd}
a_0 \arrow[r, "f_0"] \arrow[d, "f_2"'] & a_1 \arrow[d, "f_1"] \\
a_2 \arrow[r, "f_3"'] & a_3
\end{tikzcd}$$

be a co-cartesian diagram in $\mathcal{A}$. Let a be an object of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ is co-cartesian.

$$\begin{tikzcd}
a \otimes a_0 \arrow[r, "a \otimes f_0"] \arrow[d, "a \otimes f_2"'] & a \otimes a_1 \arrow[d, "a \otimes f_1"] \\
a \otimes a_2 \arrow[r, "a \otimes f_3"'] & a \otimes a_3
\end{tikzcd}$$

If a_0, a_1, a_2, and a_3 are exponentiable with respect to $\otimes$, the following diagram in $\mathcal{A}$ is cartesian.

$$\begin{tikzcd}
a^{a_3} \arrow[r, "a^{f_3}"] \arrow[d, "a^{f_1}"'] & a^{a_2} \arrow[d, "a^{f_2}"] \\
a^{a_1} \arrow[r, "a^{f_0}"'] & a^{a_0}
\end{tikzcd}$$

VI. Structures upon an interval

We define the notion of an interval in a monoidal category $\mathcal{A}$. We introduce — exactly in parallel with III — structures with which this interval may be able to be equipped.

An interval $\widehat{\mathsf{I}}$ in $\mathcal{A}$ gives rise, under natural hypotheses, to a cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ and a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$. Structures upon $\widehat{\mathsf{I}}$ pass to structures upon $\underline{\mathsf{Cyl}}(\mathsf{I})$ and $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$.

Moreover, $\underline{\mathsf{Cyl}}(\mathsf{I})$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$. Whereas in an abstract setting we work with cylinders and co-cylinders, in practise we often construct a cylinder and co-cylinder via an interval in a monoidal category.

In parallel with III once more, we introduce a strictness of left identities hypothesis, a strictness of right identities hypothesis, and a strictness of left inverses hypothesis. If these hypotheses hold for $\widehat{\mathsf{I}}$, then they hold for $\underline{\mathsf{Cyl}}(\mathsf{I})$ and $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$.

We refer the reader to V for our conventions regarding exponential objects in a monoidal category, and for other preliminary observations to which we shall appeal.

Assumption VI.1. Let $(\mathcal{A}, \otimes)$ be a monoidal category. Let 1 denote its unit object, and let λ denote its natural isomorphism

$$- \otimes 1 \longrightarrow -.$$

Let α denote its natural isomorphism

$$(- \otimes -) \otimes - \longrightarrow - \otimes (- \otimes -).$$

Definition VI.2. An *interval* in $\mathcal{A}$ is an object I of $\mathcal{A}$, together with a pair of arrows

$$\begin{tikzcd} 1 \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & I \end{tikzcd}$$

of $\mathcal{A}$.

Remark VI.3. We let

$$\mathcal{A} \xrightarrow{(-)^1} \mathcal{A}$$

denote the identity functor, by virtue of Remark V.3.

We shall frequently implicitly identify the functor

$$\mathcal{A} \xrightarrow{- \otimes 1} \mathcal{A}$$

with the identity functor, via the natural isomorphism λ.

Definition VI.4. Let $\widehat{\mathsf{I}} = (I, i_0, i_1)$ be an interval in $\mathcal{A}$. Regarding $\mathcal{A}$ as an object of the 2-category of categories, we denote by $\underline{\mathsf{Cyl}}(\mathsf{I})$ the cylinder in $\mathcal{A}$ defined by the functor

$$\mathcal{A} \xrightarrow{- \otimes I} \mathcal{A}$$

and the natural transformations

$$\begin{tikzcd} \mathsf{id}_{\mathcal{A}} \arrow[r, shift left, "- \otimes i_0"] \arrow[r, shift right, "- \otimes i_1"'] & - \otimes I. \end{tikzcd}$$

If I is exponentiable with respect to $\otimes$, we denote by $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ the co-cylinder in $\mathcal{A}$ defined by the functor

$$\mathcal{A} \xrightarrow{(-)^I} \mathcal{A}$$

and the natural transformations

$$\begin{tikzcd} (-)^I \arrow[r, shift left, "(-)^{i_0}"] \arrow[r, shift right, "(-)^{i_1}"'] & \mathsf{id}_{\mathcal{A}}. \end{tikzcd}$$

Proposition VI.5. *Let $\widehat{\mathsf{I}} = (I, i_0, i_1)$ be an interval in $\mathcal{A}$. Suppose that I is exponentiable with respect to $\otimes$. Then the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ is left adjoint to the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$.*

Proof. Let

$$a_0 \otimes I \xrightarrow{h} a_1$$

be an arrow of $\mathcal{A}$. Since I is exponentiable with respect to $\otimes$, we have a natural isomorphism

$$\mathsf{Hom}_{\mathcal{A}}(- \otimes I, a_1) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{A}}(-, (a_1)^I).$$

In particular, we have an isomorphism

$$\mathsf{Hom}_{\mathcal{A}}(a_0 \otimes I, a_1) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{A}}(a_0, (a_1)^I).$$

Let us denote this isomorphism by adj.

By definition of

$$(a_1)^I \xrightarrow{(a_1)^{i_0}} a_1,$$

the following diagram in the category of sets commutes.

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{A}}(a_0, (a_1)^I) \arrow[r, "\mathsf{adj}^{-1}"] \arrow[d, "\mathsf{Hom}_{\mathcal{A}}(a_0, (a_1)^{i_0})"'] & \mathsf{Hom}_{\mathcal{A}}(a_0 \otimes I, a_1) \arrow[d, "\mathsf{Hom}_{\mathcal{A}}(a_0 \otimes i_0, a_1)"] \\
\mathsf{Hom}_{\mathcal{A}}(a_0, a_1) & \mathsf{Hom}_{\mathcal{A}}(a_0 \otimes 1, a_1) \arrow[l, "\mathsf{Hom}_{\mathcal{A}}(\lambda^{-1}(a_0), a_1)"]
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "a_0 \otimes i_0"] \arrow[d, "\mathsf{adj}(h)"'] & a_0 \otimes I \arrow[d, "h"] \\
(a_1)^I \arrow[r, "(a_1)^{i_0}"'] & a_1
\end{tikzcd}$$

Similarly, by definition of

$$(a_1)^I \xrightarrow{(a_1)^{i_1}} a_1,$$

the following diagram in the category of sets commutes.

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{A}}(a_0, (a_1)^I) \arrow[r, "\mathsf{adj}^{-1}"] \arrow[d, "\mathsf{Hom}_{\mathcal{A}}(a_0, (a_1)^{i_1})"'] & \mathsf{Hom}_{\mathcal{A}}(a_0 \otimes I, a_1) \arrow[d, "\mathsf{Hom}_{\mathcal{A}}(a_0 \otimes i_1, a_1)"] \\
\mathsf{Hom}_{\mathcal{A}}(a_0, a_1) & \mathsf{Hom}_{\mathcal{A}}(a_0 \otimes 1, a_1) \arrow[l, "\mathsf{Hom}_{\mathcal{A}}(\lambda^{-1}(a_0), a_1)"]
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "a_0 \otimes i_1"] \arrow[d, "\mathsf{adj}(h)"'] & a_0 \otimes I \arrow[d, "h"] \\
a_1^I \arrow[r, "a_1^{i_1}"'] & a_1
\end{tikzcd}$$

$\square$

Definition VI.6. Let $\widehat{\mathsf{I}} = (I, i_0, i_1)$ be an interval in $\mathcal{A}$. A *contraction structure* with respect to $\widehat{\mathsf{I}}$ is an arrow

$$I \xrightarrow{p} 1$$

of $\mathcal{A}$, such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd} 1 \arrow[r, "i_0"] \arrow[dr, "id"'] & I \arrow[d, "p"] \\ & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "i_1"] \arrow[dr, "id"'] & I \arrow[d, "p"] \\ & 1 \end{tikzcd}$$

Remark VI.7. Let $\widehat{\mathsf{I}} = (I, i_0, i_1)$ be an interval in $\mathcal{A}$. Suppose that 1 is a final object of $\mathcal{A}$. This is the case, for example, if the monoidal structure upon $\mathcal{A}$ is cartesian. Then the canonical arrow

$$I \longrightarrow 1$$

of $\mathcal{A}$ defines a contraction structure with respect to $\widehat{\mathsf{I}}$.

Remark VI.8. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p. Then the natural transformation

$$- \otimes I \xrightarrow{- \otimes p} \mathsf{id}_{\mathcal{A}}$$

equips the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ with a contraction structure.

If I is exponentiable with respect to $\otimes$, the natural transformation

$$\mathsf{id}_{\mathcal{A}} \xrightarrow{(-)^p} (-)^I$$

equips the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ with a contraction structure.

Proposition VI.9. *Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p. Suppose that I is exponentiable with respect to $\otimes$. We regard the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ as equipped with the contraction structure defined by $- \otimes p$, and regard the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ as equipped with the contraction structure defined by $(-)^p$.*

Recall by Proposition VI.5 that $\underline{\mathsf{Cyl}}(\mathsf{I})$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$. We moreover have that the adjunction between $- \otimes I$ and $(-)^I$ is compatible with $- \otimes p$ and $(-)^p$.

Proof. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$. Since I is exponentiable with respect to $\otimes$, we have a natural isomorphism

$$\mathsf{Hom}_{\mathcal{A}}(- \otimes I, a_1) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{A}}(-, (a_1)^I).$$

In particular, we have an isomorphism

$$\mathsf{Hom}_{\mathcal{A}}(a_0 \otimes I, a_1) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{A}}(a_0, (a_1)^I).$$

Let us denote this isomorphism by adj. By definition of

$$a_1 \xrightarrow{(a_1)^p} (a_1)^I,$$

the following diagram in the category of sets commutes.

$$\begin{tikzcd}
\mathsf{Hom}_{\mathcal{A}}(a_0, a_1) \arrow[r, "\mathsf{Hom}_{\mathcal{A}}(\lambda(a_0), a_1)"] \arrow[d, "\mathsf{Hom}_{\mathcal{A}}(a_0, (a_1)^p)"'] & \mathsf{Hom}_{\mathcal{A}}(a_0 \otimes 1, a_1) \arrow[d, "\mathsf{Hom}_{\mathcal{A}}(a_0 \otimes p, a_1)"] \\
\mathsf{Hom}_{\mathcal{A}}(a_0, (a_1)^I) & \mathsf{Hom}_{\mathcal{A}}(a_0 \otimes I, a_1) \arrow[l, "\mathsf{adj}"]
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "f"] \arrow[dr, "\mathsf{adj}(f \circ (a_0 \otimes p))"'] & a_1 \arrow[d, "(a_1)^p"] \\
 & (a_1)^I
\end{tikzcd}$$

□

Definition VI.10. Let $\widehat{\mathsf{I}} = (I, i_0, i_1)$ be an interval in $\mathcal{A}$. An *involution structure* with respect to $\widehat{\mathsf{I}}$ is an arrow

$$I \xrightarrow{v} I$$

of $\mathcal{A}$, such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd}
1 \arrow[r, "i_0"] \arrow[dr, "i_1"'] & I \arrow[d, "v"] \\
 & I
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "i_1"] \arrow[dr, "i_0"'] & I \arrow[d, "v"] \\
 & I
\end{tikzcd}$$

Remark VI.11. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, v)$ be an interval in $\mathcal{A}$ equipped with an involution structure v. Then the natural transformation

$$- \otimes I \xrightarrow{- \otimes v} - \otimes I$$

equips the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ with an involution structure.

If I is exponentiable with respect to $\otimes$, the natural transformation

$$(-)^I \xrightarrow{(-)^v} (-)^I$$

equips the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ with an involution structure.

Definition VI.12. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p. An involution structure v with respect to $\widehat{\mathsf{I}}$ is *compatible with* p if the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} I \arrow[r, "v"] \arrow[dr, "p"'] & I \arrow[d, "p"] \\ & 1 \end{tikzcd}$$

Remark VI.13. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, v)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p and an involution structure v compatible with p. Then the involution structure $- \otimes v$ with respect to the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ is compatible with the contraction structure $- \otimes p$.

If I is exponentiable with respect to $\otimes$, the involution structure $(-)^v$ with respect to the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ is compatible with the contraction structure $(-)^p$.

Definition VI.14. Let $\widehat{\mathsf{I}} = (I, i_0, i_1)$ be an interval in $\mathcal{A}$. A *subdivision structure* with respect to $\widehat{\mathsf{I}}$ is an object S of $\mathcal{A}$, together with a pair of arrows

$$\begin{tikzcd} I \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & S \end{tikzcd}$$

of $\mathcal{A}$, such that the square

$$\begin{tikzcd} 1 \arrow[r, "i_0"] \arrow[d, "i_1"'] & I \arrow[d, "r_0"] \\ I \arrow[r, "r_1"'] & S \end{tikzcd}$$

in $\mathcal{A}$ is co-cartesian, and an arrow

$$I \xrightarrow{s} S$$

of $\mathcal{A}$, such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd} 1 \arrow[r, "i_0"] \arrow[d, "i_0"'] & I \arrow[d, "s"] \\ I \arrow[r, "r_1"'] & S \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "i_1"] \arrow[d, "i_1"'] & I \arrow[d, "s"] \\ I \arrow[r, "r_0"'] & S \end{tikzcd}$$

Requirement VI.15. Let $\widehat{\mathsf{I}} = (I, i_0, i_1)$ be an interval in $\mathcal{A}$, and let (S, r_0, r_1, s) be a subdivision structure with respect to $\widehat{\mathsf{I}}$. Then for any object a of $\mathcal{A}$, the square

$$\begin{tikzcd} a \arrow[r, "a \otimes i_0"] \arrow[d, "a \otimes i_1"'] & a \otimes I \arrow[d, "a \otimes r_0"] \\ a \otimes I \arrow[r, "a \otimes r_1"'] & a \otimes S \end{tikzcd}$$

in $\mathcal{A}$ is co-cartesian.

Remark VI.16. Requirement VI.15 is satisfied whenever the monoidal structure on $\mathcal{A}$ is closed. It is also satisfied, for example, by the usual interval in the category of topological spaces, equipped with its cartesian monoidal structure. This monoidal structure is not closed.

Since by Proposition VI.5 we have that $\underline{\mathsf{Cyl}}(\mathsf{I})$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$, Requirement VI.15 is equivalent to the dual requirement that, for any object a of $\mathcal{A}$, the square

$$\begin{tikzcd} a^S \arrow[r, "a^{r_0}"] \arrow[d, "a^{r_1}"'] & a^I \arrow[d, "a^{i_0}"] \\ a^I \arrow[r, "a^{i_1}"'] & a \end{tikzcd}$$

in $\mathcal{A}$ is cartesian.

Remark VI.17. Let $\widehat{\mathsf{I}} = (I, i_0, i_1)$ be an interval in $\mathcal{A}$, and let (S, r_0, r_1, s) be a subdivision structure with respect to $\widehat{\mathsf{I}}$, such that Requirement VI.15 holds. Then the functor

$$\mathcal{A} \xrightarrow{- \otimes S} \mathcal{A}$$

and the natural transformations

$$
\begin{tikzcd}
- \otimes I \arrow[r, shift left, "- \otimes r_0"] \arrow[r, shift right, "- \otimes r_1"'] & - \otimes S
\end{tikzcd}
$$

and

$$
- \otimes I \xrightarrow{- \otimes s} - \otimes S
$$

define a subdivision structure with respect to the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$. Moreover, $- \otimes I$ preserves subdivision with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$ and $(- \otimes S, - \otimes r_0, - \otimes r_1, - \otimes s)$.

If both I and S are exponentiable with respect to $\otimes$ then the functor

$$
\mathcal{A} \xrightarrow{(-)^S} \mathcal{A}
$$

and the natural transformations

$$
\begin{tikzcd}
(-)^S \arrow[r, shift left, "(-)^{r_0}"] \arrow[r, shift right, "(-)^{r_1}"'] & (-)^I
\end{tikzcd}
$$

and

$$
(-)^S \xrightarrow{(-)^s} (-)^I
$$

defines a subdivision structure with respect to the co-cylinder $\underline{\mathsf{co}\text{-}\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$. Moreover $(-)^I$ preserves subdivision with respect to $\underline{\mathsf{co}\text{-}\mathsf{Cyl}}(\mathsf{I})$ and $((-)^S, (-)^{r_0}, (-)^{r_1}, (-)^s)$.

Definition VI.18. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Let

$$
S \xrightarrow{\overline{p}} 1
$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
1 \arrow[r, "i_0"] \arrow[d, "i_1"'] & I \arrow[d, "r_0"] \arrow[ddr, bend left, "p"] & \\
I \arrow[r, "r_1"'] \arrow[drr, bend right, "p"'] & S \arrow[dr, "\overline{p}"] & \\
 & & 1
\end{tikzcd}
$$

The subdivision structure (S, r_0, r_1, s) is *compatible with* p if the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} I \arrow[r, "s"] \arrow[dr, "p"'] & S \arrow[d, "\overline{p}"] \\ & 1 \end{tikzcd}$$

Remark VI.19. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$ compatible with p. Suppose that Requirement VI.15 holds. Then the subdivision structure $(- \otimes S, - \otimes r_0, i - \otimes r_1, - \otimes s)$ with respect to the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ is compatible with the contraction structure $- \otimes p$.

If both I and S are exponentiable with respect to $\otimes$, then the subdivision structure $((-)^S, (-)^{r_0}, (-)^{r_1}, (-)^s)$ with respect to the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ is compatible with the contraction structure $(-)^p$.

Notation VI.20. Let (I, i_0, i_1) be an interval in $\mathcal{A}$. We denote by I^2 the object $I \otimes I$ of $\mathcal{A}$.

Remark VI.21. Let (I, i_0, i_1) be an interval in $\mathcal{A}$. We shall frequently implicitly identify the functor $(- \otimes I) \otimes I$ with the functor $- \otimes I^2$, via the natural isomorphism α. Similarly, we shall frequently implicitly identify the functor $(- \otimes I) \otimes 1$ with the functor $- \otimes (I \otimes 1)$, via α.

Definition VI.22. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p. An *upper left connection structure* with respect to $\widehat{\mathsf{I}}$ is an arrow

$$I^2 \xrightarrow{\Gamma_{ul}} I$$

of $\mathcal{A}$, such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd} I \arrow[r, "i_0 \otimes I"] \arrow[dr, "id"'] & I^2 \arrow[d, "\Gamma_{ul}"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "I \otimes i_0"] \arrow[dr, "id"'] & I^2 \arrow[d, "\Gamma_{ul}"] \\ & I \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "i_1 \otimes I"] \arrow[d, "p"'] & I^2 \arrow[d, "\Gamma_{ul}"] \\ 1 \arrow[r, "i_1"'] & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "I \otimes i_1"] \arrow[d, "p"'] & I^2 \arrow[d, "\Gamma_{ul}"] \\ 1 \arrow[r, "i_1"'] & I \end{tikzcd}$$

Remark VI.23. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, \Gamma_{ul})$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, and an upper left connection structure Γ_{ul}. Let us regard the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ as equipped with the contraction structure $-\otimes p$, Then the natural transformation

$$- \otimes I^2 \xrightarrow{- \otimes \Gamma_{ul}} - \otimes I$$

equips $\underline{\mathsf{Cyl}}(\mathsf{I})$ with an upper left connection structure.

Suppose that I is exponentiable with respect to $\otimes$. Let us regard the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ as equipped with the contraction structure $(-)^p$. Then the natural transformation

$$(-)^I \xrightarrow{(-)^{\Gamma_{ul}}} (-)^{I^2}$$

equips $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ with an upper left connection structure.

Definition VI.24. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p. A *lower right connection structure* with respect to $\widehat{\mathsf{I}}$ is an arrow

$$I^2 \xrightarrow{\Gamma_{lr}} I$$

of $\mathcal{A}$, such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd} I \arrow[r, "i_1 \otimes I"] \arrow[dr, "id"'] & I^2 \arrow[d, "\Gamma_{lr}"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "I \otimes i_1"] \arrow[dr, "id"'] & I^2 \arrow[d, "\Gamma_{lr}"] \\ & I \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "i_0 \otimes I"] \arrow[d, "p"'] & I^2 \arrow[d, "\Gamma_{lr}"] \\ 1 \arrow[r, "i_0"'] & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "I \otimes i_0"] \arrow[d, "p"'] & I^2 \arrow[d, "\Gamma_{lr}"] \\ 1 \arrow[r, "i_0"'] & I \end{tikzcd}$$

Remark VI.25. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, \Gamma_{lr})$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, and a lower right connection structure Γ_{lr}. Let us regard the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ as equipped with the contraction structure $-\otimes p$. Then the natural transformation

$$- \otimes I^2 \xrightarrow{- \otimes \Gamma_{lr}} - \otimes I$$

equips $\underline{\mathsf{Cyl}}(\mathsf{I})$ with a lower right connection structure.

Suppose that I is exponentiable with respect to $\otimes$. Let us regard the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ as equipped with the contraction structure $(-)^p$. Then the natural transformation

$$(-)^I \xrightarrow{(-)^{\Gamma_{lr}}} (-)^{I^2}$$

equips $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ with a lower right connection structure.

Definition VI.26. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p. A lower right connection structure Γ_{lr} with respect to $\widehat{\mathsf{I}}$ is *compatible with* p if the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
I^2 \arrow[r, "\Gamma_{lr}"] \arrow[d, "I \otimes p"'] & I \arrow[d, "p"] \\
I \arrow[r, "p"'] & 1
\end{tikzcd}$$

Remark VI.27. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, \Gamma_{lr})$ be an interval in $\mathcal{A}$ equipped with a contraction structure p and a lower right connection structure Γ_{lr}. If Γ_{lr} is compatible with p, then the lower right connection structure $- \otimes \Gamma_{lr}$ with respect to the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ is compatible with the contraction structure $- \otimes p$.

Suppose that I is exponentiable with respect to $\otimes$. Then the lower right connection structure $(-)^{\Gamma_{lr}}$ with respect to the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ is compatible with the contraction structure $(-)^p$.

Remark VI.28. We shall not need to consider compatibility of an upper left connection structure upon an interval with a contraction structure, or compatibility of an upper right connection structure upon an interval, which we shall define next, with a contraction structure.

Definition VI.29. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, v)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, and an involution structure v. An *upper right connection structure* with respect to $\widehat{\mathsf{I}}$ is an arrow

$$I^2 \xrightarrow{\Gamma_{ur}} I$$

of $\mathcal{A}$, such that the following diagrams in $\mathcal{A}$ commute.

$$
\begin{tikzcd}
I \arrow[r, "I \otimes i_0"] \arrow[dr, "id"'] & I^2 \arrow[d, "\Gamma_{ur}"] \\
& I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "i_1 \otimes I"] \arrow[dr, "v"'] & I^2 \arrow[d, "\Gamma_{ur}"] \\
& I
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \arrow[r, "i_0 \otimes I"] \arrow[d, "p"'] & I^2 \arrow[d, "\Gamma_{ur}"] \\
1 \arrow[r, "i_0"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "I \otimes i_1"] \arrow[d, "p"'] & I^2 \arrow[d, "\Gamma_{ur}"] \\
1 \arrow[r, "i_0"'] & I
\end{tikzcd}
$$

Remark VI.30. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, v, \Gamma_{ur})$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, an involution structure v, and an upper right connection structure Γ_{ur}. Let us regard the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ as equipped with the contraction structure $- \otimes p$ and the involution structure $- \otimes v$. Then the natural transformation

$$- \otimes I^2 \xrightarrow{- \otimes \Gamma_{ur}} - \otimes I$$

equips $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ with an upper right connection structure.

Suppose that I is exponentiable with respect to $\otimes$. Let us regard the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ as equipped with the contraction structure $(-)^p$ and the involution structure $(-)^v$. Then the natural transformation

$$(-)^I \xrightarrow{(-)^{\Gamma_{ur}}} (-)^{I^2}$$

equips $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ with an upper right connection structure.

Remark VI.31. Analogously, one can define a *lower left connection structure* with respect to an interval. Everything concerning upper and lower right connections below can equally be carried out for upper and lower left connections.

Definition VI.32. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, v, S, r_0, r_1, s, \Gamma_{lr}, \Gamma_{ur})$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, an involution structure v, a subdivision structure (S, r_0, r_1, s), a lower right connection structure Γ_{lr}, and an upper right connection structure Γ_{ur}. Let

$$I \otimes S \xrightarrow{x} I$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
I \arrow[r, "I \otimes i_0"] \arrow[d, "I \otimes i_1"'] & I^2 \arrow[d, "I \otimes r_0"] \arrow[dr, bend left, "\Gamma_{lr}"] & \\
I^2 \arrow[r, "I \otimes r_1"'] \arrow[rr, bend right, "\Gamma_{ur}"'] & I \otimes S \arrow[r, "x"] & I
\end{tikzcd}$$

Then Γ_{lr} and Γ_{ur} are *compatible with* (S, r_0, r_1, s) if the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
I^2 \arrow[r, "I \otimes s"] \arrow[dr, "I \otimes p"'] & I \otimes S \arrow[d, "x"] \\
& I
\end{tikzcd}$$

Remark VI.33. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, v, S, r_0, r_1, s, \Gamma_{lr}, \Gamma_{ur})$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, an involution structure v, a subdivision structure (S, r_0, r_1, s), a lower right connection structure Γ_{lr}, and an upper right connection structure Γ_{ur}. Suppose that Γ_{lr} and Γ_{ur} are compatible with (S, r_0, r_1, s), and that Requirement VI.15 holds.

Then the right connections $-\otimes \Gamma_{lr}$ and $-\otimes \Gamma_{ur}$ are compatible with the subdivision structure $(-\otimes S, -\otimes r_0, -\otimes r_1, -\otimes s)$ upon the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ equipped with the contraction structure $-\otimes p$.

Suppose that both I and S are exponentiable with respect to $\otimes$. Then the right connections $(-)^{\Gamma_{lr}}$ and $(-)^{\Gamma_{ur}}$ are compatible with the subdivision structure $((-)^S, (-)^{r_0}, (-)^{r_1}, (-)^s)$ upon the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ equipped with the contraction structure $(-)^p$.

Definition VI.34. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, and a subdivision structure (S, r_0, r_1, s). Let

$$S \xrightarrow{q_l} I$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
1 \arrow[r, "i_0"] \arrow[d, "i_1"'] & I \arrow[d, "r_0"] \arrow[ddr, bend left, "id"] & \\
I \arrow[r, "r_1"'] \arrow[drr, bend right, "i_0 \circ p"'] & S \arrow[dr, "q_l"] & \\
& & I
\end{tikzcd}$$

Then $\widehat{\mathsf{I}}$ has *strictness of left identities* if the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
I \arrow[r, "s"] \arrow[dr, "id"'] & S \arrow[d, "q_l"] \\
& I
\end{tikzcd}$$

Remark VI.35. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p and a subdivision structure (S, r_0, r_1, s). Suppose that Requirement VI.15 holds and that $\widehat{\mathsf{I}}$ has strictness of left identities.

Then the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ equipped with the contraction structure $- \otimes p$ and the subdivision structure $(-\otimes S, - \otimes r_0, - \otimes r_1, - \otimes s)$ has strictness of left identities.

If both I and S are exponentiable with respect to $\otimes$, then the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ equipped with the contraction structure $(-)^p$ and the subdivision structure $((-)^S, (-)^{r_0}, (-)^{r_1}, (-)^s)$ has strictness of left identities.

Definition VI.36. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p, and a subdivision structure (S, r_0, r_1, s). Let

$$S \xrightarrow{q_r} I$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
1 \arrow[r, "i_0"] \arrow[d, "i_1"'] & I \arrow[d, "r_0"] \arrow[ddr, bend left, "i_1 \circ p"] & \\
I \arrow[r, "r_1"'] \arrow[drr, bend right, "id"'] & S \arrow[dr, "q_r"] & \\
& & I
\end{tikzcd}$$

Then $\widehat{\mathsf{I}}$ has *strictness of right identities* if the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} I \arrow[r, "s"] \arrow[dr, "id"'] & S \arrow[d, "q_r"] \\ & I \end{tikzcd}$$

Remark VI.37. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p and a subdivision structure (S, r_0, r_1, s). Suppose that Requirement VI.15 holds, and that $\widehat{\mathsf{I}}$ has strictness of right identities.

Then the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ equipped with the contraction structure $- \otimes p$ and the subdivision structure $(- \otimes S, - \otimes r_0, - \otimes r_1, - \otimes s)$ has strictness of right identities.

If both I and S are exponentiable with respect to $\otimes$, then the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ equipped with the contraction structure $(-)^p$ and the subdivision structure $((-)^S, (-)^{r_0}, (-)^{r_1}, (-)^s)$ has strictness of right identities.

Definition VI.38. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, p, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with a contraction structure p and a subdivision structure (S, r_0, r_1, S). Then $\widehat{\mathsf{I}}$ has *strictness of identities* if it has both strictness of left identities and strictness of right identities.

Definition VI.39. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, v, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with an involution structure v, and a subdivision structure (S, r_0, r_1, s). Let

$$S \xrightarrow{w} I$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} 1 \arrow[r, "i_0"] \arrow[d, "i_1"'] & I \arrow[d, "r_0"] \arrow[ddr, bend left, "id"] & \\ I \arrow[r, "r_1"'] \arrow[drr, bend right, "v"'] & S \arrow[dr, "w"] & \\ & & I \end{tikzcd}$$

Then $\widehat{\mathsf{I}}$ has *strictness of left inverses* if the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} I \arrow[r, "s"] \arrow[d, "p"'] & S \arrow[d, "w"] \\ 1 \arrow[r, "i_1"'] & I \end{tikzcd}$$

Remark VI.40. Let $\widehat{\mathsf{I}} = (I, i_0, i_1, v, S, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with an involution structure v and a subdivision structure (S, r_0, r_1, s). Suppose that Requirement VI.15 holds. Then the cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ in $\mathcal{A}$ equipped with the involution structure $-\otimes v$ and the subdivision structure $\overline{(-\otimes S}, -\otimes r_0, -\otimes r_1, -\otimes s)$ has strictness of left inverses.

If both I and S are exponentiable with respect to $\otimes$, then the co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$ equipped with the involution structure $(-)^v$ and the subdivision structure $(\overline{(-)^S}, (-)^{r_0}, (-)^{r_1}, (-)^s)$ has strictness of left inverses.

Remark VI.41. We shall not need to consider strictness of right inverses.

VII. Homotopy and relative homotopy

In III we introduced a notion of a cylinder or a co-cylinder in a formal category $\mathcal{A}$. We now explain that a cylinder gives rise to a notion of homotopy between arrows of $\mathcal{A}$. In a dual manner, a co-cylinder gives rise to a notion of homotopy between arrows of $\mathcal{A}$. Thus a cylinder or a co-cylinder allows us to define a notion of homotopy equivalence in $\mathcal{A}$.

If $\mathcal{A}$ admits both a cylinder and a co-cylinder, as we shall assume later in this work, it will be crucial for us to know that the corresponding notions of homotopy coincide. In IV we mentioned that an adjunction between the cylinder and the co-cylinder ensures that this holds. We shall now be able to observe this.

Homotopy theory with respect to a cylinder or a co-cylinder alone is rather spartan. The structures upon a cylinder or co-cylinder defined in III allow for a much richer theory, which we shall explore in the remainder of this work.

At first, all of our constructions will be abstractions from the homotopy theory of topological spaces. Later on, for instance in IX when we shall require that the strictness of identities hypotheses introduced in III hold, topological spaces will no longer be our guide.

A contraction structure allows us to construct identity homotopies. An involution structure allows us to reverse homotopies. A subdivision structure allows to compose homotopies. In the presence of both an involution structure and a subdivision structure, homotopy equivalences in $\mathcal{A}$ have the two out of three property.

Given a pair of commutative diagrams

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "f'"'] & a_0 \arrow[d, "f"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "f'"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$, where r_0 is a retraction of g_0 and r_1 is a retraction of g_1, we demonstrate that if f is a homotopy equivalence then so is f'. We make a technical observation concerning homotopy inverses, which we shall appeal to in our proof in XI of Dold's theorem.

We introduce the notion of a double homotopy with respect to a cylinder, and explain a pictorial notation. Double homotopies will play an indispensable role throughout this work. Our three flavours of connection structures allow us to construct double homotopies with specific boundary homotopies.

With respect to a cylinder or a co-cylinder equipped with a contraction structure, we define a notion of homotopy under or over an object of $\mathcal{A}$. If $\mathcal{A}$ admits both a cylinder

$\underline{\mathsf{Cyl}}$ equipped with a contraction structure p, and a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}$ equipped with a contraction structure c, then an adjunction between $\underline{\mathsf{Cyl}}$ and $\underline{\mathsf{co\text{-}Cyl}}$ which is compatible with p and c allows us to observe that the notion of a homotopy equivalence under (respectively over) an object with respect to $\underline{\mathsf{Cyl}}$ coincides with that of a homotopy equivalence under (respectively over) an object with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Identity homotopies are also identity homotopies under or over an object. An involution structure which is compatible with contraction allows us to construct reverse homotopies under or over an object. A subdivision structure which is compatible with contraction allows us to compose homotopies under or over an object.

We conclude by introducing the notion of a strong deformation retraction with respect to a cylinder or a co-cylinder. All consideration of homotopies under or over an object in this work relates to strong deformation retractions.

As discussed in I, homotopy with respect to a cylinder in an abstract setting was first considered by Kan in [24]. The insight that further structure upon a cylinder can give rise to a richer theory is due to Kamps, presented in works such as [21] from around 1970.

In a setting closer to that in which we are working, the observation that a subdivision structure upon a cylinder allows one to compose homotopies was first explored, to the author's knowledge, by Grandis in papers such as [13], written in the 1990s. The abstract notion of homotopy under and over an object is for example discussed for in the book [23] of Kamps and Porter, which also treats much of the rest of this section.

Assumption VII.1. Let $\mathcal{C}$ be a 2-category with a final object. Suppose that pushouts and pullbacks of 2-arrows of $\mathcal{C}$ give rise to pushouts and pullbacks in formal categories, in the sense of Definition II.14. Let $\mathcal{A}$ be an object of $\mathcal{C}$. Recall that we view $\mathcal{A}$ as a formal category, writing of objects and arrows of $\mathcal{A}$. This terminology and all other formal category theory preliminaries can be found in II.

Definition VII.2. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1,)$ be a cylinder in $\mathcal{A}$, and let

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$. A *homotopy* from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$ is an arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

of $\mathcal{A}$, such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[dr, "f_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\ & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "i_1(a_0)"] \arrow[dr, "f_1"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\ & a_1 \end{tikzcd}$$

Definition VII.3. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$, and let

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$.

A *homotopy* from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is an arrow

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$, such that the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd} a_0 \arrow[r, "h"] \arrow[dr, "f_0"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "e_0(a_1)"] \\ & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "h"] \arrow[dr, "f_1"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "e_1(a_1)"] \\ & a_1 \end{tikzcd}$$

Remark VII.4. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$, and let

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1$$

of $\mathcal{A}$ is a homotopy from f_0 to f_1 if and only if h^{op} is a homotopy from f_0^{op} to f_1^{op} with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Proposition VII.5. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$. Let*

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the corresponding natural isomorphism, adopting the shorthand of Recollection II.13.

Let

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$, and suppose that

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

defines a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$. Then the arrow

$$a_0 \xrightarrow{\mathsf{adj}(h)} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$ defines a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from the fact that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$. $\square$

Corollary VII.6. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$. Let*

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the corresponding natural isomorphism, adopting the shorthand of Recollection II.13.

Let

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$, and suppose that

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1)$$

defines a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$. Then the arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{\mathsf{adj}^{-1}(h)} a_1$$

of $\mathcal{A}$ defines a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$.

Proof. Follows immediately from Proposition VII.5 by duality. $\square$

Proposition VII.7. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$. Then the arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{f \circ p(a_0)} a_1$$

of $\mathcal{A}$ defines a homotopy from f to itself with respect to $\underline{\mathsf{Cyl}}$.

Proof. Follows immediately from the fact that p defines a contraction structure with respect to $\underline{\mathsf{Cyl}}$. $\square$

Remark VII.8. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Given an arrow f of $\mathcal{A}$, we refer to the corresponding homotopy of Proposition VII.7 from f to itself as the *identity homotopy* from f to itself with respect to $\underline{\mathsf{Cyl}}$.

Proposition VII.9. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, v)$ be a cylinder in $\mathcal{A}$ equipped with an involution structure v. Let*

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$, and let

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

be a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$. The arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{h \circ v(a_0)} a_1$$

of $\mathcal{A}$ defines a homotopy from f_1 to f_0 with respect to $\underline{\mathsf{Cyl}}$.

Proof. Follows immediately from the fact that v defines an involution structure with respect to $\underline{\mathsf{Cyl}}$. $\square$

Remark VII.10. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, v)$ be a cylinder in $\mathcal{A}$ equipped with an involution structure v. Given a homotopy h with respect to $\underline{\mathsf{Cyl}}$ between a pair of arrows of $\mathcal{A}$, we refer to the corresponding homotopy of Proposition VII.9 as the *reverse* of h, and denote it by h^{-1}.

Proposition VII.11. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, \mathsf{S}, r_0, r_1, s)$ be an interval in $\mathcal{A}$ equipped with a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Let*

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

be a homotopy with respect to $\underline{\mathsf{Cyl}}$ from an arrow

$$a_0 \xrightarrow{f_0} a_1$$

of $\mathcal{A}$ to an arrow

$$a_0 \xrightarrow{f_1} a_1$$

of $\mathcal{A}$. Let

$$\mathsf{Cyl}(a_0) \xrightarrow{k} a_1$$

be a homotopy with respect to $\underline{\mathsf{Cyl}}$ from f_1 to a third arrow

$$a_0 \xrightarrow{f_2} a_1$$

of $\mathcal{A}$.

Let

$$\mathsf{S}(a_0) \xrightarrow{r} a_1$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "k"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "h"'] & \mathsf{S}(a_0) \arrow[dr, "r"] & \\
 & & a_1
\end{tikzcd}$$

Then the arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{r \circ s(a_0)} a_1$$

of $\mathcal{A}$ defines a homotopy from f_0 to f_2 with respect to Cyl.

Proof. The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_0(a_0)"] \arrow[dd, bend right=60, "f_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "s(a_0)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[d, "h"] & \mathsf{S}(a_0) \arrow[dl, "r"] \\
a_1 &
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_1(a_0)"] \arrow[d, "i_1(a_0)"] \arrow[dd, bend right=60, "f_2"'] & \mathsf{Cyl}(a_0) \arrow[d, "s(a_0)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "r_0(a_0)"'] \arrow[d, "k"] & \mathsf{S}(a_0) \arrow[dl, "r"] \\
a_1 &
\end{tikzcd}$$

□

Remark VII.12. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Let f_0, f_1, and f_2 be arrows of $\mathcal{A}$, let h be a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$, and let k be a homotopy from f_1 to f_2 with respect to $\underline{\mathsf{Cyl}}$.

We denote by $h + k$ the corresponding homotopy of Proposition VII.11 from f_0 to f_2 with respect to $\underline{\mathsf{Cyl}}$, and refer to it as a *composite homotopy*.

Remark VII.13. Thus if $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s)$ is a cylinder in $\mathcal{A}$ equipped with a contraction structure p, an involution structure v, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, then homotopy with respect to $\underline{\mathsf{Cyl}}$ defines an equivalence relation upon the arrows of $\mathcal{A}$.

Definition VII.14. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$, and let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$. A *homotopy inverse* of f with respect to $\underline{\mathsf{Cyl}}$ is an arrow

$$a_1 \xrightarrow{f^{-1}} a_0$$

of $\mathcal{A}$, together with a homotopy from $f^{-1}f$ to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$, and a homotopy from ff^{-1} to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$.

Definition VII.15. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is a *homotopy equivalence* with respect to $\underline{\mathsf{Cyl}}$ if it admits a homotopy inverse with respect to $\underline{\mathsf{Cyl}}$.

Definition VII.16. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is a *homotopy equivalence* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if f^{op} is a homotopy equivalence with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Proposition VII.17. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$. Then an arrow*

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$ if and only if it is a homotopy equivalence with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition VII.5 and Corollary VII.6. □

Lemma VII.18. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Suppose that we have four arrows*

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] & a_1 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_2 \arrow[r, "g_1"] & a_3 \end{tikzcd}$$

of $\mathcal{A}$, and a homotopy

$$\mathsf{Cyl}(a_1) \xrightarrow{h} a_2$$

from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$. Then the arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{g_1 \circ h \circ \mathsf{Cyl}(g_0)} a_3$$

of $\mathcal{A}$ defines a homotopy from $g_1 f_0 g_0$ to $g_1 f_1 g_0$ with respect to $\underline{\mathsf{Cyl}}$.

Proof. The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "g_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(g_0)"] \\ a_1 \arrow[r, "i_0(a_1)"] \arrow[dr, "f_0"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\ & a_2 \end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[dr, "g_1 \circ f_0 \circ g_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "g_1 \circ h \circ \mathsf{Cyl}(g_0)"] \\ & a_3 \end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_1(a_0)"] \arrow[d, "g_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(g_0)"] \\ a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "f_1"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\ & a_2 \end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_1(a_0)"] \arrow[dr, "g_1 \circ f_1 \circ g_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "g_1 \circ h \circ \mathsf{Cyl}(g_0)"] \\ & a_3 \end{tikzcd}$$

□

Lemma VII.19. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, v, \mathsf{S}, r_0, r_1, s)$ *be a cylinder in* $\mathcal{A}$ *equipped with an involution structure* v *and a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$. *Suppose that we have a commutative diagram in* $\mathcal{A}$ *as follows.*

$$\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[dr, "f_2"'] & a_1 \arrow[d, "f_1"] \\ & a_2 \end{tikzcd}$$

If f_1 *and* f_2 *are homotopy equivalences with respect to* $\underline{\mathsf{Cyl}}$, *then* f_0 *is a homotopy equivalence with respect to* $\underline{\mathsf{Cyl}}$.

Proof. Let f_1^{-1} be a homotopy inverse of f_1 with respect to $\underline{\mathsf{Cyl}}$, and let

$$\mathsf{Cyl}(a_1) \xrightarrow{h_1} a_1$$

denote the corresponding homotopy from $f_1^{-1}f_1$ to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$. Let f_2^{-1} be a homotopy inverse of f_2 with respect to $\underline{\mathsf{Cyl}}$, and let

$$\mathsf{Cyl}(a_2) \xrightarrow{h_2} a_2$$

denote the corresponding homotopy from $f_2f_2^{-1}$ to $id(a_2)$ with respect to $\underline{\mathsf{Cyl}}$. We claim that the arrow

$$a_1 \xrightarrow{f_2^{-1} \circ f_1} a_0$$

of $\mathcal{A}$ defines a homotopy inverse to f_0 with respect to $\underline{\mathsf{Cyl}}$.

Let

$$\mathsf{Cyl}(a_1) \xrightarrow{k_0} a_1$$

denote the arrow $h_1^{-1} \circ \mathsf{Cyl}(f_0f_2^{-1}f_1)$ of $\mathcal{A}$. By Lemma VII.18, we have that k_0 defines a homotopy from $f_0f_2^{-1}f_1$ to $f_1^{-1}f_1f_0f_2^{-1}f_1$ with respect to $\underline{\mathsf{Cyl}}$. We have that

$$f_1^{-1}f_1f_0f_2^{-1}f_1 = f_1^{-1}f_2f_2^{-1}f_1.$$

Let

$$\mathsf{Cyl}(a_1) \xrightarrow{k_1} a_1$$

denote the arrow $f_1^{-1} \circ h_2 \circ \mathsf{Cyl}(f_1)$ of $\mathcal{A}$. Appealing again to Lemma VII.18, we have that k_1 defines a homotopy from $f_1^{-1}f_2f_2^{-1}f_1$ to $f_1^{-1}f_1$ with respect to $\underline{\mathsf{Cyl}}$.

We deduce that the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{(k_0 + k_1) + h_1} a_1$$

of $\mathcal{A}$ defines a homotopy from $f_0f_2^{-1}f_1$ to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$.

We also have that h_2^{-1} defines a homotopy from

$$f_2^{-1}f_1f_0 = f_2^{-1}f_2$$

to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$. This completes the proof of the claim. □

Lemma VII.20. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, v, \mathsf{S}, r_0, r_1, s)$ *be a cylinder in* $\mathcal{A}$ *equipped with an involution structure* v *and a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$. *Suppose that we have a commutative diagram in* $\mathcal{A}$ *as follows.*

$$\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[dr, "f_2"'] & a_1 \arrow[d, "f_1"] \\ & a_2 \end{tikzcd}$$

If f_0 and f_2 are homotopy equivalences with respect to $\underline{\mathsf{Cyl}}$, then f_1 is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$.

Proof. Let f_0^{-1} be a homotopy inverse of f_0 with respect to $\underline{\mathsf{Cyl}}$, and let

$$\mathsf{Cyl}(a_1) \xrightarrow{h_0} a_1$$

denote the corresponding homotopy from $f_0 f_0^{-1}$ to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$. Let f_2^{-1} be a homotopy inverse of f_2 with respect to $\underline{\mathsf{Cyl}}$, and let

$$\mathsf{Cyl}(a_2) \xrightarrow{h_2} a_2$$

denote the corresponding homotopy from $f_2 f_2^{-1}$ to $id(a_2)$ with respect to $\underline{\mathsf{Cyl}}$. We claim that the arrow

$$a_2 \xrightarrow{f_0 \circ f_1^{-1}} a_1$$

of $\mathcal{A}$ defines a homotopy inverse to f_1 with respect to $\underline{\mathsf{Cyl}}$.

Let

$$\mathsf{Cyl}(a_1) \xrightarrow{k_0} a_1$$

denote the arrow $f_0 f_2^{-1} f_1 \circ h_0^{-1}$ of $\mathcal{A}$. By Lemma VII.18, we have that k_0 defines a homotopy from $f_0 f_2^{-1} f_1$ to $f_0 f_2^{-1} f_1 f_0 f_0^{-1}$ with respect to $\underline{\mathsf{Cyl}}$. We also have that

$$f_0 f_2^{-1} f_1 f_0 f_0^{-1} = f_0 f_2^{-1} f_2 f_0^{-1}.$$

Let

$$\mathsf{Cyl}(a_1) \xrightarrow{k_1} a_1$$

denote the arrow $f_0 \circ h_2 \circ \mathsf{Cyl}(f_0^{-1})$ of $\mathcal{A}$. Appealing again to Lemma VII.18, we have that k_1 defines a homotopy from $f_0 f_2^{-1} f_2 f_0^{-1}$ to $f_0 f_0^{-1}$ with respect to $\underline{\mathsf{Cyl}}$. We deduce that the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{(k_0 + k_1) + h_0} a_1$$

of $\mathcal{A}$ defines a homotopy from $f_0 f_2^{-1} f_1$ to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$.

We also have that h_2 defines a homotopy from

$$f_1 f_0 f_2^{-1} = f_2 f_2^{-1}$$

to $id(a_2)$ with respect to $\underline{\mathsf{Cyl}}$. This completes the proof of the claim. □

Proposition VII.21. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, v, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with an involution structure v and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Then homotopy equivalences with respect to $\underline{\mathsf{Cyl}}$ have the two-out-of-three property.*

Proof. Follows immediately from Lemma VII.19, Lemma VII.20, and Proposition VII.11. ☐

Proposition VII.22. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ which is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$. Suppose that we have commutative diagrams

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "f'"'] & a_0 \arrow[d, "f"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "f"'] & a_2 \arrow[d, "f'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$, such that r_0 is a retraction of g_0, and such that r_1 is a retraction of g_1. Then f' is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$.

Proof. Let

$$a_1 \xrightarrow{f^{-1}} a_0$$

be a homotopy inverse of f. Let h_0 be a homotopy from $f^{-1}f$ to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$, and let h_1 be a homotopy from ff^{-1} to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$.

Let

$$a_3 \xrightarrow{(f')^{-1}} a_2$$

denote the arrow $r_0 \circ f^{-1} \circ g_1$ of $\mathcal{A}$. Let

$$\mathsf{Cyl}(a_2) \xrightarrow{h_0'} a_2$$

denote the arrow $r_0 \circ h_0 \circ \mathsf{Cyl}(g_0)$ of $\mathcal{A}$. We claim that h_0' defines a homotopy from $(f')^{-1}f'$ to $id(a_2)$ with respect to $\underline{\mathsf{Cyl}}$.

Firstly, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[rr, "i_0(a_2)"] \arrow[d, "f'"'] \arrow[dr, "g_0"] & & \mathsf{Cyl}(a_2) \arrow[d, "\mathsf{Cyl}(g_0)"] \\
a_3 \arrow[d, "g_1"'] & a_0 \arrow[r, "i_0(a_0)"] \arrow[dl, "f"] & \mathsf{Cyl}(a_0) \arrow[d, "h_0"] \\
a_1 \arrow[rr, "f^{-1}"'] & & a_0
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[r, "i_0(a_2)"] \arrow[d, "f'"'] & \mathsf{Cyl}(a_2) \arrow[d, "r_0 \circ h_0 \circ \mathsf{Cyl}(g_0)"] \\
a_3 \arrow[r, "r_0 \circ f^{-1} \circ \mathsf{Cyl}(g_0)"'] & a_2
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[r, "i_0(a_2)"] \arrow[d, "f'"'] & \mathsf{Cyl}(a_2) \arrow[d, "h_0'"] \\
a_3 \arrow[r, "(f')^{-1}"'] & a_2
\end{tikzcd}$$

Secondly, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[r, "i_1(a_2)"] \arrow[d, "g_0"'] & \mathsf{Cyl}(a_2) \arrow[d, "\mathsf{Cyl}(g_0)"] \\
a_0 \arrow[r, "i_1(a_0)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_0"] \\
 & a_0
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[r, "i_1(a_2)"] \arrow[d, "id"'] \arrow[dr, "g_0"'] & \mathsf{Cyl}(a_2) \arrow[d, "h_0 \circ \mathsf{Cyl}(g_0)"] \\
a_2 & a_0 \arrow[l, "r_0"]
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[r, "i_1(a_2)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_2) \arrow[d, "h'_0"] \\
& a_2
\end{tikzcd}$$

This completes the proof of the claim.

Let

$$\mathsf{Cyl}(a_3) \xrightarrow{h'_1} a_3$$

denote the arrow $r_1 \circ h_1 \circ \mathsf{Cyl}(g_1)$ of $\mathcal{A}$. We claim that h'_1 defines a homotopy from $(f')^{-1}f'$ to $id(a_3)$ with respect to $\underline{\mathsf{Cyl}}$.

Firstly, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "f^{-1}"'] & \mathsf{Cyl}(a_1) \arrow[d, "h_1"] \\
a_0 \arrow[r, "f"'] \arrow[d, "r_0"'] & a_1 \arrow[d, "r_1"] \\
a_2 \arrow[r, "f'"'] & a_3
\end{tikzcd}$$

Hence, appealing to the commutativity of the diagram

$$\begin{tikzcd}
a_3 \arrow[r, "i_0(a_3)"] \arrow[d, "g_1"'] & \mathsf{Cyl}(a_3) \arrow[d, "\mathsf{Cyl}(g_1)"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

in $\mathcal{A}$, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_3 \arrow[r, "i_0(a_3)"] \arrow[d, "r_0 \circ f^{-1} \circ g_1"'] & \mathsf{Cyl}(a_3) \arrow[d, "r_1 \circ h_1 \circ \mathsf{Cyl}(g_1)"] \\
a_2 \arrow[r, "f'"'] & a_3
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_3 \arrow[r, "i_0(a_3)"] \arrow[d, "(f')^{-1}"'] & cyl(a_3) \arrow[d, "h_1'"] \\
a_2 \arrow[r, "f'"'] & a_3
\end{tikzcd}$$

Secondly, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_3 \arrow[r, "i_1(a_3)"] \arrow[d, "g_1"'] & \mathsf{Cyl}(a_3) \arrow[d, "\mathsf{Cyl}(g_1)"] \\
a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_1) \arrow[d, "h_1"] \\
 & a_1
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_3 \arrow[r, "i_1(a_3)"] \arrow[d, "id"'] \arrow[dr, "g_1"'] & \mathsf{Cyl}(a_3) \arrow[d, "h_1 \circ \mathsf{Cyl}(g_1)"] \\
a_3 & a_1 \arrow[l, "r_1"]
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_3 \arrow[r, "i_1(a_3)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_3) \arrow[d, "h_1'"] \\
 & a_3
\end{tikzcd}$$

This completes the proof of the claim.

$\square$

Lemma VII.23. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, v, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with an involution structure v, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ which is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$, and let

$$a_1 \xrightarrow{g} a_0$$

be an arrow of $\mathcal{A}$ such that there is a homotopy

$$\mathsf{Cyl}(a_1) \xrightarrow{h} a_1$$

from fg to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$. Then g is a homotopy inverse of f with respect to $\underline{\mathsf{Cyl}}$.

Proof. Let

$$a_1 \xrightarrow{f^{-1}} a_0$$

be a homotopy inverse of f with respect to $\underline{\mathsf{Cyl}}$, and let

$$\mathsf{Cyl}(a_0) \xrightarrow{k} a_0$$

denote the corresponding homotopy from $f^{-1}f$ to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$. By Lemma VII.18, we have that the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{k \circ \mathsf{Cyl}(g)} a_0$$

of $\mathcal{A}$ defines a homotopy from $f^{-1}fg$ to g with respect to $\underline{\mathsf{Cyl}}$.

By Lemma VII.18 once more, we also have that the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{f^{-1} \circ h^{-1}} a_0$$

of $\mathcal{A}$ defines a homotopy from f^{-1} to $f^{-1}fg$ with respect to $\underline{\mathsf{Cyl}}$. Hence the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{(k \circ \mathsf{Cyl}(g)) + (f^{-1} \circ h^{-1})} a_0$$

of $\mathcal{A}$ defines a homotopy from f^{-1} to g with respect to $\underline{\mathsf{Cyl}}$. Let us denote it by l for brevity.

Appealing again to Lemma VII.18, we have that the arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{l \circ \mathsf{Cyl}(f)} a_0$$

of $\mathcal{A}$ defines a homotopy from $f^{-1}f$ to gf with respect to $\underline{\mathsf{Cyl}}$. Then

$$\mathsf{Cyl}(a_0) \xrightarrow{(l \circ \mathsf{Cyl}(f)) + k^{-1}} a_0$$

defines a homotopy from $id(a_0)$ to gf with respect to $\underline{\mathsf{Cyl}}$.

Thus

$$\mathsf{Cyl}(a_0) \xrightarrow{\left((l \circ \mathsf{Cyl}(f)) + k^{-1}\right)^{-1}} a_0$$

defines a homotopy from gf to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$. □

Remark VII.24. An analogous argument shows that if

$$a_0 \xrightarrow{f} a_1$$

satisfies the hypotheses of Lemma VII.23, and if g is an arrow of $\mathcal{A}$ such that there is a homotopy from gf to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$, then g is a homotopy inverse of f with respect to $\underline{\mathsf{Cyl}}$. We shall not need this.

Definition VII.25. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Let a_0 and a_1 be objects of $\mathcal{A}$. We refer to an arrow

$$\mathsf{Cyl}^2(a_0) \xrightarrow{\sigma} a_1$$

of $\mathcal{A}$ as a *double homotopy* with respect to $\underline{\mathsf{Cyl}}$.

Definition VII.26. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Let

$$\mathsf{Cyl}^2(a_0) \xrightarrow{\sigma} a_1$$

be a double homotopy with respect to $\underline{\mathsf{Cyl}}$. We refer to the arrows h_0, h_1, h_2, and h_3 of $\mathcal{A}$ defined by the commutative diagrams below as the *boundary homotopies* of σ with respect to $\underline{\mathsf{Cyl}}$.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "i_0(\mathsf{Cyl}(a_0))"] \arrow[dr, "h_0"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\sigma"] \\ & a_1 \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_1(a_0))"] \arrow[dr, "h_1"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\sigma"] \\ & a_1 \end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[dr, "h_2"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\sigma"] \\
& a_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "i_1(\mathsf{Cyl}(a_0))"] \arrow[dr, "h_3"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\sigma"] \\
& a_1
\end{tikzcd}$$

Remark VII.27. The following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_0(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_2"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h_0"'] & a_1
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_1"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h_0"'] & a_1
\end{tikzcd}$$

$$\begin{tikzcd}
a_0 \arrow[r, "i_1(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_1"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h_3"'] & a_1
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[r, "i_1(a_0)"] \arrow[d, "i_0(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_2"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h_3"'] & a_1
\end{tikzcd}$$

We denote by f_0, f_1, f_2, and f_3 the arrows of $\mathcal{A}$ obtained by taking either route through each of these four commutative diagrams, proceeding clockwise respectively from the top left diagram. Thus the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd}
a_0) \arrow[r, "i_0(a_0)"] \arrow[d, "i_0(a_0)"'] \arrow[dr, "f_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_2"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h_0"'] & a_1
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] \arrow[dr, "f_1"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_1"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h_0"'] & a_1
\end{tikzcd}$$

$$\begin{tikzcd}
a_0 \arrow[r, "i_1(a_0)"] \arrow[d, "i_1(a_0)"'] \arrow[dr, "f_3"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_1"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h_3"'] & a_1
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[r, "i_1(a_0)"] \arrow[d, "i_0(a_0)"'] \arrow[dr, "f_2"'] & \mathsf{Cyl}(a_0) \arrow[d, "h_2"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h_3"'] & a_1
\end{tikzcd}$$

In summary:

(i) h_0 defines a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$,

(ii) h_1 defines a homotopy from f_1 to f_3 with respect to $\underline{\mathsf{Cyl}}$,

(iii) h_2 defines a homotopy from f_0 to f_2 with respect to $\underline{\mathsf{Cyl}}$,

(iv) h_3 defines a homotopy from f_2 to f_3 with respect to $\underline{\mathsf{Cyl}}$.

To express (i)–(iv) we shall frequently depict σ as follows.

$$\begin{tikzcd} f_0 \arrow[r, "h_0"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\sigma"] & f_1 \arrow[d, "h_1"] \\ f_2 \arrow[r, "h_3"'] & f_3 \end{tikzcd}$$

Proposition VII.28. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \Gamma_{ul})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and an upper left connection structure Γ_{ul}. Let*

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$, and let

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

be a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$. Let

$$\mathsf{Cyl}^2(a_0) \xrightarrow{\sigma} a_1$$

denote the arrow $h \circ \Gamma_{ul}(a_0)$ of $\mathcal{A}$. Then σ defines a double homotopy with respect to $\underline{\mathsf{Cyl}}$, with the boundary homotopies depicted below.

$$\begin{tikzcd} f_0 \arrow[r, "h"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma"] & f_1 \arrow[d, "id"] \\ f_1 \arrow[r, "id"'] & f_1 \end{tikzcd}$$

Proof. Follows immediately from the fact that Γ_{ul} defines an upper left connection structure with respect to $\underline{\mathsf{Cyl}}$. $\square$

Proposition VII.29. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \Gamma_{lr})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and a lower right connection structure Γ_{lr}. Let*

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$, and let

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

be a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$. Let

$$\mathsf{Cyl}^2(a_0) \xrightarrow{\sigma} a_1$$

denote the arrow $h \circ \Gamma_{lr}(a_0)$ of $\mathcal{A}$. Then σ defines a double homotopy with respect to $\underline{\mathsf{Cyl}}$, with the boundary homotopies depicted below.

$$\begin{tikzcd} f_0 \arrow[r, "id"] \arrow[d, "id"'] \arrow[dr, phantom, "\sigma"] & f_0 \arrow[d, "h"] \\ f_0 \arrow[r, "h"'] & f_1 \end{tikzcd}$$

Proof. Follows immediately from the fact that Γ_{lr} defines a lower right connection structure with respect to $\underline{\mathsf{Cyl}}$. $\square$

Proposition VII.30. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \Gamma_{ur})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and an upper right connection structure Γ_{ur}. Let*

$$\begin{tikzcd} a_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & a_1 \end{tikzcd}$$

be arrows of $\mathcal{A}$, and let

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

be a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$. Let

$$\mathsf{Cyl}^2(a_0) \xrightarrow{\sigma} a_1$$

denote the arrow $h \circ \Gamma_{ur}(a_0)$ of $\mathcal{A}$. Then σ defines a double homotopy with respect to $\underline{\mathsf{Cyl}}$, with the boundary homotopies depicted below.

$$\begin{tikzcd} f_0 \arrow[r, "h"] \arrow[d, "id"'] \arrow[dr, phantom, "\sigma"] & f_1 \arrow[d, "h^{-1}"] \\ f_0 \arrow[r, "id"'] & f_0 \end{tikzcd}$$

Proof. Follows immediately from the fact that Γ_{ur} defines an upper right connection structure with respect to $\underline{\mathsf{Cyl}}$. $\square$

Definition VII.31. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_0"] \\ & a_1 \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_1"] \\ & a_1 \end{tikzcd}$$

A *homotopy under a* from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1) is a homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a) \arrow[r, "p(a)"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a \arrow[d, "j_1"] \\ \mathsf{Cyl}(a_0) \arrow[r, "h"'] & a_1 \end{tikzcd}$$

Definition VII.32. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[d, "f_0"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[d, "f_1"'] \arrow[dr, "j_1"] & \\ a_1 \arrow[r, "j_0"'] & a \end{tikzcd}$$

A *homotopy over a* from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1) is a homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

Definition VII.33. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_0"] \\
& a_1
\end{tikzcd} \qquad \begin{tikzcd}
a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_1"] \\
& a_1
\end{tikzcd}$$

A *homotopy under a* from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$ and (j_0, j_1) is a homotopy

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1)$$

from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a \arrow[r, "c(a)"] \arrow[d, "j_0"'] & \mathsf{co\text{-}Cyl}(a) \arrow[d, "\mathsf{co\text{-}Cyl}(j_1)"] \\
a_0 \arrow[r, "h"'] & \mathsf{co\text{-}Cyl}(a_1)
\end{tikzcd}$$

Remark VII.34. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_0"] \\
& a_1
\end{tikzcd} \qquad \begin{tikzcd}
a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_1"] \\
& a_1
\end{tikzcd}$$

Then an arrow

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$ is a homotopy under a from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if and only if h^{op} is a homotopy over a from f_0^{op} to f_1^{op} with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op}, and with respect to the arrows (j_1^{op}, j_0^{op}) of $\mathcal{A}^{op}$.

Definition VII.35. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[d, "f_0"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[d, "f_1"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

A *homotopy over a* from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$ and (j_0, j_1) is a homotopy

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1)$$

from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "h"] \arrow[d, "j_0"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "\mathsf{co\text{-}Cyl}(j_1)"] \\ a \arrow[r, "c(a)"'] & \mathsf{co\text{-}Cyl}(a) \end{tikzcd}$$

Remark VII.36. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[d, "f_0"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[d, "f_1"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

An arrow

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$ is a homotopy over a from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$ and (j_0, j_1) if and only if h^{op} is a homotopy under a from f_0^{op} to f_1^{op} with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}^{op}}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op}, and with respect to the arrows (j_1^{op}, j_0^{op}) of $\mathcal{A}^{op}$.

Proposition VII.37. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*. Let*

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the corresponding natural isomorphism, adopting the shorthand of Recollection II.13. Suppose that the adjunction between Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ *as follows.*

$$\begin{tikzcd} a_0 \arrow[d, "f_0"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[d, "f_1"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

If an arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

of $\mathcal{A}$ *defines a homotopy over* a *from* f_0 *to* f_1 *with respect to* $\underline{\mathsf{Cyl}}$ *and* (j_0, j_1)*, then the arrow*

$$a_0 \xrightarrow{\mathsf{adj}(h)} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$ *defines a homotopy over* a *from* f_0 *to* f_1 *with respect to* $\underline{\mathsf{co\text{-}Cyl}}$ *and* (j_0, j_1)*.*

Proof. Firstly, by Proposition VII.5 we have that if h is a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$, then $\mathsf{adj}(h)$ is a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Secondly, since h is a homotopy over a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1), the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd}$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j_0)"] \arrow[d, "j_0"'] & \mathsf{Cyl}(a) \arrow[d, "p(a)"] \\
a_0 \arrow[r, "p(a_0)"'] & a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "h"] \arrow[d, "p(a_0)"'] & a_1 \arrow[d, "j_1"] \\
a_0 \arrow[r, "j_0"'] & a
\end{tikzcd}$$

Thus we have that

$$\mathsf{adj}(j_1 \circ h) = \mathsf{adj}(j_0 \circ p(a_0)).$$

Moreover, by the naturality of the isomorphism adj, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "\mathsf{adj}(h)"] \arrow[dr, "\mathsf{adj}(j_1 \circ h)"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "\mathsf{co\text{-}Cyl}(j_1)"] \\
& \mathsf{co\text{-}Cyl}(a)
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "\mathsf{adj}(h)"] \arrow[dr, "\mathsf{adj}(j_0 \circ p(a_0))"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "\mathsf{co\text{-}Cyl}(j_1)"] \\
& \mathsf{co\text{-}Cyl}(a)
\end{tikzcd}$$

Since the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
a_0 \arrow[d, "j_0"'] \arrow[dr, "\mathsf{adj}(j_0 \circ p(a_0))"] & \\
a \arrow[r, "c(a)"'] & \mathsf{co\text{-}Cyl}(a)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "\mathsf{adj}(h)"] \arrow[d, "j_0"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "\mathsf{co\text{-}Cyl}(j_1)"] \\ a \arrow[r, "c(a)"'] & \mathsf{co\text{-}Cyl}(a) \end{tikzcd}$$

□

Proposition VII.38. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*. Let*

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the corresponding natural isomorphism, adopting the shorthand of Recollection II.13. Suppose that the adjunction between Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ *as follows.*

$$\begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_0"] \\ & a_1 \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_1"] \\ & a_1 \end{tikzcd}$$

If an arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

of $\mathcal{A}$ *defines a homotopy under* a *from* f_0 *to* f_1 *with respect to* $\underline{\mathsf{Cyl}}$ *and* (j_0, j_1)*, then the arrow*

$$a_0 \xrightarrow{\mathsf{adj}(h)} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$ *defines a homotopy under* a *from* f_0 *to* f_1 *with respect to* $\underline{\mathsf{co\text{-}Cyl}}$ *and* (j_0, j_1)*.*

Proof. Firstly, by Proposition VII.5 we have that if h is a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$, then $\mathsf{adj}(h)$ is a homotopy from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Secondly, since h is a homotopy under a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1), the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a) \arrow[r, "p(a)"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a \arrow[d, "j_1"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h"'] & a_1
\end{tikzcd}$$

Thus we have that

$$\mathsf{adj}(h \circ \mathsf{Cyl}(j_0)) = \mathsf{adj}(j_1 \circ p(a)).$$

Moreover, by the naturality of the isomorphism adj, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a \arrow[d, "j_0"'] \arrow[dr, "\mathsf{adj}(h \circ \mathsf{Cyl}(j_0))"] & \\
a_0 \arrow[r, "\mathsf{adj}(h)"'] & \mathsf{co\text{-}Cyl}(a_1)
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a \arrow[d, "j_0"'] \arrow[dr, "\mathsf{adj}(j_1 \circ p(a))"] & \\
a_0 \arrow[r, "\mathsf{adj}(h)"'] & \mathsf{co\text{-}Cyl}(a_1)
\end{tikzcd}$$

Since the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
a \arrow[r, "c(a)"] \arrow[dr, "\mathsf{adj}(j_1 \circ p(a))"'] & \mathsf{co\text{-}Cyl}(a) \arrow[d, "\mathsf{co\text{-}Cyl}(j_1)"] \\
& \mathsf{co\text{-}Cyl}(a_1)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a \arrow[r, "c(a)"] \arrow[d, "j_0"'] & \mathsf{co\text{-}Cyl}(a) \arrow[d, "\mathsf{co\text{-}Cyl}(j_1)"] \\
a_0 \arrow[r, "\mathsf{adj}(h)"'] & \mathsf{co\text{-}Cyl}(a_1)
\end{tikzcd}$$

$\square$

Corollary VII.39. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*. Let*

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the corresponding natural isomorphism, adopting the shorthand of Recollection II.13. Suppose that the adjunction between Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ *as follows.*

$$\begin{tikzcd} a_0 \arrow[d, "f_0"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[d, "f_1"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

If an arrow

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$ *defines a homotopy over* a *from* f_0 *to* f_1 *with respect to* $\underline{\mathsf{co\text{-}Cyl}}$ *and* (j_0, j_1)*, then the arrow*

$$\mathsf{Cyl}(a_0) \xrightarrow{\mathsf{adj}^{-1}(h)} a_1$$

of $\mathcal{A}$ *defines a homotopy over* a *from* f_0 *to* f_1 *with respect to* $\underline{\mathsf{Cyl}}$ *and* (j_0, j_1)*.*

Proof. Follows immediately from Proposition VII.38 by duality. $\square$

Corollary VII.40. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*. Let*

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the corresponding natural isomorphism, adopting the shorthand of Recollection II.13. Suppose that the adjunction between Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ *as follows.*

$$\begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_0"] \\ & a_1 \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f_1"] \\ & a_1 \end{tikzcd}$$

If an arrow

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$ defines a homotopy under a from f_0 to f_1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$ and (j_0, j_1), then the arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{\mathsf{adj}^{-1}(h)} a_1$$

of $\mathcal{A}$ defines a homotopy under a from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1).

Proof. Follows immediately from Proposition VII.37 by duality. □

Proposition VII.41. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.*

$$\begin{tikzcd} a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

Then the identity homotopy from f to itself with respect to $\underline{\mathsf{Cyl}}$ is moreover a homotopy over a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1).

Proof. Let h denote the identity homotopy from f to itself with respect to $\underline{\mathsf{Cyl}}$. By definition of h, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, "h"'] & a_0 \arrow[d, "f"] \\ & a_1 \end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[rr, "h"] \arrow[dr, "p(a_0)"] \arrow[dd, "\mathsf{Cyl}(j_0)"'] & & a_1 \arrow[dd, "j_1"] \\ & a_0 \arrow[ur, "f"'] \arrow[dr, "j_0"'] & \\ \mathsf{Cyl}(a) \arrow[rr, "p(a)"'] & & a \end{tikzcd}$$

□

Proposition VII.42. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and an involution structure v compatible with p.*

Suppose that we have arrows

$$a_0 \xrightarrow{j_0} a$$

and

$$a_1 \xrightarrow{j_1} a$$

of $\mathcal{A}$, and a homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

with respect to $\underline{\mathsf{Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "h^{-1}"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

Proof. By definition of h^{-1}, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "v(a_0)"] \arrow[dr, "h^{-1}"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\
& a_1
\end{tikzcd}$$

Thus we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "h^{-1}"] \arrow[d, "\mathsf{Cyl}(j_0)"'] \arrow[dr, "v(a_0)"] & & a_1 \arrow[dd, "j_1"] \\
\mathsf{Cyl}(a) \arrow[d, "v(a)"'] & \mathsf{Cyl}(a_0) \arrow[ur, "h"'] \arrow[dl, "\mathsf{Cyl}(j_0)"] & \\
\mathsf{Cyl}(a) \arrow[rr, "p(a)"'] & & a
\end{tikzcd}$$

Since v is compatible with p, we also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a) \arrow[r, "v(a)"] \arrow[dr, "p(a)"'] & \mathsf{Cyl}(a) \arrow[d, "p(a)"] \\
& a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "h^{-1}"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

$\square$

Corollary VII.43. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and an involution structure v compatible with p.*

Suppose that we have a pair of commutative diagrams in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a_0 \arrow[d, "f_0"'] \arrow[dr, "j_0"] & \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[d, "f_1"'] \arrow[dr, "j_0"] & \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

Let

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

be a homotopy over a from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1). Then the reverse homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{h^{-1}} a_1$$

from f_1 to f_0 with respect to $\underline{\mathsf{Cyl}}$ is moreover a homotopy under a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1) .

Proof. Follows immediately from Proposition VII.42. ☐

Proposition VII.44. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$ compatible with p.*

Suppose that we have arrows

$$a_0 \xrightarrow{j_0} a$$

and

$$a_1 \xrightarrow{j_1} a$$

of $\mathcal{A}$, and homotopies

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

and

$$\mathsf{Cyl}(a_0) \xrightarrow{k} a_1$$

with respect to $\underline{\mathsf{Cyl}}$, such that the diagrams

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "k"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd}$$

and

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "k"] \\ \mathsf{Cyl}(a_0) \arrow[r, "h"'] & a_1 \end{tikzcd}$$

in $\mathcal{A}$ commute. Then the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "h+k"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

Proof. Appealing to the universal property of $\mathsf{S}(a_1)$, there is an arrow

$$\mathsf{S}(a_0) \xrightarrow{r} a_1$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "h"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "k"'] & \mathsf{S}(a_0) \arrow[dr, "r"] & \\
 & & a_1
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "r_0(a_0)"] \arrow[d, "p(a) \circ \mathsf{Cyl}(j_0)"'] \arrow[dr, "h"] & \mathsf{S}(a_0) \arrow[d, "r"] \\
a & a_1 \arrow[l, "j_1"]
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"] \arrow[d, "p(a) \circ \mathsf{Cyl}(j_0)"'] \arrow[dr, "k"] & \mathsf{S}(a_0) \arrow[d, "r"] \\
a & a_1 \arrow[l, "j_1"]
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "p(a_0) \circ \mathsf{Cyl}(j_0)"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "p(a_0) \circ \mathsf{Cyl}(j_0)"'] & \mathsf{S}(a_0) \arrow[dr, "j_1 \circ r"'] & \\
& & a
\end{tikzcd}
$$

Let

$$
\mathsf{S} \xrightarrow{\overline{p}} \mathsf{id}_{\mathcal{A}}
$$

denote the canonical 2-arrow of $\mathcal{C}$ of Definition III.13. The following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "p(a_0)"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "p(a_0)"'] & \mathsf{S}(a_0) \arrow[dr, "\overline{p}(a_0)"'] & \\
& & a_0
\end{tikzcd}
$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "j_0 \circ p(a_0)"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "j_0 \circ p(a_0)"'] & \mathsf{S}(a_0) \arrow[dr, "j_0 \circ \overline{p}(a_0)"'] & \\
& & a
\end{tikzcd}
$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_0 \arrow[d, "j_0"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "p(a) \circ \mathsf{Cyl}(j_0)"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "p(a) \circ \mathsf{Cyl}(j_0)"'] & \mathsf{S}(a_0) \arrow[dr, "j_0 \circ \overline{p}(a_0)"'] & \\
& & a
\end{tikzcd}$$

Appealing to the universal property of $\mathsf{S}(a_0)$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{S}(a_0) \arrow[r, "r"] \arrow[d, "\overline{p}(a_0)"'] & a_1 \arrow[d, "j_1"] \\
a_0 \arrow[r, "j_0"'] & a
\end{tikzcd}$$

Since the subdivision structure $(\mathsf{S}, r_0, r_1, s)$ is compatible with p, we also have that that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "s(a_0)"] \arrow[dr, "p(a_0)"'] & \mathsf{S}(a_0) \arrow[d, "\overline{p}(a_0)"] \\
& a_0
\end{tikzcd}$$

By definition of $h + k$, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "s(a_0)"] \arrow[dr, "h+k"'] & \mathsf{S}(a_0) \arrow[d, "r"] \\
& a_1
\end{tikzcd}$$

Putting the last two observations together we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "h+k"] \arrow[dr, "s(a_0)"] \arrow[dd, "p(a_0)"'] & & a_1 \arrow[dd, "j_1"] \\
& \mathsf{S}(a_0) \arrow[ur, "r"'] \arrow[dl, "\overline{p}(a_0)"] & \\
a_0 \arrow[rr, "j_0"'] & & a
\end{tikzcd}$$

Appealing once more to the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_0 \arrow[d, "j_0"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

in $\mathcal{A}$, we conclude that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "h+k"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

$\square$

Corollary VII.45. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*.*

Suppose that we have three commutative diagrams in $\mathcal{A}$ *as follows.*

$$\begin{tikzcd} a_0 \arrow[d, "f_0"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[d, "f_1"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[d, "f_2"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

Let

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

be a homotopy over a from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1). Let

$$\mathsf{Cyl}(a_0) \xrightarrow{k} a_1$$

be a homotopy under a from f_1 to f_2 with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1). Then the homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{h+k} a_1$$

from f_0 to f_2 with respect to $\underline{\mathsf{Cyl}}$ is moreover a homotopy under a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1).

Proof. Follows immediately from Proposition VII.44. $\square$

Remark VII.46. Analogues of Proposition VII.41, Corollary VII.43, and Corollary VII.45 for homotopies under an object can all be proven. Moreover, all of these results dualise to the setting of over and under homotopies with respect to a co-cylinder. We shall not need any of this.

Definition VII.47. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f"] \\ & a_1 \end{tikzcd}$$

A *homotopy inverse under a* of f with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1) is an arrow

$$a_1 \xrightarrow{f^{-1}} a_0$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a \arrow[r, "j_1"] \arrow[dr, "j_0"'] & a_1 \arrow[d, "f^{-1}"] \\ & a_0 \end{tikzcd}$$

in $\mathcal{A}$ commutes, together with a homotopy under a from $f^{-1}f$ to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_0), and a homotopy under a from ff^{-1} to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_1, j_1).

Definition VII.48. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f"] \\ & a_1 \end{tikzcd}$$

Then f is a *homotopy equivalence under* a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1) if it admits a homotopy inverse under a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1).

Definition VII.49. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

A *homotopy inverse over* a of f with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1) is an arrow

$$a_1 \xrightarrow{f^{-1}} a_0$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_1 \arrow[d, "f^{-1}"'] \arrow[dr, "j_1"] & \\ a_0 \arrow[r, "j_0"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes together with a homotopy over a from $f^{-1}f$ to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_0) and a homotopy over a from ff^{-1} to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_1, j_1).

Definition VII.50. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

Then f is a *homotopy equivalence over* a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1) if it admits a homotopy inverse over a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1).

Definition VII.51. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f"] \\
& a_1
\end{tikzcd}$$

Then f is a *homotopy equivalence under* a with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if f^{op} is a homotopy equivalence over a with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op}, and (j_1^{op}, j_0^{op}).

Definition VII.52. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

Then f is a *homotopy equivalence over* a with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if f^{op} is a homotopy equivalence under a with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op}, and (j_1^{op}, j_0^{op}).

Proposition VII.53. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$ and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.*

Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f"] \\ & a_1 \end{tikzcd}$$

Then f is a homotopy equivalence under a with respect to $\underline{\mathsf{Cyl}}$ if and only if it is a homotopy equivalence under a with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition VII.38 and Corollary VII.40. $\square$

Proposition VII.54. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.*

Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "j_1"] & \\ a_0 \arrow[r, "j_0"'] & a_1 \end{tikzcd}$$

Then f is a homotopy equivalence over a with respect to $\underline{\mathsf{Cyl}}$ if and only if it is a homotopy equivalence over a with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition VII.37 and Corollary VII.40. $\square$

Definition VII.55. Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is a *retraction* of j if the diagram

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "id"'] & a_1 \arrow[d, "f"] \\ & a_0 \end{tikzcd}$$

in $\mathcal{A}$ commutes.

Definition VII.56. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$.

An arrow

$$a_1 \xrightarrow{f} a_0$$

of $\mathcal{A}$ is a *strong deformation retraction* of j with respect to $\underline{\mathsf{Cyl}}$ if f is a retraction of j, and if there is a homotopy under a_0 from jf to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$ and (j, j).

Definition VII.57. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ *admits a strong deformation retraction* with respect to $\underline{\mathsf{Cyl}}$ if there is an arrow

$$a_1 \xrightarrow{f} a_0$$

of $\mathcal{A}$ which defines a strong deformation retraction of j with respect to $\underline{\mathsf{Cyl}}$.

Definition VII.58. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$.

An arrow

$$a_1 \xrightarrow{f} a_0$$

of $\mathcal{A}$ is a *strong deformation retraction* of j with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if f is a retraction of j, and if there is a homotopy over a_0 from jf to $id(a_1)$ with respect to $\underline{\mathsf{co\text{-}Cyl}}$ and (f, f).

Remark VII.59. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$.

An arrow

$$a_1 \xrightarrow{f} a_0$$

of $\mathcal{A}$ is a *strong deformation retraction* of j with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if and only if j^{op} defines a strong deformation retraction of f^{op} with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}$, equipped with the contraction structure defined by c^{op}.

Definition VII.60. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ *admits a strong deformation retraction* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if there is an arrow

$$a_1 \xrightarrow{f} a_0$$

of $\mathcal{A}$ which defines a strong deformation retraction of j with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proposition VII.61. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$*. Then an arrow*

$$a_1 \xrightarrow{f} a_0$$

of $\mathcal{A}$ *is a strong deformation retraction of* j *with respect to* $\underline{\mathsf{co\text{-}Cyl}}$ *if and only if the following conditions are satisfied:*

(i) f *is a retraction of* j*,*

(ii) there exists a homotopy over a_0 *from* jf *to* $id(a_1)$ *with respect to* $\underline{\mathsf{Cyl}}$ *and* (f, f)*.*

Proof. Follows immediately from Corollary VII.39. □

Lemma VII.62. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure. Suppose that we have four commutative diagrams in* $\mathcal{A}$ *as follows.*

$$\begin{tikzcd} a_0 \arrow[d, "g_0"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_1 \arrow[d, "f_0"'] \arrow[dr, "j_1"] & \\ a_2 \arrow[r, "j_2"'] & a \end{tikzcd}$$

$$\begin{tikzcd} a_1 \arrow[d, "f_1"'] \arrow[dr, "j_1"] & \\ a_2 \arrow[r, "j_2"'] & a \end{tikzcd} \qquad \begin{tikzcd} a_2 \arrow[d, "g_1"'] \arrow[dr, "j_2"] & \\ a_3 \arrow[r, "j_3"'] & a \end{tikzcd}$$

Suppose also that we have a homotopy

$$\mathsf{Cyl}(a_1) \xrightarrow{h} a_2$$

over a *from* f_0 *to* f_1 *with respect to* $\underline{\mathsf{Cyl}}$ *and* (j_1, j_2). *Then the arrow*

$$\mathsf{Cyl}(a_0) \xrightarrow{g_1 \circ h \circ \mathsf{Cyl}(g_0)} a_3$$

of $\mathcal{A}$ *defines a homotopy over* a *from* $g_1 f_0 g_0$ *to* $g_1 f_1 g_0$ *with respect to* $\underline{\mathsf{Cyl}}$ *and* (j_0, j_3).

Proof. That the arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{g_1 \circ h \circ \mathsf{Cyl}(g_0)} a_3$$

of $\mathcal{A}$ defines a homotopy from $g_1 f_0 g_0$ to $g_1 f_1 g_0$ with respect to $\underline{\mathsf{Cyl}}$ is Lemma VII.18.

In addition, since h defines a homotopy over a from f_0 to f_1 with respect to $\underline{\mathsf{Cyl}}$ and (j_1, j_2), the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_2 \arrow[d, "j_2"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd}$$

Since the diagram

$$\begin{tikzcd} a_0 \arrow[d, "g_0"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, we also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(g_0)"] \arrow[dr, "\mathsf{Cyl}(j_0)"'] & \mathsf{Cyl}(a_1) \arrow[d, "\mathsf{Cyl}(j_1)"] \\
& \mathsf{Cyl}(a)
\end{tikzcd}
$$

Moreover, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_2 \arrow[r, "g_1"] \arrow[dr, "j_2"'] & a_3 \arrow[d, "j_3"] \\
& a
\end{tikzcd}
$$

Putting the last three observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "g_1 \circ h \circ \mathsf{Cyl}(g_0)"] \arrow[d, "\mathsf{Cyl}(j_0)"'] & a_3 \arrow[d, "j_3"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}
$$

$\square$

Remark VII.63. An analogous result holds for homotopies under an object with respect to $\underline{\mathsf{Cyl}}$. We shall not need this.

VIII. Cofibrations and fibrations

We introduce the notion of a cofibration with respect to a cylinder or a co-cylinder in a formal category $\mathcal{A}$, and the dual notion of a fibration with respect to a cylinder or a co-cylinder in $\mathcal{A}$. Given both a cylinder $\underline{\mathsf{Cyl}}$ and a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}$ in $\mathcal{A}$ which are adjoint, we characterise fibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$ as fibrations with respect to $\underline{\mathsf{Cyl}}$, and characterise cofibrations with respect to $\underline{\mathsf{Cyl}}$ as cofibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Thus far, this is as for the homotopy theory of topological spaces. In an abstract setting, cofibrations and fibrations are treated in [21] and [23], for example. We must go a little further.

Let us assume once more that we have a cylinder $\underline{\mathsf{Cyl}}$ and a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}$ in a formal category $\mathcal{A}$. We introduce the notion of a normally cloven fibration with respect to a cylinder $\underline{\mathsf{Cyl}}$ or $\underline{\mathsf{co\text{-}Cyl}}$, which is a strengthening of the notion of a fibration. Roughly speaking, we impose two requirements upon the lifts of homotopies that define a fibration: that these lifts are compatible, and that identity homotopies lift to identity homotopies. If $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, we characterise normally cloven fibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$ exactly as normally cloven fibrations with respect to $\underline{\mathsf{Cyl}}$.

We introduce the dual notion of a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$ or $\underline{\mathsf{co\text{-}Cyl}}$. If $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, we characterise normally cloven cofibrations with respect to $\underline{\mathsf{Cyl}}$ exactly as normally cloven cofibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

A composition of cofibrations is a cofibration, and a composition of fibrations is a fibration. Moreover, suppose that we have commutative diagrams

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "j'"'] & a_0 \arrow[d, "j"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "j"'] & a_2 \arrow[d, "j'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ such that r_0 is a retraction of g_0, and such that r_1 is a retraction of g_1. If j is a cofibration, then j' is also a cofibration. All this holds equally for normally cloven cofibrations, and dually for fibrations and normally cloven fibrations.

In XVI, we shall construct a homotopy theory of categories by means of our abstract theory. A normally cloven fibration in our sense with respect to this homotopy theory is exactly what is known as a normally cloven iso-fibration. This motivates our choice of terminology.

In an abstract setting, van den Berg and Garner recently discussed the notion of a normally cloven fibration in [36], independently of the author. We refer the reader to around Proposition 6.1.5. We shall present further ideas from this paper in IX.

Assumption VIII.1. Let $\mathcal{C}$ be a 2-category with a final object. Suppose that pushouts and pullbacks of 2-arrows of $\mathcal{C}$ give rise to pushouts and pullbacks in formal categories, in the sense of Definition II.14. Let $\mathcal{A}$ be an object of $\mathcal{C}$. As before, we view $\mathcal{A}$ as a formal category, writing of objects and arrows of $\mathcal{A}$.

Definition VIII.2. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ is a *cofibration* with respect to $\underline{\mathsf{Cyl}}$ if, for any commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\ a_1 \arrow[r, "f"'] & a_2 \end{tikzcd}$$

in $\mathcal{A}$, there is an arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{k} a_2$$

with respect to $\underline{\mathsf{Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(j)"] \arrow[ddr, bend left, "h"] & \\ a_1 \arrow[r, "i_0(a_1)"'] \arrow[drr, bend right, "f"'] & \mathsf{Cyl}(a_1) \arrow[dr, "k"] & \\ & & a_2 \end{tikzcd}$$

Definition VIII.3. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is a *fibration* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if f^{op} is a cofibration with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Definition VIII.4. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ is a *trivial cofibration* with respect to $\underline{\mathsf{Cyl}}$ if it is both a cofibration and a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$.

Definition VIII.5. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ is a *trivial fibration* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if it is both a fibration and a homotopy equivalence with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Definition VIII.6. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. An arrow

$$a_1 \xrightarrow{f} a_2$$

of $\mathcal{A}$ is a *fibration* with respect to $\underline{\mathsf{Cyl}}$ if, for any commutative diagram

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h"'] & a_2
\end{tikzcd}$$

in $\mathcal{A}$, there is a homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{l} a_1$$

with respect to $\underline{\mathsf{Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h"'] \arrow[ur, "l"'] & a_2
\end{tikzcd}$$

Proposition VIII.7. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ *be a cylinder in* $\mathcal{A}$*, and let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ *be a co-cylinder in* $\mathcal{A}$*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*. An arrow*

$$a_1 \xrightarrow{f} a_2$$

is a fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$ *if and only if it is a fibration with respect to* $\underline{\mathsf{Cyl}}$.

Proof. We first prove that if f is a fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$, then it is a fibration with respect to $\underline{\mathsf{Cyl}}$. To this end, suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\ \mathsf{Cyl}(a_0) \arrow[r, "h"'] & a_2 \end{tikzcd}$$

Adopting the shorthand of Recollection II.13, let

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the natural isomorphism which the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ gives rise to.

Since $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "\mathsf{adj}(h)"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\ \mathsf{co\text{-}Cyl}(a_2) \arrow[r, "e_0(a_2)"'] & a_2 \end{tikzcd}$$

By the commutativity of the last two diagrams, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "a_2"] \arrow[d, "f"'] & a_1 \arrow[d, "g"] \\ a_2 \arrow[r, "\mathsf{adj}(h)"'] & \mathsf{co\text{-}Cyl}(a_2) \end{tikzcd}$$

Thus there is a homotopy

$$a_0 \xrightarrow{k} \mathsf{co\text{-}Cyl}(a_1)$$

with respect to $\underline{\mathsf{co\text{-}Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes, since f is a fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

$$
\begin{tikzcd}
a_0 \arrow[dr, "k"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "\mathsf{adj}(h)"'] & & \\
& \mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"] \arrow[d, "\mathsf{co\text{-}Cyl}(f)"'] & a_1 \arrow[d, "f"] \\
& \mathsf{co\text{-}Cyl}(a_2) \arrow[r, "e_0(a_2)"'] & a_2
\end{tikzcd}
$$

By the naturality of adj, we also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "k"] \arrow[dr, "\mathsf{adj}\left(f \circ \mathsf{adj}^{-1}(k)\right)"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "\mathsf{co\text{-}Cyl}(f)"] \\
& \mathsf{co\text{-}Cyl}(a_2)
\end{tikzcd}
$$

We deduce that $\mathsf{adj}(f \circ \mathsf{adj}^{-1}(k)) = \mathsf{adj}(h)$, and hence that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{adj}^{-1}(k)"] \arrow[dr, "h"'] & a_1 \arrow[d, "f"] \\
& a_2
\end{tikzcd}
$$

Moreover, since $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$ the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "k"] \arrow[d, "i_0(a_0)"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "e_0(a_1)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{adj}^{-1}(k)"'] & a_1
\end{tikzcd}
$$

Hence, by the commutativity of the diagram which defines k, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[d, "i_0(a_0)"'] \arrow[dr, "g"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{adj}^{-1}(k)"'] & a_1
\end{tikzcd}$$

Putting this all together, we have shown that the following diagram in $\mathcal{A}$ commutes, concluding this direction of the proof.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[ur, "\mathsf{adj}^{-1}(k)"'] \arrow[r, "h"'] & a_2
\end{tikzcd}$$

We now prove that if f is a fibration with respect to $\underline{\mathsf{Cyl}}$, then f is a fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$. To this end, suppose now that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "h"'] & a_1 \arrow[d, "f"] \\
\mathsf{co\text{-}Cyl}(a_2) \arrow[r, "e_0(a_2)"'] & a_2
\end{tikzcd}$$

Adopting again the shorthand of Recollection II.13, let

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the natural isomorphism which the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ gives rise to. Since $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "h"] \arrow[d, "i_0(a_0)"'] & \mathsf{co\text{-}Cyl}(a_2) \arrow[d, "e_0(a_2)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{adj}^{-1}(h)"'] & a_2
\end{tikzcd}$$

By the commutativity of the last two diagrams, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{adj}^{-1}(h)"'] & a_2
\end{tikzcd}$$

Thus there is a homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{l} a_1$$

with respect to $\underline{\mathsf{Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes, since f is a fibration with respect to $\underline{\mathsf{Cyl}}$.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{adj}^{-1}(h)"'] \arrow[ur, "l"'] & a_2
\end{tikzcd}$$

By the naturality of adj, we also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "\mathsf{adj}(l)"] \arrow[dr, "\mathsf{adj}(f \circ l)"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "\mathsf{co\text{-}Cyl}(f)"] \\
& \mathsf{co\text{-}Cyl}(a_2)
\end{tikzcd}$$

We deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "\mathsf{adj}(l)"] \arrow[dr, "h"'] & \mathsf{co\text{-}Cyl}(a_1) \arrow[d, "\mathsf{co\text{-}Cyl}(f)"] \\
& \mathsf{co\text{-}Cyl}(a_2)
\end{tikzcd}$$

Moreover, since $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "\mathsf{adj}(l)"'] & \mathsf{Cyl}(a_0) \arrow[d, "k"] \\
\mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"'] & a_1
\end{tikzcd}$$

Hence, by the commutativity of the diagram which defines l, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[d, "\mathsf{adj}(l)"'] \arrow[dr, "g"] & \\
\mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"'] & a_1
\end{tikzcd}$$

Putting this all together, we have shown that the following diagram in $\mathcal{A}$ commutes, concluding this direction of the proof.

$$\begin{tikzcd}
a_0 \arrow[dr, "\mathsf{adj}(l)"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\
& \mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"] \arrow[d, "\mathsf{co\text{-}Cyl}(f)"'] & a_1 \arrow[d, "f"] \\
& \mathsf{co\text{-}Cyl}(a_2) \arrow[r, "e_0(a_2)"'] & a_2
\end{tikzcd}$$

$\square$

Definition VIII.8. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, i_0, i_1)$ be a co-cylinder in $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ is a *cofibration* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if, for any commutative diagram

$$\begin{tikzcd}
a_0 \arrow[r, "h"] \arrow[d, "j"'] & \mathsf{co\text{-}Cyl}(a_2) \arrow[d, "e_0(a_2)"] \\
a_1 \arrow[r, "g"'] & a_3
\end{tikzcd}$$

in $\mathcal{A}$, there is a homotopy

$$a_1 \xrightarrow{l} \mathsf{co\text{-}Cyl}(a_2)$$

with respect to $\underline{\mathsf{co\text{-}Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "h"] \arrow[d, "j"'] & \mathsf{co\text{-}Cyl}(a_2) \arrow[d, "e_0(a_2)"] \\ a_1 \arrow[ur, "l"'] \arrow[r, "g"'] & a_2 \end{tikzcd}$$

Remark VIII.9. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, i_0, i_1)$ be a co-cylinder in $\mathcal{A}$. An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ is a cofibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if and only if j^{op} is a fibration with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Corollary VIII.10. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ *be a cylinder in* $\mathcal{A}$*, and let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ *be a co-cylinder in* $\mathcal{A}$*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

An arrow

$$a_0 \xrightarrow{j} a_1$$

is a cofibration with respect to $\underline{\mathsf{Cyl}}$ *if and only if it is a cofibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

Proof. Follows immediately from Proposition VIII.7 by duality. □

Proposition VIII.11. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ *be a cylinder in* $\mathcal{A}$*. Then for any object* a *of* $\mathcal{A}$*, the arrow*

$$a \xrightarrow{id} a$$

of $\mathcal{A}$ *is a cofibration with respect to* $\underline{\mathsf{Cyl}}$*.*

Proof. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a \arrow[r, "i_0(a)"] \arrow[d, "id"'] & \mathsf{Cyl}(a) \arrow[d, "h"] \\ a \arrow[r, "g"'] & a' \end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a \arrow[r, "i_0(a)"] \arrow[d, "id"'] & \mathsf{Cyl}(a) \arrow[d, "id"] \arrow[ddr, bend left, "h"] & \\
a \arrow[r, "i_0(a)"'] \arrow[drr, bend right, "g"'] & \mathsf{Cyl}(a) \arrow[dr, "h"] & \\
& & a'
\end{tikzcd}$$

□

Proposition VIII.12. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ *be a cylinder in* $\mathcal{A}$*. Let*

$$a_0 \xrightarrow{j_0} a_1$$

and

$$a_1 \xrightarrow{j_1} a_2$$

be arrows of $\mathcal{A}$ *which are cofibrations with respect to* $\underline{\mathsf{Cyl}}$*. Then* $j_1 \circ j_0$ *is a cofibration with respect to* $\underline{\mathsf{Cyl}}$*.*

Proof. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j_1 \circ j_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\
a_2 \arrow[r, "g"'] & a_3
\end{tikzcd}$$

Since j_0 is a cofibration with respect to $\underline{\mathsf{Cyl}}$, there is an arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{k_0} a_3$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(j_0)"] \arrow[ddr, bend left, "h"] & \\
a_2 \arrow[r, "i_0(a_1)"'] \arrow[drr, bend right, "g \circ j_1"'] & \mathsf{Cyl}(a_2) \arrow[dr, "k_0"] & \\
 & & a_3
\end{tikzcd}$$

Since j_1 is a cofibration with respect to $\underline{\mathsf{Cyl}}$, there is an arrow

$$\mathsf{Cyl}(a_2) \xrightarrow{k_1} a_3$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "j_1"'] & \mathsf{Cyl}(a_1) \arrow[d, "\mathsf{Cyl}(j_1)"] \arrow[ddr, bend left, "k_0"] & \\
a_2 \arrow[r, "i_0(a_2)"'] \arrow[drr, bend right, "g"'] & \mathsf{Cyl}(a_2) \arrow[dr, "k_1"] & \\
 & & a_3
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j_1 \circ j_0"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(j_1 \circ j_0)"] \arrow[ddr, bend left, "h"] & \\
a_2 \arrow[r, "i_0(a_2)"'] \arrow[drr, bend right, "g"'] & \mathsf{Cyl}(a_2) \arrow[dr, "k_1"] & \\
 & & a_3
\end{tikzcd}$$

□

Corollary VIII.13. *Let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ *be a co-cylinder in* $\mathcal{A}$*. Let*

$$a_0 \xrightarrow{f_0} a_1$$

and

$$a_1 \xrightarrow{f_1} a_2$$

be arrows of $\mathcal{A}$ *which are fibrations with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*. Then* $f_1 \circ f_0$ *is a fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

Proof. Follows immediately from Proposition VIII.12 by duality. $\square$

Proposition VIII.14. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ *be a cylinder in* $\mathcal{A}$*. Let*

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ *which is a cofibration with respect to* $\underline{\mathsf{Cyl}}$*.*

Suppose that we have commutative diagrams

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "j'"'] & a_0 \arrow[d, "j"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "j"'] & a_2 \arrow[d, "j'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, such that* r_0 *is a retraction of* g_0*, and such that* r_1 *is a retraction of* g_1*. Then* j' *is a cofibration with respect to* $\underline{\mathsf{Cyl}}$*.*

Proof. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_2 \arrow[r, "i_0(a_2)"] \arrow[d, "j'"'] & \mathsf{Cyl}(a_2) \arrow[d, "h"] \\ a_3 \arrow[r, "f"'] & a_4 \end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[rr, "i_0(a_0)"] \arrow[d, "j"'] \arrow[dr, "r_0"] & & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(r_0)"] \\
a_1 \arrow[d, "r_1"'] & a_2 \arrow[r, "i_0(a_2)"] \arrow[dl, "j'"] & \mathsf{Cyl}(a_2) \arrow[d, "h"] \\
a_3 \arrow[rr, "f"'] & & a_4
\end{tikzcd}$$

Thus, since j is a cofibration with respect to $\underline{\mathsf{Cyl}}$, there is an arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{k} a_4$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(j)"] \arrow[ddr, bend left, "h \circ \mathsf{Cyl}(r_0)"] & \\
a_1 \arrow[r, "i_0(a_1)"'] \arrow[drr, bend right, "f \circ r_1"'] & \mathsf{Cyl}(a_1) \arrow[dr, "k"] & \\
& & a_4
\end{tikzcd}$$

Let

$$\mathsf{Cyl}(a_3) \xrightarrow{l} a_4$$

denote the arrow $k \circ \mathsf{Cyl}(g_1)$ of $\mathcal{A}$. We claim that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[r, "i_0(a_2)"] \arrow[d, "j'"'] & \mathsf{Cyl}(a_2) \arrow[d, "\mathsf{Cyl}(j')"] \arrow[ddr, bend left, "h"] & \\
a_3 \arrow[r, "i_0(a_3)"'] \arrow[drr, bend right, "f"'] & \mathsf{Cyl}(a_3) \arrow[dr, "l"] & \\
& & a_4
\end{tikzcd}$$

Firstly, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_3 \arrow[rrr, "i_0(a_1)"] \arrow[dd, "id"'] \arrow[dr, "g_1"] & & & \mathsf{Cyl}(a_3) \arrow[dl, "\mathsf{Cyl}(g_1)"'] \arrow[dd, "l"] \\
& a_1 \arrow[r, "i_0(a_2)"] \arrow[dl, "r_1"] & \mathsf{Cyl}(a_1) \arrow[dr, "k"] & \\
a_3 \arrow[rrr, "f"'] & & & a_4
\end{tikzcd}$$

Secondly, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_2) \arrow[rrr, "\mathsf{Cyl}(j')"] \arrow[dd, "id"'] \arrow[dr, "\mathsf{Cyl}(g_0)"] & & & \mathsf{Cyl}(a_3) \arrow[dl, "\mathsf{Cyl}(g_1)"'] \arrow[dd, "l"] \\
& \mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"] \arrow[dl, "\mathsf{Cyl}(r_0)"] & \mathsf{Cyl}(a_1) \arrow[dr, "k"] & \\
\mathsf{Cyl}(a_3) \arrow[rrr, "h"'] & & & a_4
\end{tikzcd}$$

□

Corollary VIII.15. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ *be a cylinder in* $\mathcal{A}$*. Let*

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ *which is a trivial cofibration with respect to* $\underline{\mathsf{Cyl}}$*.*

Suppose that we have commutative diagrams

$$\begin{tikzcd}
a_2 \arrow[r, "g_0"] \arrow[d, "j'"'] & a_0 \arrow[d, "j"] \\
a_3 \arrow[r, "g_1"'] & a_1
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[r, "r_0"] \arrow[d, "j"'] & a_2 \arrow[d, "j'"] \\
a_1 \arrow[r, "r_1"'] & a_3
\end{tikzcd}$$

in $\mathcal{A}$*, such that* r_0 *is a retraction of* g_0*, and such that* r_1 *is a retraction of* g_1*. Then* j' *is a trivial cofibration with respect to* $\underline{\mathsf{Cyl}}$*.*

Proof. Follows immediately from Proposition VIII.14 and Proposition VII.22. □

Corollary VIII.16. *Let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ *be a co-cylinder in* $\mathcal{A}$*. Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ *which is a fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*. Suppose that we have commutative diagrams*

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "f'"'] & a_0 \arrow[d, "f"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "f"'] & a_2 \arrow[d, "f'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, such that* r_0 *is a retraction of* g_0 *and such that* r_1 *is a retraction of* g_1*. Then* f' *is a fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

Proof. Follows immediately from Proposition VIII.14 by duality. □

Corollary VIII.17. *Let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ *be a co-cylinder in* $\mathcal{A}$*. Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ *which is a trivial fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

Suppose that we have commutative diagrams

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "f'"'] & a_0 \arrow[d, "f"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "f"'] & a_2 \arrow[d, "f'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, such that* r_0 *is a retraction of* g_0*, and such that* r_1 *is a retraction of* g_1*. Then* f' *is a trivial fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

Proof. Follows immediately from Corollary VIII.16 and Proposition VII.22. □

Notation VIII.18. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$, and let a_2 be an object of $\mathcal{A}$.

Let $\Sigma^{\underline{\mathsf{Cyl}}}_{j,a_2}$ denote the set of pairs (g, h) consisting of an arrow

$$a_1 \xrightarrow{g} a_2$$

of $\mathcal{A}$ and a homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_2$$

with respect to $\underline{\mathsf{Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\
a_1 \arrow[r, "g"'] & a_2
\end{tikzcd}$$

Let $\Upsilon^{\mathsf{Cyl}}_{j,a_2}$ denote the set of homotopies

$$\mathsf{Cyl}(a_1) \xrightarrow{k} a_2$$

with respect to $\underline{\mathsf{Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(j)"] \arrow[ddr, bend left, "h"] & \\
a_1 \arrow[r, "i_0(a_1)"'] \arrow[drr, bend right, "g"'] & \mathsf{Cyl}(a_1) \arrow[dr, "k"] & \\
 & & a_2
\end{tikzcd}$$

Definition VIII.19. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. A *cofibration equipped with a cleavage* with respect to $\underline{\mathsf{Cyl}}$ is an arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$, together with a map

$$\Sigma^{\mathsf{Cyl}}_{j,a_2} \xrightarrow{k_{a_2}} \Upsilon^{\mathsf{Cyl}}_{j,a_2}$$

for every object a_2 of $\mathcal{A}$.

Definition VIII.20. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. A *normally cloven cofibration* with respect to $\underline{\mathsf{Cyl}}$ is an arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ which is a cofibration equipped with a cleavage with respect to $\underline{\mathsf{Cyl}}$, satisfying the following conditions, for which we denote by

$$\Sigma^{\mathsf{Cyl}}_{j,a_2} \xrightarrow{k_{a_2}} \Upsilon^{\mathsf{Cyl}}_{j,a_2}$$

the map of the cleavage corresponding to an object a_2 of $\mathcal{A}$.

(i) Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[d, "j"'] \arrow[dr, "g_0"] & \\ a_1 \arrow[r, "g_1"'] & a_2 \end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "p(a_1)"] \arrow[dr, "k_{a_2}(g_1, g_0 \circ p(a_0))"'] & a_1 \arrow[d, "g_1"] \\ & a_2 \end{tikzcd}$$

(ii) Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "h"] \\ a_1 \arrow[r, "g_1"'] & a_2 \end{tikzcd}$$

Then for any arrow

$$a_2 \xrightarrow{g_2} a_3$$

of $\mathcal{A}$, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "k_{a_2}(g_1, h)"] \arrow[dr, "k_{a_3}(g_2 \circ g_1, g_2 \circ h)"'] & a_2 \arrow[d, "g_2"] \\
& a_3
\end{tikzcd}$$

Remark VIII.21. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Let j be an arrow of $\mathcal{A}$ which is a cofibration equipped with a cleavage with respect to $\underline{\mathsf{Cyl}}$.

We shall refer to condition (i) of Definition VIII.20 as *lifting of identities*, and to condition (ii) of Definition VIII.20 as *compatibility of liftings*.

Remark VIII.22. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. If an arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ is a cofibration with respect to $\underline{\mathsf{Cyl}}$, we can think of $\mathsf{Cyl}(a_1)$ as a weak pushout of j along the arrow

$$a_0 \xrightarrow{i_0(a_0)} \mathsf{Cyl}(a_0)$$

of $\mathcal{A}$.

The lifting of identities and compatibility of liftings conditions bring $\mathsf{Cyl}(a_1)$ closer to an actual pushout of j along $i_0(a_0)$.

Terminology VIII.23. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Let

$$a_0 \xrightarrow{j} a_1$$

be a cofibration equipped with a cleavage with respect to $\underline{\mathsf{Cyl}}$, which is moreover a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

We shall typically refer to j as a normally cloven cofibration, without explicitly mentioning its cleavage.

Notation VIII.24. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Let

$$a_1 \xrightarrow{f} a_2$$

be an arrow of $\mathcal{A}$, and let a_0 be an object of $\mathcal{A}$.

Let $\Sigma^{\underline{\mathsf{co\text{-}Cyl}}}_{f,a_0}$ denote the set of pairs (g, h) consisting of an arrow

$$a_0 \xrightarrow{g} a_1$$

of $\mathcal{A}$ and a homotopy

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_2)$$

with respect to $\underline{\mathsf{co\text{-}Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "h"'] & a_1 \arrow[d, "f"] \\
\mathsf{co\text{-}Cyl}(a_2) \arrow[r, "e_0(a_2)"'] & a_2
\end{tikzcd}$$

Let $\Upsilon^{\underline{\mathsf{co\text{-}Cyl}}}_{f,a_0}$ denote the set of homotopies

$$a_0 \xrightarrow{k} \mathsf{co\text{-}Cyl}(a_1)$$

with respect to $\underline{\mathsf{co\text{-}Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[dr, "k"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & \\
& \mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"] \arrow[d, "\mathsf{co\text{-}Cyl}(f)"'] & a_1 \arrow[d, "f"] \\
& \mathsf{co\text{-}Cyl}(a_2) \arrow[r, "e_0(a_2)"'] & a_2
\end{tikzcd}$$

Definition VIII.25. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. A *fibration equipped with a cleavage* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is an arrow

$$a_1 \xrightarrow{f} a_2$$

of $\mathcal{A}$, together with a map

$$\Sigma^{\underline{\mathsf{co\text{-}Cyl}}}_{f,a_0} \xrightarrow{k_{a_0}} \Upsilon^{\underline{\mathsf{co\text{-}Cyl}}}_{f,a_0}$$

for every object a_0 of $\mathcal{A}$.

Remark VIII.26. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$, and let

$$a_1 \xrightarrow{f} a_2$$

be a fibration equipped with a cleavage with respect to $\underline{\mathsf{co\text{-}Cyl}}$. Let a_0 be an object of $\mathcal{A}$, and let

$$\Sigma^{\underline{\mathsf{co\text{-}Cyl}}}_{f,a_0} \xrightarrow{k_{a_0}} \Upsilon^{\underline{\mathsf{co\text{-}Cyl}}}_{f,a_0}$$

denote the corresponding map of the cleavage.

Associating to a pair (g^{op}, h^{op}) in $\Sigma^{\underline{\mathsf{co\text{-}Cyl}}^{op}}_{f^{op},a_0}$ the arrow $(k_{a_0}(g,h))^{op}$ of $\mathcal{A}^{op}$, defines a map

$$\Sigma^{\underline{\mathsf{co\text{-}Cyl}}^{op}}_{f^{op},a_0} \longrightarrow \Upsilon^{\underline{\mathsf{co\text{-}Cyl}}^{op}}_{f^{op},a_0}$$

, which we shall denote by $(k^{op})_{a_0}$. Thus a cleavage with respect to f and $\underline{\mathsf{co\text{-}Cyl}}$ gives rise to a cleavage with respect to f^{op} and the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Definition VIII.27. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. A *normally cloven fibration* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is an arrow

$$a_1 \xrightarrow{f} a_2$$

of $\mathcal{A}$ which is a fibration equipped with a cleavage with respect to $\underline{\mathsf{co\text{-}Cyl}}$, such that f^{op} equipped with the cleavage of Remark VIII.26 is a normally cloven cofibration with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op}.

Terminology VIII.28. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Let

$$a_1 \xrightarrow{f} a_2$$

be a fibration equipped with a cleavage with respect to $\underline{\mathsf{co\text{-}Cyl}}$, which is moreover a normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

We shall typically refer to f as a normally cloven fibration, without explicitly mentioning its cleavage.

Definition VIII.29. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ is a *trivial normally cloven cofibration* with respect to $\underline{\mathsf{Cyl}}$ if it is both a normally cloven cofibration and a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$.

Definition VIII.30. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. An arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is a *trivial normally cloven fibration* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if it is both a normally cloven fibration and a homotopy equivalence with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Notation VIII.31. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Let

$$a_1 \xrightarrow{f} a_2$$

be an arrow of $\mathcal{A}$, and let a_0 be an object of $\mathcal{A}$.

Let $\Delta^{\mathsf{Cyl}}_{f,a_0}$ denote the set of pairs (g, h) consisting of an arrow

$$a_0 \xrightarrow{g} a_1$$

of $\mathcal{A}$ and a homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{h} a_1$$

with respect to $\underline{\mathsf{Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h"'] & a_2
\end{tikzcd}$$

Let $\Omega^{\mathsf{Cyl}}_{f,a_0}$ denote the set of homotopies

$$\mathsf{Cyl}(a_0) \xrightarrow{l} a_1$$

with respect to $\underline{\mathsf{Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[r, "h"'] \arrow[ur, "l"] & a_2
\end{tikzcd}$$

Definition VIII.32. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. A *fibration equipped with a cleavage* with respect to $\underline{\mathsf{Cyl}}$ is an arrow

$$a_1 \xrightarrow{f} a_2$$

of $\mathcal{A}$, together with a map

$$\Delta^{\underline{\mathsf{Cyl}}}_{f,a_0} \xrightarrow{l_{a_0}} \Omega^{\underline{\mathsf{Cyl}}}_{f,a_0}$$

for every object a_0 of $\mathcal{A}$.

Definition VIII.33. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. A *normally cloven fibration* with respect to $\underline{\mathsf{Cyl}}$ is an arrow

$$a_1 \xrightarrow{f} a_2$$

of $\mathcal{A}$ which is a fibration equipped with a cleavage with respect to $\underline{\mathsf{Cyl}}$, satisfying the following conditions, for which we denote by

$$\Delta^{\underline{\mathsf{Cyl}}}_{f,a_0} \xrightarrow{l_{a_0}} \Omega^{\underline{\mathsf{Cyl}}}_{f,a_0}$$

the map of the cleavage corresponding to an object a_0 of $\mathcal{A}$.

(i) Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[d, "g_1"'] \arrow[dr, "g_2"] & \\ a_1 \arrow[r, "f"'] & a_2 \end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, "l_{a_0}(g_1, g_2 \circ p(a_0))"'] & a_0 \arrow[d, "g_1"] \\ & a_1 \end{tikzcd}$$

(ii) Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[r, "g_1"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\ \mathsf{Cyl}(a_0) \arrow[r, "h"'] & a_2 \end{tikzcd}$$

Then for any arrow

$$a_{-1} \xrightarrow{g_0} a_0$$

of $\mathcal{A}$ the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_{-1}) \arrow[r, "\mathsf{Cyl}(g_0)"] \arrow[dr, "l_{a_{-1}}(g_1 \circ g_0, h \circ \mathsf{Cyl}(g_0))"'] & \mathsf{Cyl}(a_0) \arrow[d, "l_{a_0}(g_1, h)"] \\ & a_1 \end{tikzcd}$$

Terminology VIII.34. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure c. Let

$$a_1 \xrightarrow{f} a_2$$

be a fibration equipped with a cleavage with respect to $\underline{\mathsf{Cyl}}$, which is moreover a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$.

We shall typically refer to f as a normally cloven fibration, without explicitly mentioning its cleavage.

Notation VIII.35. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$, and let a_2 be an object of $\mathcal{A}$.

Let $\Delta^{\underline{\mathsf{co\text{-}Cyl}}}_{j,a_2}$ denote the set of pairs (g, h) consisting of an arrow

$$a_1 \xrightarrow{g} a_2$$

of $\mathcal{A}$ and a homotopy

$$a_0 \xrightarrow{h} \mathsf{co\text{-}Cyl}(a_2)$$

with respect to $\underline{\mathsf{co\text{-}Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "h"] \arrow[d, "j"'] & \mathsf{co\text{-}Cyl}(a_2) \arrow[d, "e_0(a_2)"] \\ a_1 \arrow[r, "g"'] & a_2 \end{tikzcd}$$

Let $\Omega^{\underline{\mathsf{co\text{-}Cyl}}}_{j,a_2}$ denote the set of homotopies

$$a_1 \xrightarrow{l} \mathsf{co\text{-}Cyl}(a_2)$$

with respect to $\underline{\mathsf{co\text{-}Cyl}}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "h"] \arrow[d, "j"'] & \mathsf{co\text{-}Cyl}(a_2) \arrow[d, "e_0(a_2)"] \\ a_1 \arrow[ur] \arrow[r, "l"'] & a_2 \end{tikzcd}$$

Definition VIII.36. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. A *cofibration equipped with a cleavage* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is an arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$, together with a map

$$\Delta^{\underline{\mathsf{co\text{-}Cyl}}}_{j,a_2} \xrightarrow{l_{a_2}} \Omega^{\underline{\mathsf{co\text{-}Cyl}}}_{j,a_2}$$

for every object a_2 of $\mathcal{A}$.

Remark VIII.37. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$, and let

$$a_0 \xrightarrow{j} a_1$$

be a cofibration equipped with a cleavage with respect to $\underline{\mathsf{co\text{-}Cyl}}$. Let a_2 be an object of $\mathcal{A}$, and let

$$\Delta^{\underline{\mathsf{co\text{-}Cyl}}}_{j,a_2} \xrightarrow{l_{a_2}} \Omega^{\underline{\mathsf{co\text{-}Cyl}}}_{j,a_2}$$

denote the corresponding map of the cleavage.

Associating to a pair (g^{op}, h^{op}) in $\Delta^{\underline{\mathsf{co\text{-}Cyl}}^{op}}_{j^{op},a_2}$ the arrow $(l_{a_2}(g,h))^{op}$ of $\mathcal{A}^{op}$, defines a map

$$\Delta^{\underline{\mathsf{co\text{-}Cyl}}^{op}}_{j^{op},a_2} \longrightarrow \Omega^{\underline{\mathsf{co\text{-}Cyl}}^{op}}_{j^{op},a_2},$$

which we shall denote by $(l^{op})_{a_2}$. Thus a cleavage with respect to j and $\underline{\mathsf{co\text{-}Cyl}}$ gives rise to a cleavage with respect to j^{op} and the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Definition VIII.38. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. A *normally cloven cofibration* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ is an arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ which is a cofibration equipped with a cleavage with respect to $\underline{\mathsf{co\text{-}Cyl}}$, such that j^{op} equipped with the cleavage of Remark VIII.37 is a normally cloven fibration with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$ equipped with the contraction structure c^{op}.

Terminology VIII.39. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, i_0, i_1, c)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure c. Let

$$a_0 \xrightarrow{j} a_1$$

be a cofibration equipped with a cleavage with respect to $\underline{\mathsf{co\text{-}Cyl}}$, which is moreover a normally cloven cofibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

We shall typically refer to j as a normally cloven fibration, without explicitly mentioning its cleavage.

Proposition VIII.40. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c.*

Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c. Then an arrow

$$a_1 \xrightarrow{f} a_2$$

of $\mathcal{A}$ is a normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if and only if it is a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$.

Proof. We first prove that if f is a normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$, then it is a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$. To this end, adopting the shorthand of Recollection II.13, let

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{Cyl}(-), -) \xrightarrow{\mathsf{adj}} \mathsf{Hom}_{\mathcal{A}}(-, \mathsf{co\text{-}Cyl}(-))$$

denote the natural isomorphism which the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ gives rise to. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a \arrow[r, "g_1"] \arrow[d, "i_0(a)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a) \arrow[r, "h"'] & a_2
\end{tikzcd}$$

As in the proof of Proposition VIII.7, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a \arrow[r, "g_1"] \arrow[d, "\mathsf{adj}(h)"'] & a_1 \arrow[d, "f"] \\
a_2 \arrow[r, "e_0(a_2)"'] & \mathsf{co\text{-}Cyl}(a_2)
\end{tikzcd}$$

Let

$$\Sigma^{\mathsf{co\text{-}Cyl}}_{f,a} \xrightarrow{k_a} \Upsilon^{\mathsf{Cyl}}_{f,a}$$

denote the map, of the cleavage with which f is equipped, corresponding to the object a of $\mathcal{A}$. Then the homotopy

$$a \xrightarrow{k_a(g_1)} \mathsf{co\text{-}Cyl}(a_1)$$

with respect to $\underline{\mathsf{co\text{-}Cyl}}$ fits into a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a \arrow[dr, "k_a(g_1, \mathsf{adj}(h))"] \arrow[drr, bend left, "g_1"] \arrow[ddr, bend right, "\mathsf{adj}(h)"'] & & \\
& \mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"] \arrow[d, "\mathsf{co\text{-}Cyl}(f)"'] & a_1 \arrow[d, "f"] \\
& \mathsf{co\text{-}Cyl}(a_2) \arrow[r, "e_0(a_2)"'] & a_2
\end{tikzcd}$$

Let $l_a(g_1, h)$ denote the arrow

$$\mathsf{Cyl}(a) \xrightarrow{\mathsf{adj}^{-1}(k_a(g_1, h))} a_1$$

of $\mathcal{A}$. Following again the proof of Proposition VIII.7, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a \arrow[rr, "g_1"] \arrow[d, "i_0(a)"'] & & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a) \arrow[rr, "h"'] \arrow[urr, "l_a(g_1, h)"'] & & a_2
\end{tikzcd}$$

We claim that the cleavage given by the maps

$$\Delta^{\mathsf{Cyl}}_{f,a} \xrightarrow{l_a} \Omega^{\mathsf{Cyl}}_{f,a}$$

defined by $(g_1, h) \mapsto l_a(g_1, h)$, for an object a of $\mathcal{A}$, equips f with the structure of a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$.

Indeed, suppose that we have a commutative triangle in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, "h"'] & a_0 \arrow[d, "g_2"] \\
& a_2
\end{tikzcd}$$

Then

$$\mathsf{adj}(h) = \mathsf{adj}(g_2 \circ p(a_0)).$$

Moreover, the following diagram in $\mathcal{A}$ commutes since the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

$$\begin{tikzcd}
a_0 \arrow[r, "g_2"] \arrow[dr, "\mathsf{adj}(g_2 \circ p(a_0))"'] & a_2 \arrow[d, "c(a_2)"] \\
& \mathsf{co\text{-}Cyl}(a_2)
\end{tikzcd}$$

We deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g_2"] \arrow[dr, "\mathsf{adj}(h)"'] & a_2 \arrow[d, "c(a_2)"] \\
& \mathsf{co\text{-}Cyl}(a_2)
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes, since f satisfies the lifting of identities condition of Definition VIII.27.

$$\begin{tikzcd} a_0 \arrow[r, "g_1"] \arrow[dr, "k_{a_0}(g_1, \mathsf{adj}(h))"'] & a_1 \arrow[d, "c(a_1)"] \\ & \mathsf{co\text{-}Cyl}(a_1) \end{tikzcd}$$

Thus we have that

$$l_{a_0}(g_1, h) = \mathsf{adj}^{-1}(c(a_1) \circ g_1).$$

Moreover, the following diagram in $\mathcal{A}$ commutes since the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, "\mathsf{adj}^{-1}(c(a_1) \circ g_1)"'] & a_0 \arrow[d, "g_1"] \\ & a_1 \end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, "l_{a_0}(g_1, h)"'] & a_0 \arrow[d, "g_1"] \\ & a_1 \end{tikzcd}$$

This proves that f satisfies the lifting of identities condition of Definition VIII.33.

Let h instead be arbitrary, and let

$$a_{-1} \xrightarrow{g_0} a_0$$

be an arrow of $\mathcal{A}$. The following diagram in $\mathcal{A}$ commutes, since f satisfies the compatibility of lifts condition of Definition VIII.27.

$$\begin{tikzcd} a_{-1} \arrow[r, "g_0"] \arrow[dr, "k_{a_{-1}}(g_1 \circ g_0, \mathsf{adj}(h) \circ g_0)"'] & a_0 \arrow[d, "k_{a_0}(g_1, \mathsf{adj}(h))"] \\ & \mathsf{co\text{-}Cyl}(a_1) \end{tikzcd}$$

We also have that the following diagram in $\mathcal{A}$ commutes, by the naturality of adj.

$$\begin{tikzcd}
a_{-1} \arrow[r, "g_0"] \arrow[dr, "\mathsf{adj}(h \circ \mathsf{Cyl}(g_0))"'] & a_0 \arrow[d, "\mathsf{adj}(h)"] \\
& \mathsf{co\text{-}Cyl}(a_2)
\end{tikzcd}$$

Putting the last two observations together, we have that

$$l_{a_{-1}}(g_1 \circ g_0, h \circ \mathsf{Cyl}(g_0)) = \mathsf{adj}^{-1}\Big(k_{a_0}(g_1, \mathsf{adj}(h)) \circ g_0\Big).$$

Moreover, the following diagram in $\mathcal{A}$ commutes, by the naturality of adj^{-1}.

$$\begin{tikzcd}
\mathsf{Cyl}(a_{-1}) \arrow[r, "\mathsf{Cyl}(g_0)"] \arrow[dr, "\mathsf{adj}^{-1}\big(k_{a_0}(g_1, \mathsf{adj}(h)) \circ g_0\big)"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{adj}^{-1}\big(k_{a_0}(g_1, \mathsf{adj}(h))\big)"] \\
& a_1
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_{-1}) \arrow[r, "\mathsf{Cyl}(g_0)"] \arrow[dr, "l_{a_{-1}}(g_1 \circ g_0, h \circ \mathsf{Cyl}(g_0))"'] & \mathsf{Cyl}(a_0) \arrow[d, "l_{a_0}(g_1, h)"] \\
& a_1
\end{tikzcd}$$

This proves that f satisfies the compatibility of liftings condition of Definition VIII.33, and concludes this direction of the proof.

We now prove that if f is a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$, then f is a normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$. To this end, suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a \arrow[r, "g_1"] \arrow[d, "h"'] & a_1 \arrow[d, "f"] \\
\mathsf{co\text{-}Cyl}(a_2) \arrow[r, "e_0(a_2)"'] & a_2
\end{tikzcd}$$

As in the proof of Proposition VIII.7, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a \arrow[r, "g_1"] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "f"] \\ \mathsf{Cyl}(a) \arrow[r, "\mathsf{adj}^{-1}(h)"'] & a_2 \end{tikzcd}$$

Let

$$\Delta^{\mathsf{Cyl}}_{\underline{f,a}} \xrightarrow{l_a} \Omega^{\mathsf{Cyl}}_{\underline{f,a}}$$

denote the map, of the cleavage with which f is equipped, corresponding to the object a of $\mathcal{A}$. Then the homotopy

$$\mathsf{Cyl}(a) \xrightarrow{l_a(g_1, \mathsf{adj}^{-1}(h))} a_1$$

with respect to $\underline{\mathsf{Cyl}}$ fits into a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a \arrow[r, "g_1"] \arrow[d, "i_0(a)"'] & a_1 \arrow[d, "f"] \\ \mathsf{Cyl}(a) \arrow[r, "\mathsf{adj}^{-1}(h)"'] \arrow[ur, "l_a(g_1, \mathsf{adj}^{-1}(h))"'] & a_2 \end{tikzcd}$$

Let $k_a(g_1, h)$ denote the arrow

$$a \xrightarrow{\mathsf{adj}\left(l_a(g_1, \mathsf{adj}^{-1}(h))\right)} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$.

We claim that the cleavage defined by the maps

$$\Sigma^{\mathsf{co\text{-}Cyl}}_{\underline{f,a}} \xrightarrow{k_a} \Upsilon^{\mathsf{co\text{-}Cyl}}_{\underline{f,a}}$$

defined by $(g_1, h) \mapsto k_a(g_1, h)$, for an object a of $\mathcal{A}$, equips f with the structure of a normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Indeed, suppose that we have a commutative triangle in $\mathcal{A}$ as follows.

$$\begin{tikzcd} a_0 \arrow[r, "g_2"] \arrow[dr, "h"'] & a_2 \arrow[d, "c(a_2)"] \\ & \mathsf{co\text{-}Cyl}(a_2) \end{tikzcd}$$

Then we have that
$$\mathsf{adj}^{-1}(h) = \mathsf{adj}^{-1}(c(a_2) \circ g_2).$$

Moreover, the following diagram in $\mathcal{A}$ commutes, since the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, swap, "\mathsf{adj}^{-1}(c(a_2) \circ g_2)"] & a_0 \arrow[d, "g_2"] \\ & a_2 \end{tikzcd}$$

We deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, swap, "\mathsf{adj}^{-1}(h)"] & a_0 \arrow[d, "g_2"] \\ & a_2 \end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes, since f satisfies the lifting of identities condition of Definition VIII.33.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, swap, "l_{a_0}(g_1, \mathsf{adj}^{-1}(h))"] & a_0 \arrow[d, "g_1"] \\ & a_1 \end{tikzcd}$$

Thus we have that
$$k_{a_0}(g_1, h) = \mathsf{adj}(g_1 \circ p(a_0)).$$

Moreover, the following diagram in $\mathcal{A}$ commutes, since the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

$$\begin{tikzcd} a_0 \arrow[r, "g_1"] \arrow[dr, swap, "\mathsf{adj}(g_1 \circ p(a_0))"] & a_1 \arrow[d, "c(a_1)"] \\ & \mathsf{co\text{-}Cyl}(a_1) \end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "g_1"] \arrow[dr, "k_{a_0}(g_1, h)"'] & a_1 \arrow[d, "c(a_1)"] \\ & \mathsf{co\text{-}Cyl}(a_1) \end{tikzcd}$$

This proves that f satisfies the lifting of identities condition of Definition VIII.27.

Let h instead be arbitrary, and let

$$a_{-1} \xrightarrow{g_0} a_0$$

be an arrow of $\mathcal{A}$. The following diagram in $\mathcal{A}$ commutes, since f satisfies the compatibility of lifts condition of Definition VIII.33.

$$\begin{tikzcd} \mathsf{Cyl}(a_{-1}) \arrow[r, "\mathsf{Cyl}(g_0)"] \arrow[dr, "l_{a_{-1}}(g_1 \circ g_0, \mathsf{adj}^{-1}(h) \circ \mathsf{Cyl}(g_0))"'] & \mathsf{Cyl}(a_0) \arrow[d, "l_{a_0}(g_1, \mathsf{adj}^{-1}(h))"] \\ & a_1 \end{tikzcd}$$

We also have that the following diagram in $\mathcal{A}$ commutes, by the naturality of adj^{-1}.

$$\begin{tikzcd} \mathsf{Cyl}(a_{-1}) \arrow[r, "\mathsf{Cyl}(g_0)"] \arrow[dr, "\mathsf{adj}^{-1}(h \circ g_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{adj}^{-1}(h)"] \\ & a_2 \end{tikzcd}$$

Putting the last two observations together, we have that

$$k_{a_{-1}}(g_1 \circ g_0, h \circ g_0) = \mathsf{adj}\Big(l_{a_0}(g_1, \mathsf{adj}^{-1}(h)) \circ \mathsf{Cyl}(g_0)\Big).$$

Moreover, the following diagram in $\mathcal{A}$ commutes, by the naturality of adj.

$$\begin{tikzcd} a_{-1} \arrow[r, "g_0"] \arrow[dr, "\mathsf{adj}\big(l_{a_0}(g_1, \mathsf{adj}^{-1}(h)) \circ \mathsf{Cyl}(g_0)\big)"'] & a_0 \arrow[d, "\mathsf{adj}\big(l_{a_0}(g_1, \mathsf{adj}^{-1}(h))\big)"] \\ & \mathsf{co\text{-}Cyl}(a_1) \end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_{-1} \arrow[r, "g_0"] \arrow[dr, "k_{a_{-1}}(g_1 \circ g_0, h \circ g_0)"'] & a_0 \arrow[d, "k_{a_0}(g_1, h)"] \\
& \mathsf{co\text{-}Cyl}(a_1)
\end{tikzcd}$$

This proves that f satisfies the compatibility of liftings condition of Definition VIII.27, and concludes this direction of the proof. □

Corollary VIII.41. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.*

An arrow

$$a_0 \xrightarrow{j} a_1$$

is a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$ if and only if it is a normally cloven cofibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition VIII.40 by duality. □

Proposition VIII.42. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Let*

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ which is a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

Suppose that we have commutative diagrams

$$\begin{tikzcd}
a_2 \arrow[r, "g_0"] \arrow[d, "j'"'] & a_0 \arrow[d, "j"] \\
a_3 \arrow[r, "g_1"'] & a_1
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[r, "r_0"] \arrow[d, "j"'] & a_2 \arrow[d, "j'"] \\
a_1 \arrow[r, "r_1"'] & a_3
\end{tikzcd}$$

in $\mathcal{A}$, such that r_0 is a retraction of g_0, and such that r_1 is a retraction of g_1. Then j' is a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

Proof. For any object a_4 of $\mathcal{A}$, let

$$\Sigma^{\mathsf{Cyl}}_{j,a_4} \xrightarrow{k_{a_4}} \Upsilon^{\mathsf{Cyl}}_{j,a_4}$$

denote the corresponding map of the cleavage with which j is equipped. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a_2 \arrow[r, "i_0(a_2)"] \arrow[d, "j'"'] & \mathsf{Cyl}(a_2) \arrow[d, "h"] \\
a_3 \arrow[r, "f"'] & a_4
\end{tikzcd}$$

As in the proof of Proposition VIII.14, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "h \circ \mathsf{Cyl}(r_0)"] \\
a_1 \arrow[r, "f \circ r_1"'] & a_4
\end{tikzcd}$$

Let us define a map

$$\Sigma^{\underline{\mathsf{Cyl}}}_{j',a_4} \xrightarrow{k'_{a_4}} \Upsilon^{\underline{\mathsf{Cyl}}}_{j',a_4}$$

by

$$(f, h) \mapsto k_{a_4}(f \circ r_1, h \circ \mathsf{Cyl}(r_0)) \circ \mathsf{Cyl}(g_1).$$

We claim that the maps k'_a, for a an object of $\mathcal{A}$, define a cleavage exhibiting j' to be a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

After the proof of Proposition VIII.14, it remains to demonstrate that conditions (i) and (ii) of Definition VIII.20 are satisfied.

Let us first show that condition (i) holds. To this end, suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a_2 \arrow[d, "j'"'] \arrow[dr, "f_0"] & \\
a_3 \arrow[r, "f_1"'] & a_4
\end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "j"] \arrow[d, "r_0"'] & a_1 \arrow[d, "r_1"] \\
a_2 \arrow[r, "j'"] \arrow[dr, "f_0"'] & a_3 \arrow[d, "f_1"] \\
 & a_4
\end{tikzcd}
$$

Since the cleavage defined by the maps k_a, for a an object of $\mathcal{A}$, exhibits j to be a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$, this cleavage satisfies condition (i) of Definition VIII.20. We deduce that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "p(a_1)"] \arrow[dr, "k_{a_4}(f_1 \circ r_1, f_0 \circ r_0 \circ p(a_0))"'] & a_1 \arrow[d, "f_1 \circ r_1"] \\
 & a_4
\end{tikzcd}
$$

Thus, by the commutativity of the diagram

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[d, "\mathsf{Cyl}(r_0)"'] & a_0 \arrow[d, "r_0"] \\
\mathsf{Cyl}(a_2) \arrow[r, "p(a_2)"'] & a_2
\end{tikzcd}
$$

in $\mathcal{A}$, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "p(a_1)"] \arrow[dr, "k_{a_4}(f_1 \circ r_1, f_0 \circ p(a_2) \circ \mathsf{Cyl}(r_0))"'] & a_1 \arrow[d, "f_1 \circ r_1"] \\
 & a_4
\end{tikzcd}
$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_3) \arrow[r, "\mathsf{Cyl}(g_1)"] \arrow[d, "p(a_3)"'] & \mathsf{Cyl}(a_1) \arrow[d, "p(a_1)"] \\
a_3 \arrow[r, "g_1"] \arrow[dr, "id"'] & a_1 \arrow[d, "r_1"] \\
& a_3
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_3) \arrow[d, "\mathsf{Cyl}(g_1)"'] \arrow[dr, "p(a_3)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1 \circ p(a_1)"'] \arrow[dr, "k_{a_4}(f_1 \circ r_1, f_0 \circ p(a_2) \circ \mathsf{Cyl}(r_0))"'] & a_3 \arrow[d, "f_1"] \\
& a_4
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd}
\mathsf{Cyl}(a_3) \arrow[r, "p(a_3)"] \arrow[dr, "k'_{a_4}(f_1, f_0 \circ p(a_2))"'] & a_3 \arrow[d, "f_1"] \\
& a_4
\end{tikzcd}$$

Let us now prove that condition (ii) of Definition VIII.20 holds. Let

$$a_4 \xrightarrow{f'} a_5$$

be an arrow of $\mathcal{A}$. Since the cleavage defined by the maps k_a, for a an object of $\mathcal{A}$, exhibits j to be a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$, this cleavage satisfies condition (ii) of Definition VIII.20. We deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "k_{a_4}(f \circ r_1, h \circ \mathsf{Cyl}(r_0))"] \arrow[dr, "k_{a_5}(f' \circ f \circ r_1, f' \circ h \circ \mathsf{Cyl}(r_0))"'] & a_4 \arrow[d, "f'"] \\
& a_5
\end{tikzcd}$$

By definition of $k'_{a_4}(f, h)$ and $k'_{a_5}(f' \circ f, f' \circ h)$, the diagrams

$$\begin{tikzcd}
\mathsf{Cyl}(a_3) \arrow[r, "\mathsf{Cyl}(g_1)"] \arrow[dr, swap, "k'_{a_4}(f,h)"] & a_4 \arrow[d, "k_{a_4}(f \circ r_1, h \circ \mathsf{Cyl}(r_0))"] \\
 & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{Cyl}(a_3) \arrow[r, "\mathsf{Cyl}(g_1)"] \arrow[dr, swap, "k'_{a_5}(f' \circ f, f' \circ h)"] & \mathsf{Cyl}(a_1) \arrow[d, "k_{a_5}(f' \circ f \circ r_1, f' \circ h \circ \mathsf{Cyl}(r_0))"] \\
 & a_5
\end{tikzcd}$$

in $\mathcal{A}$ commute. Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd}
\mathsf{Cyl}(a_3) \arrow[r, "k'_{a_4}(f,h)"] \arrow[dr, swap, "k'_{a_5}(f' \circ f, f' \circ h)"] & a_4 \arrow[d, "f'"] \\
 & a_5
\end{tikzcd}$$

$\square$

Corollary VIII.43. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Let*

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ which is a trivial normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

Suppose that we have commutative diagrams

$$\begin{tikzcd}
a_2 \arrow[r, "g_0"] \arrow[d, swap, "j'"] & a_0 \arrow[d, "j"] \\
a_3 \arrow[r, swap, "g_1"] & a_1
\end{tikzcd} \qquad \begin{tikzcd}
a_0 \arrow[r, "r_0"] \arrow[d, swap, "j"] & a_2 \arrow[d, "j'"] \\
a_1 \arrow[r, swap, "r_1"] & a_3
\end{tikzcd}$$

in $\mathcal{A}$, such that r_0 is a retraction of g_0, and such that r_1 is a retraction of g_1. Then j' is a trivial normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

Proof. Follows immediately from Proposition VIII.42 and Proposition VII.22. $\square$

Corollary VIII.44. *Let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ *which is a normally cloven fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$.

Suppose that we have commutative diagrams

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "f'"'] & a_0 \arrow[d, "f"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "f"'] & a_2 \arrow[d, "f'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, such that* r_0 *is a retraction of* g_0*, and such that* r_1 *is a retraction of* g_1*. Then* f' *is a normally cloven fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition VIII.42 by duality. $\square$

Corollary VIII.45. *Let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ *which is a trivial normally cloven fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$.

Suppose that we have commutative diagrams

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "f'"'] & a_0 \arrow[d, "f"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "f"'] & a_2 \arrow[d, "f'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, such that* r_0 *is a retraction of* g_0*, and such that* r_1 *is a retraction of* g_1*. Then* f' *is a trivial normally cloven fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Corollary VIII.44 and Proposition VII.22. $\square$

IX. Mapping cylinders and mapping co-cylinders

Let $\mathcal{A}$ be a formal category. With respect to a cylinder $\underline{\mathsf{Cyl}}$ in $\mathcal{A}$, we introduce the notion of a mapping cylinder $a_f^{\underline{\mathsf{Cyl}}}$ of an arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$. There is a canonical arrow $m_f^{\underline{\mathsf{Cyl}}}$ from $a_f^{\underline{\mathsf{Cyl}}}$ to $\mathsf{Cyl}(a_1)$. This arrow admits a retraction if f is a cofibration with respect to $\underline{\mathsf{Cyl}}$.

We introduce the dual notion of a mapping co-cylinder $a_f^{\underline{\mathsf{co\text{-}Cyl}}}$ of f with respect to a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}$ in $\mathcal{A}$. There is a canonical arrow from $\mathsf{co\text{-}Cyl}(a_0)$ to $a_f^{\underline{\mathsf{co\text{-}Cyl}}}$. This arrow admits a section if f is a fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

In XII, we shall prove that if $\underline{\mathsf{Cyl}}$ admits sufficient structure, then the retraction of $m_f^{\underline{\mathsf{Cyl}}}$ for f a cofibration is a strong deformation retraction with respect to $\underline{\mathsf{Cyl}}$. This will be a vital step towards establishing the lifting axioms for a model structure.

To return to the current section, we present a proof, given by van den Berg and Garner in §6 of [36], that if $\underline{\mathsf{Cyl}}$ has strictness of right identities, then every arrow f of $\mathcal{A}$ factors through $a_f^{\underline{\mathsf{Cyl}}}$ into a normally cloven cofibration followed by a strong deformation retraction.

If $\underline{\mathsf{co\text{-}Cyl}}$ has strictness of right identities we deduce, dually, that f factors through $a_f^{\underline{\mathsf{co\text{-}Cyl}}}$ into an arrow admitting a strong deformation retraction followed by a normally cloven fibration.

Strictness of right identities is an indispensable hypothesis here. We shall discuss it a little more in XIII, where we shall build upon our work here to establish the factorisation axioms for a model structure.

To carry through the ideas of van den Berg and Garner, we assume that $\underline{\mathsf{Cyl}}$ admits a lower right connection structure. Instead, van den Berg and Garner work with what is known as a strength, a structure of a slightly different nature to those which we considered in III.

The Moore co-cylinder in topological spaces is a key motivating example for van den Berg and Garner. It does not admit a lower connection structure, but does admit a strength. We refer the reader to §4.2 of [36] for more on this. Everywhere that we make use of a connection in this work, it should be possible to instead make use of a strength.

In abstract homotopy theory, the observation that the mapping cylinder of an arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ gives rise to a factorisation into a cofibration followed by a strong deformation retraction goes back to Kamps. It can be found in §4 of [21], and is also Theorem 5.11 in the book [23] of Kamps and Porter.

However, the cleavage of the arrow

$$a_0 \xrightarrow{j} a_f^{\underline{\mathsf{Cyl}}}$$

constructed in these works does not satisfy our conditions for j to be a normally cloven cofibration. The cleavage we construct after van den Berg and Garner does demonstrate j to be a normally cloven cofibration, which is crucial for us.

A different proof that the mapping cylinder of f gives rise to a factorisation into a cofibration followed by a strong deformation retraction was given by Grandis in §4.6.5 – §4.6.7 of the book [15], under the additional assumption that $\underline{\mathsf{Cyl}}$ admits what is referred to as an extended acceleration structure.

A key step in this argument of Grandis appears as Theorem 1.8 in his earlier paper [14], but under weaker hypotheses. The proof given in [14] is erroneous, but can be repaired if $\underline{\mathsf{Cyl}}$ admits a zero collapse structure, in the sense of [13], or alternatively if $\underline{\mathsf{Cyl}}$ has strictness of right identities.

We also point the reader to Theorem 3.8 of the paper [12] of Grandis.

Assumption IX.1. Let $\mathcal{C}$ be a 2-category with a final object. Suppose that pushouts and pullbacks of 2-arrows of $\mathcal{C}$ give rise to pushouts and pullbacks in formal categories, in the sense of Definition II.14. Let $\mathcal{A}$ be an object of $\mathcal{C}$. As before, we view $\mathcal{A}$ as a formal category, writing of objects and arrows of $\mathcal{A}$.

Definition IX.2. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. A *mapping cylinder* with respect to $\underline{\mathsf{Cyl}}$ of an arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is an object $a_f^{\underline{\mathsf{Cyl}}}$ of $\mathcal{A}$, together with an arrow

$$\mathsf{Cyl}(a_0) \xrightarrow{d_f^0} a_f^{\underline{\mathsf{Cyl}}}$$

of $\mathcal{A}$, and an arrow

$$a_1 \xrightarrow{d_f^1} a_f^{\underline{\mathsf{Cyl}}}$$

of $\mathcal{A}$, such that the following diagram in $\mathcal{A}$ is co-cartesian.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \\
a_1 \arrow[r, "d_f^1"'] & a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

Definition IX.3. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Then $\mathcal{A}$ *has mapping cylinders* with respect to $\underline{\mathsf{Cyl}}$ if, for every arrow f of $\mathcal{A}$, we have a mapping cylinder $(a_f^{\mathsf{Cyl}}, d_f^0, d_f^1)$ of f with respect to $\underline{\mathsf{Cyl}}$.

Definition IX.4. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. A *mapping co-cylinder* of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$ of an arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is an object $a_f^{\mathsf{co\text{-}Cyl}}$ of $\mathcal{A}$, together with an arrow

$$a_f^{\mathsf{co\text{-}Cyl}} \xrightarrow{d_f^0} a_0$$

of $\mathcal{A}$, and an arrow

$$a_f^{\mathsf{co\text{-}Cyl}} \xrightarrow{d_f^1} \mathsf{co\text{-}Cyl}(a_1)$$

of $\mathcal{A}$, such that the following diagram in $\mathcal{A}$ is cartesian.

$$\begin{tikzcd}
a_f^{\mathsf{co\text{-}Cyl}} \arrow[r, "d_f^0"] \arrow[d, "d_f^1"'] & a_0 \arrow[d, "f"] \\
\mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"'] & a_1
\end{tikzcd}$$

Remark IX.5. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$. Then $(a_f^{\mathsf{co\text{-}Cyl}}, d_f^0, d_f^1)$ defines a mapping co-cylinder of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if and only if $(a_f^{\mathsf{co\text{-}Cyl}}, (d_f^0)^{op}, (d_f^1)^{op})$ defines a mapping cylinder of f with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Definition IX.6. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Then $\mathcal{A}$ *has mapping co-cylinders* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if, for every arrow f of $\mathcal{A}$, we have a mapping co-cylinder $(a_f^{\underline{\mathsf{co\text{-}Cyl}}}, d_f^0, d_f^1)$ of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Definition IX.7. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Then Cyl *preserves mapping cylinders* with respect to $\underline{\mathsf{Cyl}}$ if, for every pair of an arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ and a mapping cylinder $(a_f^{\underline{\mathsf{Cyl}}}, d_f^0, d_f^1)$ of f with respect to $\underline{\mathsf{Cyl}}$, we have that $(\mathsf{Cyl}(a_f^{\underline{\mathsf{Cyl}}}), \mathsf{Cyl}(d_f^0), \mathsf{Cyl}(d_f^1))$ defines a mapping cylinder of $\mathsf{Cyl}(f)$ with respect to $\underline{\mathsf{Cyl}}$.

Definition IX.8. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Then $\mathsf{co\text{-}Cyl}$ *preserves mapping co-cylinders* with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if $\mathsf{co\text{-}Cyl}$ preserves mapping cylinders with respect to the cylinder $\underline{\mathsf{co\text{-}Cyl}}^{op}$ in $\mathcal{A}^{op}$.

Notation IX.9. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and suppose that $(a_f^{\underline{\mathsf{Cyl}}}, d_f^0, d_f^1)$ defines a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$.

We denote by $m_f^{\underline{\mathsf{Cyl}}}$ the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \arrow[ddr, bend left, "\mathsf{Cyl}(f)"] & \\
a_1 \arrow[r, "d_f^1"'] \arrow[drr, bend right, "i_0(a_1)"'] & a_f^{\underline{\mathsf{Cyl}}} \arrow[dr, "m_f^{\underline{\mathsf{Cyl}}}"'] & \\
 & & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Notation IX.10. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1)$ be a co-cylinder in $\mathcal{A}$. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and suppose that $(a_f^{\underline{\mathsf{co\text{-}Cyl}}}, d_f^0, d_f^1)$ defines a mapping co-cylinder of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

We denote by $m_f^{\underline{\mathsf{co\text{-}Cyl}}}$ the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{co\text{-}Cyl}(a_0) \arrow[dr, "m_f^{\underline{\mathsf{co\text{-}Cyl}}}"] \arrow[drr, bend left, "e_0(a_0)"] \arrow[ddr, bend right, "\mathsf{co\text{-}Cyl}(f)"'] & & \\
& a_f^{\underline{\mathsf{co\text{-}Cyl}}} \arrow[r, "d_f^0"] \arrow[d, "d_f^1"'] & a_0 \arrow[d, "f"] \\
& \mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"'] & a_1
\end{tikzcd}$$

Proposition IX.11. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Let*

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$, and suppose that $(a_j^{\underline{\mathsf{Cyl}}}, d_j^0, d_j^1)$ defines a mapping cylinder of j with respect to $\underline{\mathsf{Cyl}}$.

Suppose that j is a cofibration with respect to $\underline{\mathsf{Cyl}}$, and let

$$\mathsf{Cyl}(a_1) \xrightarrow{r_j^{\underline{\mathsf{Cyl}}}} a_j^{\underline{\mathsf{Cyl}}}$$

denote the corresponding arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(j)"] \arrow[ddr, bend left, "d_j^0"] & \\
a_1 \arrow[r, "i_0(a_1)"'] \arrow[drr, bend right, "d_j^1"'] & \mathsf{Cyl}(a_1) \arrow[dr, "r_j^{\underline{\mathsf{Cyl}}}"'] & \\
& & a_j^{\underline{\mathsf{Cyl}}}
\end{tikzcd}$$

Then $r_j^{\underline{\mathsf{Cyl}}}$ is a retraction of $m_j^{\underline{\mathsf{Cyl}}}$.

Proof. The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "d_j^0"] \arrow[d, "d_j^0"'] \arrow[rrd, "\mathsf{Cyl}(j)"] & & a_j^{\underline{\mathsf{Cyl}}} \arrow[d, "m_j^{\underline{\mathsf{Cyl}}}"] \\
a_j^{\underline{\mathsf{Cyl}}} & & \mathsf{Cyl}(a_1) \arrow[ll, "r_j^{\underline{\mathsf{Cyl}}}"]
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
a_1 \arrow[rr, "d_j^1"] \arrow[d, "d_j^1"'] \arrow[rrd, "i_0(a_1)"] & & a_j^{\underline{\mathsf{Cyl}}} \arrow[d, "m_j^{\underline{\mathsf{Cyl}}}"] \\
a_j^{\underline{\mathsf{Cyl}}} & & \mathsf{Cyl}(a_1) \arrow[ll, "r_j^{\underline{\mathsf{Cyl}}}"]
\end{tikzcd}$$

Putting these observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_j^0"] \arrow[rdd, bend left, "d_j^0"] & \\
a_1 \arrow[r, "d_j^1"'] \arrow[rrd, bend right, "d_j^1"'] & a_j^{\underline{\mathsf{Cyl}}} \arrow[rd, "r_j^{\underline{\mathsf{Cyl}}} \circ m_j^{\underline{\mathsf{Cyl}}}"'] & \\
& & a_j^{\underline{\mathsf{Cyl}}}
\end{tikzcd}$$

Appealing to the universal property of $a_j^{\underline{\mathsf{Cyl}}}$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_j^{\underline{\mathsf{Cyl}}} \arrow[r, "m_j^{\underline{\mathsf{Cyl}}}"] \arrow[rd, "id"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_j^{\underline{\mathsf{Cyl}}}"] \\
& a_j^{\underline{\mathsf{Cyl}}}
\end{tikzcd}$$

$\square$

Proposition IX.12. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$*, equipped with a contraction structure* p*. Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and suppose that $(a_f^{\mathsf{Cyl}}, d_f^0, d_f^1)$ defines a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$.

Let

$$a_f^{\mathsf{Cyl}} \xrightarrow{g} a_1$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \arrow[ddr, bend left, "f \circ p(a_0)"] & \\
a_1 \arrow[r, "d_f^1"'] \arrow[drr, bend right, "id"'] & a_f^{\mathsf{Cyl}} \arrow[dr, "g"'] & \\
& & a_1
\end{tikzcd}$$

Let

$$a_0 \xrightarrow{j} a_f^{\mathsf{Cyl}}$$

denote the arrow $d_f^0 \circ i_1(a_0)$ of $\mathcal{A}$.

The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\
& a_1
\end{tikzcd}$$

Proof. The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[rr, "j"] \arrow[dd, "id"'] \arrow[dr, "i_1(a_0)"] & & a_f^{\underline{\mathsf{Cyl}}} \arrow[dd, "g"] \\
& \mathsf{Cyl}(a_0) \arrow[ur, "d_f^0"'] \arrow[dl, "p(a_0)"] & \\
a_0 \arrow[rr, "f"'] & & a_1
\end{tikzcd}$$

□

Definition IX.13. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and suppose that $(a_f^{\underline{\mathsf{Cyl}}}, d_f^0, d_f^1)$ defines a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$.

We refer to the factorisation of f obtained via Proposition IX.12 as the *mapping cylinder factorisation* of f with respect to $\underline{\mathsf{Cyl}}$.

Corollary IX.14. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c. *Let*

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, *and suppose that* $(a_f^{\underline{\mathsf{co\text{-}Cyl}}}, d_f^0, d_f^1)$ *defines a mapping co-cylinder of* f *with respect to* $\underline{\mathsf{co\text{-}Cyl}}$.

Let

$$a_0 \xrightarrow{j} a_f^{\underline{\mathsf{co\text{-}Cyl}}}$$

denote the canonical arrow of $\mathcal{A}$ *such that the following diagram in* $\mathcal{A}$ *commutes.*

$$\begin{tikzcd}
a_0 \arrow[dr, "j"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "c(a_1) \circ f"'] & & \\
& a_f^{\underline{\mathsf{co\text{-}Cyl}}} \arrow[r, "d_f^0"] \arrow[d, "d_f^1"'] & a_0 \arrow[d, "f"] \\
& \mathsf{co\text{-}Cyl}(a_1) \arrow[r, "e_0(a_1)"'] & a_1
\end{tikzcd}$$

Let

$$a_f^{\underline{\mathsf{co\text{-}Cyl}}} \xrightarrow{g} a_1$$

denote the arrow $e_1(a_1) \circ d_f^1$ *of* $\mathcal{A}$.

The following diagram in $\mathcal{A}$ *commutes.*

$$\begin{tikzcd}
a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\underline{\mathsf{co\text{-}Cyl}}} \arrow[d, "g"] \\
& a_1
\end{tikzcd}$$

Proof. Follows immediately from Proposition IX.12 by duality. □

Definition IX.15. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and suppose that $(a_f^{\underline{\mathsf{co\text{-}Cyl}}}, d_f^0, d_f^1)$ defines a mapping co-cylinder of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$. We refer to the factorisation of f obtained via Proposition IX.14 as the *mapping co-cylinder factorisation* of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proposition IX.16. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \Gamma_{lr})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and a lower right connection structure* Γ_{lr}*. Suppose that* Γ_{lr} *is compatible with* p*, and that* Cyl *preserves mapping cylinders with respect to* $\underline{\mathsf{Cyl}}$.

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$*, and let* $(a_f^{\underline{\mathsf{Cyl}}}, d_f^0, d_f^1)$ *be a mapping cylinder of* f *with respect to* $\underline{\mathsf{Cyl}}$.

Let

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping cylinder factorisation of f.

There is a homotopy over a_1 from $d_f^1 \circ g$ to $id(a_f^{\mathsf{Cyl}})$ with respect to $\underline{\mathsf{Cyl}}$ and (g, g).

Proof. By construction, g is a retraction of d_f^1. The following diagram in $\mathcal{A}$ is co-cartesian, since Cyl preserves mapping cylinders with respect to $\underline{\mathsf{Cyl}}$.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_f^0)"] \\ \mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \end{tikzcd}$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[rr, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(f)"'] \arrow[dr, "p(a_0)"] & & \mathsf{Cyl}^2(a_0) \arrow[d, "\Gamma_{lr}(a_0)"] \\ \mathsf{Cyl}(a_1) \arrow[d, "p(a_1)"'] & a_0 \arrow[r, "i_0(a_0)"] \arrow[dl, "f"] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \\ a_1 \arrow[rr, "d_f^1"'] & & a_f^{\mathsf{Cyl}} \end{tikzcd}$$

Thus there is a canonical arrow

$$\mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \xrightarrow{h} a_f^{\mathsf{Cyl}}$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(f)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_f^0)"] \arrow[ddr, bend left, "d_f^0 \circ \Gamma_{lr}(a_0)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"'] \arrow[drr, bend right, "d_f^1 \circ p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[dr, "h"'] & \\
& & a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

We claim that h defines a homotopy over a_1 from $d_f^1 \circ g$ to $id(a_f^{\mathsf{Cyl}})$ with respect to $\underline{\mathsf{Cyl}}$ and (d_f^1, d_f^1).

Let us first prove that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_f^{\mathsf{Cyl}} \arrow[r, "i_0(a_f^{\mathsf{Cyl}})"] \arrow[dr, "d_f^1 \circ g"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "h"] \\
& a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[rr, "d_f^1"] \arrow[dr, "i_0(a_1)"] \arrow[dd, "id"'] & & a_f^{\mathsf{Cyl}} \arrow[d, "i_0(a_f^{\mathsf{Cyl}})"] \\
& \mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"] \arrow[dl, "p(a_1)"] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "h"] \\
a_1 \arrow[rr, "d_f^1"'] & & a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "d_f^0"] \arrow[d, "p(a_0)"'] \arrow[dr, "i_0(\mathsf{Cyl}(a_0))"] & & a_f^{\mathsf{Cyl}} \arrow[d, "i_0(a_f^{\mathsf{Cyl}})"] \\
a_0 \arrow[d, "i_0(a_0)"'] & \mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d_f^0)"] \arrow[dl, "\Gamma_{lr}(a_0)"] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "h"] \\
\mathsf{Cyl}(a_0) \arrow[rr, "d_f^0"'] & & a_f^{\mathsf{Cyl}}
\end{tikzcd}
$$

Putting these two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \arrow[ddr, bend left, "d_f^0 \circ i_0(a_0) \circ p(a_0)"] & \\
a_1 \arrow[r, "d_f^1"'] \arrow[drr, bend right, "d_f^1"'] & a_f^{\mathsf{Cyl}} \arrow[dr, "h \circ i_0(a_f^{\mathsf{Cyl}})"'] & \\
& & a_f^{\mathsf{Cyl}}
\end{tikzcd}
$$

We also have that the diagram

$$
\begin{tikzcd}
a_1 \arrow[r, "d_f^1"] \arrow[dr, "d_f^1"'] & a_f^{\mathsf{Cyl}} \arrow[d, "d_f^1 \circ g"] \\
& a_f^{\mathsf{Cyl}}
\end{tikzcd}
$$

in $\mathcal{A}$ commutes, since g is a retraction of d_f^1. Moreover, the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "d_f^0"] \arrow[d, "p(a_0)"'] & a^{\mathsf{Cyl}}_f \arrow[d, "g"] \\
a_0 \arrow[r, "f"'] \arrow[d, "i_0(a_0)"'] & a_1 \arrow[d, "d_f^1"] \\
\mathsf{Cyl}(a_0) \arrow[r, "d_f^0"'] & a^{\mathsf{Cyl}}_f
\end{tikzcd}
$$

It follows from the commutativity of the last two diagrams that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \arrow[ddr, bend left, "d_f^0 \circ i_0(a_0) \circ p(a_0)"] & \\
a_1 \arrow[r, "d_f^1"'] \arrow[drr, bend right, "d_f^1"'] & a^{\mathsf{Cyl}}_f \arrow[dr, "d_f^1 \circ g"'] & \\
& & a^{\mathsf{Cyl}}_f
\end{tikzcd}
$$

Appealing to the universal property of a^{Cyl}_f, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a^{\mathsf{Cyl}}_f \arrow[r, "i_0(a^{\mathsf{Cyl}}_f)"] \arrow[dr, "d_f^1 \circ g"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_f) \arrow[d, "h"] \\
& a^{\mathsf{Cyl}}_f
\end{tikzcd}
$$

Let us now prove that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a^{\mathsf{Cyl}}_f \arrow[r, "i_1(a^{\mathsf{Cyl}}_f)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_f) \arrow[d, "h"] \\
& a^{\mathsf{Cyl}}_f
\end{tikzcd}
$$

We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[rr, "d_f^1"] \arrow[dd, "id"'] \arrow[dr, "i_1(a_1)"] & & a_f^{\mathsf{Cyl}} \arrow[d, "i_1(a_f^{\mathsf{Cyl}})"] \\
& \mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"] \arrow[dl, "p(a_1)"] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "h"] \\
a_1 \arrow[rr, "d_f^1"'] & & a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "d_f^0"] \arrow[dd, "id"'] \arrow[dr, "i_1(\mathsf{Cyl}(a_0))"] & & a_f^{\mathsf{Cyl}} \arrow[d, "i_1(a_f^{\mathsf{Cyl}})"] \\
& \mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d_f^0)"] \arrow[dl, "\Gamma_{lr}(a_0)"] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "h"] \\
\mathsf{Cyl}(a_0) \arrow[rr, "d_f^0"'] & & a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \arrow[ddr, bend left, "d_f^0"] & \\
a_1 \arrow[r, "d_f^1"'] \arrow[drr, bend right, "d_f^1"'] & a_f^{\mathsf{Cyl}} \arrow[dr, "h \circ i_1(a_f^{\mathsf{Cyl}})"'] & \\
& & a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

Appealing to the universal property of $a^{\underline{\mathsf{Cyl}}}_{f}$, we deduce that the following diagram in $\mathcal{A}$ commutes, as we were aiming to show.

$$\begin{tikzcd}
a^{\underline{\mathsf{Cyl}}}_{f} \arrow[r, "i_1(a^{\underline{\mathsf{Cyl}}}_{f})"] \arrow[dr, "id"'] & \mathsf{Cyl}(a^{\underline{\mathsf{Cyl}}}_{f}) \arrow[d, "h"] \\
 & a^{\underline{\mathsf{Cyl}}}_{f}
\end{tikzcd}$$

We have now proven that h defines a homotopy from $d^1_f \circ g$ to $id(a^{\underline{\mathsf{Cyl}}}_{f})$ with respect to $\underline{\mathsf{Cyl}}$. It remains to demonstrate that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a^{\underline{\mathsf{Cyl}}}_{f}) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(g)"'] & a^{\underline{\mathsf{Cyl}}}_{f} \arrow[d, "g"] \\
\mathsf{Cyl}(a_1) \arrow[r, "p(a_1)"'] & a_1
\end{tikzcd}$$

Since Γ_{lr} is compatible with p, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\Gamma_{lr}(a_0)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}(a_0) \arrow[d, "p(a_0)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"'] & a_0
\end{tikzcd}$$

By definition of h and g, we thus have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[rr, "\mathsf{Cyl}(d^0_f)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] \arrow[dr, "\Gamma_{lr}(a_0)"] & & \mathsf{Cyl}(a^{\underline{\mathsf{Cyl}}}_{f}) \arrow[d, "h"] \\
\mathsf{Cyl}(a_0) \arrow[d, "p(a_0)"'] & \mathsf{Cyl}^2(a_0) \arrow[r, "d^0_f"] \arrow[dl, "p(a_0)"] & a^{\underline{\mathsf{Cyl}}}_{f} \arrow[d, "g"] \\
a_0 \arrow[rr, "f"'] & & a_1
\end{tikzcd}$$

By definition of h and g, we also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "h"] \\
a_1 \arrow[r, "d_f^1"] \arrow[dr, "id"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\
 & a_1
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_f^0)"] \arrow[ddr, bend left, "f \circ p(a_0) \circ p(\mathsf{Cyl}(a_0))"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"'] \arrow[drr, bend right, "p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[dr, "g \circ h"'] & \\
 & & a_1
\end{tikzcd}$$

By definition of g, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "\mathsf{Cyl}(g)"] \\
 & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"] \arrow[dr, "p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "p(a_1) \circ \mathsf{Cyl}(g)"] \\
 & a_1
\end{tikzcd}$$

Moreover, appealing to the definition of g, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[rr, "\mathsf{Cyl}(d_f^0)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] \arrow[dr, "\mathsf{Cyl}(p(a_0))"] & & \mathsf{Cyl}(a_f^{\underline{\mathsf{Cyl}}}) \arrow[d, "\mathsf{Cyl}(g)"] \\
\mathsf{Cyl}(a_0) \arrow[d, "p(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(f)"] \arrow[dl, "p(a_0)"] & \mathsf{Cyl}(a_1) \arrow[d, "p(a_1)"] \\
a_0 \arrow[rr, "f"'] & & a_1
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_f^0)"] \arrow[ddr, bend left, "f \circ p(a_0) \circ p(\mathsf{Cyl}(a_0))"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"'] \arrow[drr, bend right, "p(a_1)"'] & \mathsf{Cyl}(a_f^{\underline{\mathsf{Cyl}}}) \arrow[dr, "p(a_1) \circ \mathsf{Cyl}(g)"'] & \\
& & a_1
\end{tikzcd}$$

Appealing to the universal property of $\mathsf{Cyl}(a_f^{\underline{\mathsf{Cyl}}})$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_f^{\underline{\mathsf{Cyl}}}) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(g)"'] & a_f^{\underline{\mathsf{Cyl}}} \arrow[d, "g"] \\
\mathsf{Cyl}(a_1) \arrow[r, "p(a_1)"'] & a_1
\end{tikzcd}$$

This completes the proof of the claim.

□

Corollary IX.17. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \Gamma_{lr})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and a lower right connection structure Γ_{lr}. Suppose that Γ_{lr} is compatible with p. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.*

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and let $(a_f^{\mathsf{Cyl}}, d_f^0, d_f^1)$ be a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$. Let

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping cylinder factorisation of f. Then g is a strong deformation retraction of d_f^1 with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Since Cyl is left adjoint to $\mathsf{co\text{-}Cyl}$ we have that Cyl preserves mapping cylinders with respect to $\underline{\mathsf{Cyl}}$. Thus the result follows immediately from Proposition IX.16 and Proposition VII.61. $\square$

Lemma IX.18. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \Gamma_{lr})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p and a lower right connection structure Γ_{lr}. Suppose that Γ_{lr} is compatible with p, and that Cyl preserves mapping cylinders with respect to $\underline{\mathsf{Cyl}}$.*

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and let $(a_f^{\mathsf{Cyl}}, d_f^0, d_f^1)$ be a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$. Let

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping cylinder factorisation of f.

Suppose that there is an arrow

$$a_1 \xrightarrow{l} a_f^{\mathsf{Cyl}}$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "j"] \arrow[d, "f"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\
a_1 \arrow[r, "id"'] \arrow[ur, "l"'] & a_1
\end{tikzcd}
$$

Then f is a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

Proof. We first prove that f is a cofibration with respect to $\underline{\mathsf{Cyl}}$. To this end, suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "k"] \\
a_1 \arrow[r, "f'"'] & a_2
\end{tikzcd}
$$

Let

$$
\mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \xrightarrow{h} a_f^{\mathsf{Cyl}}
$$

denote the canonical arrow of $\mathcal{A}$ constructed as in the proof of Proposition IX.16. We have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(f)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_f^0)"] \arrow[ddr, bend left, "d_f^0 \circ \Gamma_{lr}(a_0)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"'] \arrow[drr, bend right, "d_f^1 \circ p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[dr, "h"] & \\
 & & a_f^{\mathsf{Cyl}}
\end{tikzcd}
$$

Let

$$
a_f^{\mathsf{Cyl}} \xrightarrow{u} a_1
$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \arrow[rd, bend left, "k"] & \\
a_1 \arrow[r, "d_f^1"'] \arrow[rrd, bend right, "f'"'] & a_f^{\mathsf{Cyl}} \arrow[rd, "u"] & \\
 & & a_2
\end{tikzcd}
$$

By definition of l, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[d, "f"'] \arrow[rd, "j"] & \\
a_1 \arrow[r, "l"'] & a_f^{\mathsf{Cyl}}
\end{tikzcd}
$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(f)"'] \arrow[rd, "\mathsf{Cyl}(j)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(l)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}})
\end{tikzcd}
$$

Moreover, by definition of j, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_1(a_0))"] \arrow[rd, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_f^0)"] \\
 & \mathsf{Cyl}(a_f^{\mathsf{Cyl}})
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_1(a_0))"] \arrow[d, "\mathsf{Cyl}(f)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d^0_f)"] \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(l)"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_f)
\end{tikzcd}
$$

By definition of h and u, we also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d^0_f)"] \arrow[d, "\Gamma_{lr}(a_0)"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_f) \arrow[d, "h"] \\
\mathsf{Cyl}(a_0) \arrow[r, "d^0_f"] \arrow[dr, "k"'] & a^{\mathsf{Cyl}}_f \arrow[d, "u"] \\
& a_2
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "\mathsf{Cyl}(f)"] \arrow[dr, "\mathsf{Cyl}(i_1(a_0))"] \arrow[dd, "id"'] & & \mathsf{Cyl}(a_1) \arrow[d, "\mathsf{Cyl}(l)"] \\
& \mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d^0_f)"] \arrow[dl, "\Gamma_{lr}(a_0)"] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_f) \arrow[d, "u \circ h"] \\
\mathsf{Cyl}(a_0) \arrow[rr, "k"'] & & a_2
\end{tikzcd}
$$

As in the proof of Proposition IX.16, we also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a^{\mathsf{Cyl}}_f \arrow[r, "i_0(a^{\mathsf{Cyl}}_f)"] \arrow[d, "g"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_f) \arrow[d, "h"] \\
a_1 \arrow[r, "d^1_f"'] & a^{\mathsf{Cyl}}_f
\end{tikzcd}
$$

Thus, appealing to the definition of u, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_f^{\mathsf{Cyl}} \arrow[r, "i_0(a_f^{\mathsf{Cyl}})"] \arrow[d, "g"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "h"] \\
a_1 \arrow[r, "d_f^1"] \arrow[dr, "f'"'] & a_f^{\mathsf{Cyl}} \arrow[d, "u"] \\
& a_2
\end{tikzcd}$$

Moreover, by definition of l, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "l"] \arrow[dr, "id"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\
& a_1
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[rr, "i_0(a_1)"] \arrow[dr, "l"] \arrow[dd, "id"'] & & \mathsf{Cyl}(a_1) \arrow[d, "\mathsf{Cyl}(l)"] \\
& a_f^{\mathsf{Cyl}} \arrow[r, "i_0(a_f^{\mathsf{Cyl}})"] \arrow[dl, "g"] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "u \circ h"] \\
a_1 \arrow[rr, "f'"'] & & a_2
\end{tikzcd}$$

We have now shown that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(f)"] \arrow[ddr, bend left, "k"] & \\
a_1 \arrow[r, "i_0(a_1)"'] \arrow[drr, bend right, "f'"'] & \mathsf{Cyl}(a_1) \arrow[dr, "u \circ h \circ \mathsf{Cyl}(l)"'] & \\
& & a_2
\end{tikzcd}$$

This completes our proof that f is a cofibration with respect to $\underline{\mathsf{Cyl}}$.

We claim, moreover, that the cleavage which associates, to an object a_2 and a pair of arrows (f', g) of $\mathcal{A}$ as above, the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{f' \circ h \circ \mathsf{Cyl}(l)} a_2$$

of $\mathcal{A}$, equips f with the structure of a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

To this end, let us prove that this cleavage has property (i) of Definition VIII.20. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, "k"'] & a_0 \arrow[d, "f''"] \\
& a_2
\end{tikzcd}$$

By definition of u, we then have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "d^0_f"] \arrow[d, "p(a_0)"'] & a^{\underline{\mathsf{Cyl}}}_f \arrow[d, "u"] \\
a_0 \arrow[r, "f''"'] & a_2
\end{tikzcd}$$

Since Γ_{lr} is compatible with p, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\Gamma_{lr}(a_0)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}(a_0) \arrow[d, "p(a_0)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"'] & a_0
\end{tikzcd}$$

Putting the last two observations together, and appealing to the definition of h, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[rr, "\mathsf{Cyl}(d^0_f)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] \arrow[dr, "\Gamma_{lr}(a_0)"] & & \mathsf{Cyl}(a^{\mathsf{Cyl}}_f) \arrow[d, "h"] \\
\mathsf{Cyl}(a_0) \arrow[d, "p(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[r, "d^0_f"] \arrow[dl, "p(a_0)"] & a^{\mathsf{Cyl}}_f \arrow[d, "u"] \\
a_0 \arrow[rr, "f''"'] & & a_2
\end{tikzcd}$$

By definition of h and u, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d^1_f)"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_f) \arrow[d, "h"] \\
a_1 \arrow[r, "d^1_f"] \arrow[dr, "f'"'] & a^{\mathsf{Cyl}}_f \arrow[d, "u"] \\
& a_2
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(f)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_f^0)"] \arrow[ddr, bend left, "f'' \circ p(a_0) \circ p(\mathsf{Cyl}(a_0))"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"'] \arrow[drr, bend right, "f' \circ p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[dr, "u \circ h"'] & \\
 & & a_2
\end{tikzcd}
$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[rr, "f"] \arrow[dr, "i_0(a_0)"] \arrow[dd, "id"'] & & a_1 \arrow[dd, "f'"] \\
 & \mathsf{Cyl}(a_0) \arrow[dl, "p(a_0)"] \arrow[dr, "k"] & \\
a_0 \arrow[rr, "f''"'] & & a_2
\end{tikzcd}
$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(f)"] \arrow[d, "p(a_0)"'] & \mathsf{Cyl}(a_1) \arrow[d, "p(a_1)"] \\
a_0 \arrow[r, "f"] \arrow[dr, "f''"'] & a_1 \arrow[d, "f'"] \\
 & a_2
\end{tikzcd}
$$

By definition of g, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d_f^0)"] \arrow[d, "\mathsf{Cyl}(p(a_0))"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "\mathsf{Cyl}(g)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(f)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[rr, "\mathsf{Cyl}(d_f^0)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] \arrow[dr, "\mathsf{Cyl}(p(a_0))"] & & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "\mathsf{Cyl}(g)"] \\
\mathsf{Cyl}(a_0) \arrow[d, "p(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(f)"] \arrow[dl, "p(a_0)"] & \mathsf{Cyl}(a_1) \arrow[d, "f' \circ p(a_1)"] \\
a_0 \arrow[rr, "f''"'] & & a_2
\end{tikzcd}$$

Moreover, by definition of g, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "\mathsf{Cyl}(g)"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"] \arrow[dr, "f' \circ p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "f' \circ p(a_1) \circ \mathsf{Cyl}(g)"] \\
& a_2
\end{tikzcd}$$

We have now shown that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(f)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_f^0)"] \arrow[rdd, bend left, "f'' \circ p(a_0) \circ p(\mathsf{Cyl}(a_0))"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_f^1)"'] \arrow[rrd, bend right, "f' \circ p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[rd, "f' \circ p(a_1) \circ \mathsf{Cyl}(g)"'] & \\
 & & a_2
\end{tikzcd}$$

Appealing to the universal property which $\mathsf{Cyl}(a_f^{\mathsf{Cyl}})$ possesses due to our assumption that Cyl preserves mapping cylinders, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[r, "h"] \arrow[rd, "f' \circ p(a_1) \circ \mathsf{Cyl}(g)"'] & a_f^{\mathsf{Cyl}} \arrow[d, "u"] \\
 & a_2
\end{tikzcd}$$

Moreover, by definition of l, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(l)"] \arrow[rd, "id"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "\mathsf{Cyl}(g)"] \\
 & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(l)"] \arrow[rd, "f' \circ p(a_1)"'] & \mathsf{Cyl}(a_f^{\mathsf{Cyl}}) \arrow[d, "u \circ h"] \\
 & a_2
\end{tikzcd}$$

This completes the proof that the cleavage satisfies property (i) of Definition VIII.20. That it moreover satisfies property (ii) of Definition VIII.20 is clear. □

Lemma IX.19. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*. Suppose that* $\underline{\mathsf{Cyl}}$ *has strictness of right identities.*

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$*, and let* $(a_f^{\mathsf{Cyl}}, d_f^0, d_f^1)$ *be a mapping cylinder of* f *with respect to* $\underline{\mathsf{Cyl}}$*. Let*

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping cylinder factorisation of f*.*

Let $(a_j^{\mathsf{Cyl}}, d_j^0, d_j^1)$ *be a mapping cylinder of* j *with respect to* $\underline{\mathsf{Cyl}}$*, and let*

$$\begin{tikzcd} a_0 \arrow[r, "j'"] \arrow[dr, "j"'] & a_j^{\mathsf{Cyl}} \arrow[d, "g'"] \\ & a_f^{\mathsf{Cyl}} \end{tikzcd}$$

denote the corresponding mapping cylinder factorisation of j*.*

There is an arrow

$$a_f^{\mathsf{Cyl}} \xrightarrow{l} a_j^{\mathsf{Cyl}}$$

of $\mathcal{A}$ *such that the following diagram in* $\mathcal{A}$ *commutes.*

$$\begin{tikzcd} a_0 \arrow[r, "j'"] \arrow[d, "j"'] & a_j^{\mathsf{Cyl}} \arrow[d, "g'"] \\ a_f^{\mathsf{Cyl}} \arrow[ur, "l"] \arrow[r, "id"'] & a_f^{\mathsf{Cyl}} \end{tikzcd}$$

Proof. Since $(a_j^{\mathsf{Cyl}}, d_j^0, d_j^1)$ defines a mapping cylinder of j with respect to $\underline{\mathsf{Cyl}}$, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "d^0_j"] \\
a^{\mathsf{Cyl}}_f \arrow[r, "d^1_j"'] & a^{\mathsf{Cyl}}_j
\end{tikzcd}$$

By definition of j, we also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_1(a_0)"] \arrow[dr, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "d^0_f"] \\
 & a^{\mathsf{Cyl}}_f
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "d^0_j"] \\
\mathsf{Cyl}(a_0) \arrow[r, "d^1_j \circ d^0_f"'] & a^{\mathsf{Cyl}}_j
\end{tikzcd}$$

Thus there is a canonical arrow

$$\mathsf{S}(a_0) \xrightarrow{u} a^{\mathsf{Cyl}}_j$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "d^0_j"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "d^1_j \circ d^0_f"'] & \mathsf{S}(a_0) \arrow[dr, "u"] & \\
 & & a^{\mathsf{Cyl}}_j
\end{tikzcd}$$

By definition of g', the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "d_j^0"] \arrow[d, "p(a_0)"'] & a_j^{\mathsf{Cyl}} \arrow[d, "g'"] \\
a_0 \arrow[r, "j"'] & a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

Appealing again to the commutativity, by definition of j, of the diagram

$$\begin{tikzcd}
a_0 \arrow[d, "i_1(a_0)"'] \arrow[dr, "j"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "d_f^0"'] & a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

in $\mathcal{A}$, we thus have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "d_j^0"] \arrow[dr, "d_f^0 \circ i_1(a_0) \circ p(a_0)"'] & a_j^{\mathsf{Cyl}} \arrow[d, "g'"] \\
& a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

Appealing to the commutativity, by definition of u, of the diagram

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "r_0(a_0)"] \arrow[dr, "d_j^0"'] & \mathsf{S}(a_0) \arrow[d, "u"] \\
& a_j^{\mathsf{Cyl}}
\end{tikzcd}$$

in $\mathcal{A}$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "r_0(a_0)"] \arrow[dr, "d_f^1 \circ i_1(a_0) \circ p(a_0)"'] & \mathsf{S}(a_0) \arrow[d, "g' \circ u"] \\
& a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

We also have that the following diagram in $\mathcal{A}$ commutes, by definition of u and g'.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"] \arrow[d, "d^0_f"'] & \mathsf{S}(a_0) \arrow[d, "u"] \\
a^{\mathsf{Cyl}}_{f} \arrow[r, "d^1_j"] \arrow[dr, "id"'] & a^{\mathsf{Cyl}}_{j} \arrow[d, "g'"] \\
& a^{\mathsf{Cyl}}_{f}
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "d^0_f \circ i_1(a_0) \circ p(a_0)"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "d^0_f"'] & \mathsf{S}(a_0) \arrow[dr, "g' \circ u"] & \\
& & a^{\mathsf{Cyl}}_{f}
\end{tikzcd}$$

Let

$$\mathsf{S} \xrightarrow{q_r} \mathsf{Cyl}$$

denote the canonical 2-arrow of $\mathcal{C}$ constructed as in Definition III.31. We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "i_1(a_0) \circ p(a_0)"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "id"'] & \mathsf{S}(a_0) \arrow[dr, "q_r(a_0)"] & \\
& & \mathsf{Cyl}(a_0)
\end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "d^0_f \circ i_1(a_0) \circ p(a_0)"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "d^0_f"'] & \mathsf{S}(a_0) \arrow[dr, "d^0_f \circ q_r(a_0)"'] & \\
 & & a^{\underline{\mathsf{Cyl}}}_f
\end{tikzcd}
$$

Appealing to the universal property of $\mathsf{S}(a_0)$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{S}(a_0) \arrow[r, "u"] \arrow[d, "q_r(a_0)"'] & a^{\underline{\mathsf{Cyl}}}_j \arrow[d, "g'"] \\
\mathsf{Cyl}(a_0) \arrow[r, "d^0_f"'] & a^{\underline{\mathsf{Cyl}}}_f
\end{tikzcd}
$$

By definition of the homotopy $(d^1_j \circ d^0_f) + d^0_j$ with respect to $\underline{\mathsf{Cyl}}$, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "s(a_0)"] \arrow[dr, "(d^1_j \circ d^0_f) + d^0_j"'] & \mathsf{S}(a_0) \arrow[d, "u"] \\
 & a^{\underline{\mathsf{Cyl}}}_j
\end{tikzcd}
$$

The following diagram in $\mathcal{A}$ also commutes, since $\underline{\mathsf{Cyl}}$ has strictness of right identities.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "s(a_0)"] \arrow[dr, "id"'] & \mathsf{S}(a_0) \arrow[d, "q_r(a_0)"] \\
 & \mathsf{Cyl}(a_0)
\end{tikzcd}
$$

Putting the last three observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "(d_j^1 \circ d_f^0) + d_j^0"] \arrow[dr, "d_f^0"'] & a_j^{\mathsf{Cyl}} \arrow[d, "g'"] \\
& a_f^{\mathsf{Cyl}}
\end{tikzcd}
$$

Since $(a_f^{\mathsf{Cyl}}, d_f^0, d_f^1)$ defines a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$, the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \\
a_1 \arrow[r, "d_f^1"'] & a_f^{\mathsf{Cyl}}
\end{tikzcd}
$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_j^1 \circ d_f^0"] \\
a_1 \arrow[r, "d_j^1 \circ d_f^1"'] & a_j^{\mathsf{Cyl}}
\end{tikzcd}
$$

We deduce that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "(d_j^1 \circ d_f^0) + d_j^0"] \\
a_1 \arrow[r, "d_j^1 \circ d_f^1"'] & a_j^{\mathsf{Cyl}}
\end{tikzcd}
$$

Thus there is a canonical arrow

$$
a_f^{\mathsf{Cyl}} \xrightarrow{l} a_j^{\mathsf{Cyl}}
$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_f^0"] \arrow[ddr, bend left, "(d_j^1 \circ d_f^0) + d_j^0"] & \\
a_1 \arrow[r, "d_f^1"'] \arrow[drr, bend right, "d_j^1 \circ d_f^1"'] & a_f^{\mathsf{Cyl}} \arrow[dr, "l"'] & \\
& & a_j^{\mathsf{Cyl}}
\end{tikzcd}$$

We claim that l fits into a commutative diagram as in the statement of the proposition.

Firstly, it follows from the commutativity of the diagrams

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "d_f^0"] \arrow[dr, "(d_j^1 \circ d_f^0) + d_j^0"'] & a_f^{\mathsf{Cyl}} \arrow[d, "l"] \\
& a_j^{\mathsf{Cyl}}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "(d_j^1 \circ d_f^0) + d_j^0"] \arrow[dr, "d_f^0"'] & a_j^{\mathsf{Cyl}} \arrow[d, "g'"] \\
& a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

in $\mathcal{A}$ that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "d_f^0"] \arrow[dr, "d_f^0"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g' \circ l"] \\
& a_f^{\mathsf{Cyl}}
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes, by definition of l and g'.

$$
\begin{tikzcd}
a_1 \arrow[r, "d^1_f"] \arrow[d, "d^1_f"'] & a^{\mathsf{Cyl}}_{f} \arrow[d, "l"] \\
a^{\mathsf{Cyl}}_{f} \arrow[r, "d^1_j"] \arrow[dr, "id"'] & a^{\mathsf{Cyl}}_{j} \arrow[d, "g'"] \\
 & a^{\mathsf{Cyl}}_{j}
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "f"'] & \mathsf{Cyl}(a_0) \arrow[d, "d^0_f"] \arrow[ddr, bend left, "d^0_f"] & \\
a_1 \arrow[r, "d^1_f"'] \arrow[drr, bend right, "d^1_f"'] & a^{\mathsf{Cyl}}_{f} \arrow[dr, "g' \circ l"'] & \\
 & & a^{\mathsf{Cyl}}_{f}
\end{tikzcd}
$$

Appealing to the universal property of a^{Cyl}_{f}, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a^{\mathsf{Cyl}}_{f} \arrow[r, "l"] \arrow[dr, "id"'] & a^{\mathsf{Cyl}}_{j} \arrow[d, "g'"] \\
 & a^{\mathsf{Cyl}}_{f}
\end{tikzcd}
$$

Secondly, the following diagram in $\mathcal{A}$ commutes, by definition of j'.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_1(a_0)"] \arrow[dr, "j'"'] & \mathsf{Cyl}(a_0) \arrow[d, "d^0_j"] \\
 & a^{\mathsf{Cyl}}_{f}
\end{tikzcd}
$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "i_1(a_0)"] \arrow[dr, swap, "j'"] & \mathsf{Cyl}(a_0) \arrow[d, "(d^1_j \circ d^0_f) + d^0_j"] \\
 & a^{\underline{\mathsf{Cyl}}}_j
\end{tikzcd}$$

By definition of j and l, we also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[rr, "j"] \arrow[dr, swap, "i_1(a_0)"] & & \mathsf{Cyl}(a_0) \arrow[dd, "l"] \\
 & \mathsf{Cyl}(a_0) \arrow[ur, swap, "d^0_f"] \arrow[dr, swap, "(d^1_j \circ d^0_f) + d^0_j"] & \\
 & & a^{\underline{\mathsf{Cyl}}}_j
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "j"] \arrow[dr, swap, "j'"] & \mathsf{Cyl}(a_0) \arrow[d, "l"] \\
 & a^{\underline{\mathsf{Cyl}}}_j
\end{tikzcd}$$

This completes the proof of the claim. $\square$

Proposition IX.20. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, and a lower right connection structure Γ_{lr}. Suppose that Γ_{lr} is compatible with p, and that $\underline{\mathsf{Cyl}}$ has strictness of right identities. Suppose moreover that Cyl preserves mapping cylinders with respect to $\underline{\mathsf{Cyl}}$.*

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and let $(a^{\underline{\mathsf{Cyl}}}_f, d^0_f, d^1_f)$ be a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$.

Let

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\underline{\mathsf{Cyl}}} \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping cylinder factorisation of f. Then j is a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$.

Proof. Follows immediately from Lemma IX.18 and Lemma IX.19. $\square$

Corollary IX.21. *Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, and a lower right connection structure Γ_{lr}. Suppose that Γ_{lr} is compatible with c, and that $\underline{\mathsf{co\text{-}Cyl}}$ has strictness of right identities. Suppose moreover that $\mathsf{co\text{-}Cyl}$ preserves mapping co-cylinders with respect to $\underline{\mathsf{co\text{-}Cyl}}$.*

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and let $(a_f^{\underline{\mathsf{co\text{-}Cyl}}}, d_f^0, d_f^1)$ be a mapping co-cylinder of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Let

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\underline{\mathsf{co\text{-}Cyl}}} \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping co-cylinder factorisation of f. Then g is a normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition IX.20 by duality. $\square$

X. Covering homotopy extension property

We introduce the covering homotopy extension property with respect to a cylinder $\underline{\mathsf{Cyl}}$ in a formal category $\mathcal{A}$, and to an arrow j of $\mathcal{A}$.

Suppose that $\underline{\mathsf{Cyl}}$ is equipped with a contraction structure p. Let $\underline{\mathsf{co\text{-}Cyl}}$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c. If an arrow f of $\mathcal{A}$ has the covering homotopy extension property with respect to $\underline{\mathsf{Cyl}}$ and j, and if f is, moreover, a strong deformation retraction with respect to $\underline{\mathsf{co\text{-}Cyl}}$, we prove that f has the right lifting property with respect to j. This will be vital to us in XII.

That the covering homotopy extension property could imply a right lifting property goes back to the proof of Theorem 9 in the paper [33] of Strøm, and was explored in §6 of the paper [18] of Hastings. Our proof is abstracted from these two papers. Already in [18], Hastings observed that an abstraction of this kind should be possible. A proof in an abstract setting is also given towards the end of §3 of Chapter II of the book [23] of Kamps and Porter.

In the category of topological spaces, Strøm proved as Theorem 4 of [32] that fibrations have the covering homotopy extension property with respect to closed cofibrations. In Theorem 2.1 of [18], Hastings proved that in the category of compactly generated Hausdorff spaces, fibrations have the covering homotopy extension property with respect to arbitrary cofibrations. Related theorems go back to around 1960, if not further.

More recently, fibrations in the category of topological spaces satisfying the covering homotopy extension property with respect to cofibrations were investigated in the paper [30] of Schwänzel and Vogt, in which they are referred to as strong fibrations. They are also discussed in Chapter 4 of the book [26] of May and Sigurdsson, in which the same terminology is adopted.

Assumption X.1. Let $\mathcal{C}$ be a 2-category with a final object. Suppose that pushouts and pullbacks of 2-arrows of $\mathcal{C}$ give rise to pushouts and pullbacks in formal categories, in the sense of Definition II.14. Let $\mathcal{A}$ be an object of $\mathcal{C}$. As before, we view $\mathcal{A}$ as a formal category, writing of objects and arrows of $\mathcal{A}$.

Definition X.2. Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1)$ be a cylinder in $\mathcal{A}$. Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$, and suppose that $(a_j^{\underline{\mathsf{Cyl}}}, d_j^0, d_j^1)$ defines a mapping cylinder of j with respect to $\underline{\mathsf{Cyl}}$.

An arrow

$$a_2 \xrightarrow{f} a_3$$

of $\mathcal{A}$ has the *covering homotopy extension property* with respect to j and $\underline{\mathsf{Cyl}}$ if, for any commutative diagram

$$\begin{tikzcd} a_j^{\underline{\mathsf{Cyl}}} \arrow[r, "g"] \arrow[d, "m_j^{\underline{\mathsf{Cyl}}}"'] & a_2 \arrow[d, "f"] \\ \mathsf{Cyl}(a_1) \arrow[r, "h"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$, there is an arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{l} a_2$$

of $\mathcal{A}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_j^{\underline{\mathsf{Cyl}}} \arrow[r, "g"] \arrow[d, "m_j^{\underline{\mathsf{Cyl}}}"'] & a_2 \arrow[d, "f"] \\ \mathsf{Cyl}(a_1) \arrow[r, "h"'] \arrow[ur, "l"] & a_3 \end{tikzcd}$$

Remark X.3. This definition is equivalent to that given towards the end of §3 of Chapter II of the book [23] of Kamps and Porter. We shall not need this.

Proposition X.4. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, and let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.*

Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$, and suppose that $(a_j^{\underline{\mathsf{Cyl}}}, d_j^0, d_j^1)$ defines a mapping cylinder of j with respect to $\underline{\mathsf{Cyl}}$. Let

$$a_2 \xrightarrow{f} a_3$$

be an arrow of $\mathcal{A}$ which has the covering homotopy extension property with respect to j and $\underline{\mathsf{Cyl}}$. Moreover, let

$$a_3 \xrightarrow{j'} a_2$$

be an arrow of $\mathcal{A}$ and suppose that f is a strong deformation retraction of j' with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Then, for any commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$, there is an arrow

$$a_1 \xrightarrow{l} a_2$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3 \end{tikzcd}$$

Proof. Since f is a strong deformation retraction of j' with respect to $\underline{\mathsf{co\text{-}Cyl}}$, we have by Proposition VII.61 that there is a homotopy

$$\mathsf{Cyl}(a_2) \xrightarrow{h} a_2$$

over a_3 from $j'f$ to $id(a_2)$ with respect to $\underline{\mathsf{Cyl}}$ and (f, f). The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[rr, "i_0(a_0)"] \arrow[d, "j"'] \arrow[dr, "g_0"] & & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(g_0)"] \\ a_1 \arrow[d, "g_1"'] & a_2 \arrow[r, "i_0(a_2)"] \arrow[dl, "f"] & \mathsf{Cyl}(a_2) \arrow[d, "h"] \\ a_3 \arrow[rr, "j'"'] & & a_2 \end{tikzcd}$$

Thus there is a canonical arrow

$$a_j^{\mathsf{Cyl}} \xrightarrow{u} a_2$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_j^0"] \arrow[ddr, bend left, "h \circ \mathsf{Cyl}(g_0)"] & \\
a_1 \arrow[r, "d_j^1"'] \arrow[drr, bend right, "j' \circ g_1"'] & a_j^{\mathsf{Cyl}} \arrow[dr, "u"'] & \\
& & a_2
\end{tikzcd}$$

By definition of h as a homotopy over a_3 with respect to $\underline{\mathsf{Cyl}}$ and (f, f), the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_2) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(f)"'] & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_3) \arrow[r, "p(a_3)"'] & a_3
\end{tikzcd}$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_2) \arrow[r, "\mathsf{Cyl}(f)"] \arrow[d, "p(a_2)"'] & \mathsf{Cyl}(a_3) \arrow[d, "p(a_3)"] \\
a_2 \arrow[r, "f"'] & a_3
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_2) \arrow[r, "h"] \arrow[d, "p(a_2)"'] & a_2 \arrow[d, "f"] \\
a_2 \arrow[r, "f"'] & a_3
\end{tikzcd}$$

We now have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "d_j^0"] \arrow[dr, "\mathsf{Cyl}(g_0)"] \arrow[ddd, "p(a_0)"'] & & a_j^{\mathsf{Cyl}} \arrow[d, "u"] \\
& \mathsf{Cyl}(a_2) \arrow[r, "h"] \arrow[d, "p(a_2)"'] & a_2 \arrow[dd, "f"] \\
& a_2 \arrow[dr, "f"] & \\
a_0 \arrow[ur, "g_0"] \arrow[r, "j"'] & a_1 \arrow[r, "g_1"'] & a_3
\end{tikzcd}
$$

The following diagram in $\mathcal{A}$ also commutes.

$$
\begin{tikzcd}
a_1 \arrow[r, "d_j^1"] \arrow[d, "g_1"'] & a_j^{\mathsf{Cyl}} \arrow[d, "u"] \\
a_3 \arrow[r, "j'"] \arrow[dr, "id"'] & a_2 \arrow[d, "f"] \\
& a_3
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_j^0"] \arrow[ddr, bend left, "g_1 \circ j \circ p(a_0)"] & \\
a_1 \arrow[r, "d_j^1"'] \arrow[drr, bend right, "g_1"'] & a_j^{\mathsf{Cyl}} \arrow[dr, "f \circ u"'] & \\
& & a_3
\end{tikzcd}
$$

The following diagram in $\mathcal{A}$ also commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "d_j^0"] \arrow[dd, "p(a_0)"'] \arrow[drr, "\mathsf{Cyl}(j)"'] & & a_j^{\mathsf{Cyl}} \arrow[d, "m_j^{\mathsf{Cyl}}"] \\
& & \mathsf{Cyl}(a_1) \arrow[d, "p(a_1)"] \\
a_0 \arrow[rr, "j"'] & & a_1
\end{tikzcd}
$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "d_j^0"] \arrow[dr, "g_1 \circ j \circ p(a_0)"'] & a_j^{\mathsf{Cyl}} \arrow[d, "g_1 \circ p(a_1) \circ m_j^{\mathsf{Cyl}}"] \\
& a_3
\end{tikzcd}
$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_1 \arrow[r, "d_j^1"] \arrow[d, "id"'] \arrow[dr, "i_0(a_1)"'] & a_j^{\mathsf{Cyl}} \arrow[d, "m_j^{\mathsf{Cyl}}"] \\
a_1 & \mathsf{Cyl}(a_1) \arrow[l, "p(a_1)"]
\end{tikzcd}
$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_1 \arrow[r, "d_j^1"] \arrow[dr, "g_1"'] & a_j^{\mathsf{Cyl}} \arrow[d, "g_1 \circ p(a_1) \circ m_j^{\mathsf{Cyl}}"] \\
& a_3
\end{tikzcd}
$$

We now have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "j"'] & \mathsf{Cyl}(a_0) \arrow[d, "d_j^0"] \arrow[ddr, bend left, "g_1 \circ j \circ p(a_0)"] & \\
a_1 \arrow[r, "d_j^1"'] \arrow[ddr, bend right, ""] \arrow[rrd, bend right, "g_1"'] & a_j^{\mathsf{Cyl}} \arrow[dr, "g_1 \circ p(a_1) \circ m_j^{\mathsf{Cyl}}"'] & \\
 & & a_3
\end{tikzcd}$$

Appealing to the universal property of a_j^{Cyl}, it follows that the diagram

$$\begin{tikzcd}
a_j^{\mathsf{Cyl}} \arrow[r, "u"] \arrow[d, "m_j^{\mathsf{Cyl}}"'] & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_1) \arrow[r, "g_1 \circ p(a_1)"'] & a_3
\end{tikzcd}$$

in $\mathcal{A}$ commutes. Since f has the covering homotopy extension property with respect to j, we deduce there is an arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{l} a_2$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_j^{\mathsf{Cyl}} \arrow[r, "u"] \arrow[d, "m_j^{\mathsf{Cyl}}"'] & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_1) \arrow[r, "g_1 \circ p(a_1)"'] \arrow[ur, "l"] & a_3
\end{tikzcd}$$

Let x denote the arrow

$$a_1 \xrightarrow{l \circ i_1(a_1)} a_2$$

of $\mathcal{A}$. We claim that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\
a_1 \arrow[ur, "x"'] \arrow[r, "g_1"'] & a_3
\end{tikzcd}$$

Firstly, note that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"] \arrow[d, "d^0_j"'] & \mathsf{Cyl}(a_1) \arrow[d, "l"] \\
a^{\underline{\mathsf{Cyl}}}_j \arrow[ur, "m^{\underline{\mathsf{Cyl}}}_j"'] \arrow[r, "u"'] & a_2
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[rr, "j"] \arrow[d, "i_1(a_0)"'] & & a_1 \arrow[dl, "i_1(a_1)"'] \arrow[dd, "x"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] \arrow[d, "d^0_j"'] & \mathsf{Cyl}(a_1) \arrow[dr, "l"'] & \\
a^{\underline{\mathsf{Cyl}}}_j \arrow[rr, "u"'] & & a_2
\end{tikzcd}$$

Moreover, the following diagram in $\mathcal{A}$ commutes, by appeal to the definition of u and h.

$$\begin{tikzcd}
a_0 \arrow[rr, "i_1(a_0)"] \arrow[d, "g_0"'] & & \mathsf{Cyl}(a_0) \arrow[dl, "\mathsf{Cyl}(g_0)"'] \arrow[d, "d^0_j"] \\
a_2 \arrow[r, "i_1(a_2)"'] \arrow[drr, bend right, "id"'] & \mathsf{Cyl}(a_2) \arrow[dr, "h"'] & a^{\underline{\mathsf{Cyl}}}_j \arrow[d, "u"] \\
& & a_2
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "j"] \arrow[dr, "g_0"'] & a_1 \arrow[d, "x"] \\
& a_2
\end{tikzcd}$$

Secondly, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[rr, "x"] \arrow[dr, "i_1(a_1)"] \arrow[dd, "id"'] & & a_2 \arrow[dd, "f"] \\
& \mathsf{Cyl}(a_1) \arrow[ur, "l"'] \arrow[dl, "p(a_1)"] & \\
a_1 \arrow[rr, "g_1"'] & & a_3
\end{tikzcd}$$

This completes the proof of the claim.

$\square$

XI. Dold's theorem

Let $\underline{\mathsf{Cyl}}$ be a cylinder in a formal category $\mathcal{A}$, equipped with a contraction structure, an involution structure compatible with contraction, a subdivision structure compatible with contraction, and an upper left connection structure. Suppose that we have a commutative diagram

$$\begin{tikzcd} a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

in $\mathcal{A}$, in which j_0 and j_1 are fibrations with respect to $\underline{\mathsf{Cyl}}$. We prove that if f is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$, then f is, moreover, a homotopy equivalence over a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1).

For topological spaces, this Theorem 3.1 of the paper [9] of Dold. Our proof is an abstraction of a hybrid of Dold's proof and the proof presented in §5 of Chapter 6 of the book [25] of May.

We exhibit a double homotopy

$$\begin{tikzcd} f_0 \arrow[r, "h_0"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\sigma"] & f_1 \arrow[d, "h_1"] \\ f_2 \arrow[r, "h_3"'] & f_3 \end{tikzcd}$$

for which h_1, h_2, and h_3 can be proven to be homotopies over a. One can then construct a homotopy over a from f_0 to f_1 by taking the indirect route around the square, after reversing h_1.

In our construction of this double homotopy, the key role is played by an upper left connection structure. The reader may observe that the map

$$I^2 \xrightarrow{(t_0,t_1) \mapsto t_0 + (1-t_0)t_1} I,$$

in which I is the unit interval, underlies the construction of the double homotopy in [25]. This map defines an upper left connection structure with respect to the topological interval.

Given a cylinder whose associated cubical set satisfies low dimensional Kan conditions, a proof of Dold's theorem was given by Kamps in §6 of [21]. A variation is presented in §6 of Chapter I of [23]. There is a fundamental difference between our proof and these two.

We demonstrate that the double homotopy can be constructed if $\underline{\mathsf{Cyl}}$ admits certain structures. By contrast, requiring that the cubical set associated to $\underline{\mathsf{Cyl}}$ satisfies low dimensional Kan conditions ensures the existence of this double homotopy, but the Kan conditions must themselves be proven to hold. To put it another way, the proofs of [21] and [23] can be thought of as a plan for the construction of the double homotopy, whereas we identify structures upon a cylinder which allow us to carry out this plan.

A different proof of Dold's theorem can be given by identifying structures upon $\underline{\mathsf{Cyl}}$ which allow one to prove that the objects, arrows, and homotopies up to homotopy of $\mathcal{A}$ with respect to $\underline{\mathsf{Cyl}}$ assemble into a 2-category. For one can then appeal to the argument presented in §1 and §2 of Chapter IV of the book [23] of Kamps and Porter.

We deduce from Dold's theorem that a trivial cofibration admits a strong deformation retraction. This will be vital for us in XII, when we establish the lifting axioms for a model structure. As observed by Dold as Satz 3.6 of [10], his theorem dualises, from which we deduce that a trivial fibration is a strong deformation retraction.

If we have strictness of identities, we shall prove in XII that trivial cofibrations are exactly sections of strong deformation retractions, and dually that trivial fibrations are exactly strong deformation retractions.

Assumption XI.1. Let $\mathcal{C}$ be a 2-category with a final object. Suppose that pushouts and pullbacks of 2-arrows of $\mathcal{C}$ give rise to pushouts and pullbacks in formal categories, in the sense of Definition II.14. Let $\mathcal{A}$ be an object of $\mathcal{C}$. As before, we view $\mathcal{A}$ as a formal category, writing of objects and arrows of $\mathcal{A}$.

Lemma XI.2. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v*, and a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*.*

Let

$$a \xrightarrow{j_0} a_0$$

be an arrow of $\mathcal{A}$ *which is a fibration with respect to* $\underline{\mathsf{Cyl}}$*. Let*

$$a \xrightarrow{j_1} a_1$$

and

$$a_0 \xrightarrow{f} a_1$$

be arrows of $\mathcal{A}$*, such that the diagram*

$$\begin{tikzcd} a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, and such that f is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$.

There is an arrow

$$a_1 \xrightarrow{g} a_0$$

of $\mathcal{A}$, and a homotopy from fg to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_1 \arrow[d, "g"'] \arrow[dr, "j_1"] & \\ a_0 \arrow[r, "j_0"'] & a \end{tikzcd}$$

Proof. Let

$$\mathsf{Cyl}(a_1) \xrightarrow{h} a_1$$

be a homotopy from ff^{-1} to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$. The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "f^{-1}"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\ a_0 \arrow[r, "f"] \arrow[dr, "j_0"'] & a_1 \arrow[d, "j_1"] \\ & a \end{tikzcd}$$

Since j_0 is a fibration with respect to $\underline{\mathsf{Cyl}}$, we deduce that there is an arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{k} a_0$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "f^{-1}"] \arrow[d, "i_0(a_1)"'] & a_0 \arrow[d, "j_0"] \\
\mathsf{Cyl}(a_1) \arrow[ur, "k"'] \arrow[r, "j_1 \circ h"'] & a
\end{tikzcd}$$

Let g denote the arrow

$$a_1 \xrightarrow{k \circ i_1(a_1)} a_0$$

of $\mathcal{A}$. The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[d, "i_1(a_1)"'] \arrow[dr, "id"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "h"'] \arrow[d, "k"'] & a_1 \arrow[d, "j_1"] \\
a_1 \arrow[r, "j_0"'] & a
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[d, "g"'] \arrow[dr, "j_1"] & \\
a_0 \arrow[r, "j_0"'] & a
\end{tikzcd}$$

It remains to construct a homotopy from fg to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$.

Firstly, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_1(a_1)"] \arrow[d, "f^{-1}"'] \arrow[dr, "i_0(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "v(a_1)"] \\
a_0 & \mathsf{Cyl}(a_1) \arrow[l, "k"]
\end{tikzcd}$$

We also have that the diagram

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "f^{-1}"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\
a_0 \arrow[r, "f"'] & a_1
\end{tikzcd}$$

in $\mathcal{A}$ commutes. Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\
\mathsf{Cyl}(a_1) \arrow[r, "f \circ k \circ v(a_1)"'] & a_1
\end{tikzcd}$$

Let us denote by

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "l"] & a_1
\end{tikzcd}$$

the homotopy $(f \circ k \circ v(a_1)) + h$ with respect to $\underline{\mathsf{Cyl}}$. The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "g"'] \arrow[dr, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "v(a_1)"] \\
a_0 & \mathsf{Cyl}(a_1) \arrow[l, "k"]
\end{tikzcd}$$

Thus, since the diagram

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_0(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "l"] \\
\mathsf{Cyl}(a_1) \arrow[r, "h"'] & a_1
\end{tikzcd}$$

in $\mathcal{A}$ commutes, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_1 \arrow[r, "i_0(a_1)"] \arrow[dr, "fg"'] & \mathsf{Cyl}(a_1) \arrow[d, "l"] \\ & a_1 \end{tikzcd}$$

In addition, since the diagram

$$\begin{tikzcd} a_1 \arrow[r, "i_1(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "l"] \\ \mathsf{Cyl}(a_1) \arrow[r, "h"'] & a_1 \end{tikzcd}$$

in $\mathcal{A}$ commutes, and since the diagram

$$\begin{tikzcd} a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\ & a_1 \end{tikzcd}$$

in $\mathcal{A}$ commutes, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_1) \arrow[d, "l"] \\ & a_1 \end{tikzcd}$$

□

Lemma XI.3. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and an upper left connection structure* Γ_{ul}*. Suppose that* Cyl *preserves subdivision with respect to* $\underline{\mathsf{Cyl}}$*.*

Let

$$a \xrightarrow{j_0} a_0$$

be an arrow of $\mathcal{A}$ *which is a fibration with respect to* $\underline{\mathsf{Cyl}}$*. Let*

$$a \xrightarrow{j_1} a_1$$

and

$$a_0 \xrightarrow{f} a_1$$

be arrows of $\mathcal{A}$, such that the diagram

$$\begin{tikzcd} a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, and such that f is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$.

Let

$$a_1 \xrightarrow{g} a_0$$

and

$$\mathsf{Cyl}(a_1) \xrightarrow{l} a_1$$

denote the arrows of $\mathcal{A}$ constructed in Lemma XI.2, such that the diagram

$$\begin{tikzcd} a_1 \arrow[d, "g"'] \arrow[dr, "j_1"] & \\ a_0 \arrow[r, "j_0"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, and such that l defines a homotopy from fg to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$.

There is an arrow

$$\mathsf{Cyl}^2(a_1) \xrightarrow{\tau} a$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[d, "l"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\tau"] \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

Proof. Let

$$\mathsf{Cyl}(a_1) \xrightarrow{k} a_0$$

denote the arrow of $\mathcal{A}$ constructed in the proof of Lemma XI.2. In particular, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "k"] \arrow[d, "h"'] & a_0 \arrow[d, "j_0"] \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

By definition of Γ_{ul} as an upper left connection structure, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[dr, "id"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ul}(a_1)"] \\ & \mathsf{Cyl}(a_1) \end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[dr, "j_0 \circ k"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "j_1 \circ h \circ \Gamma_{ul}(a_1)"] \\ & a \end{tikzcd}$$

By definition of v as an involution structure, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "i_0(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "v(a_1)"] \\ & \mathsf{Cyl}(a_1) \end{tikzcd}$$

Thus, appealing once more to the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[dr, "id"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ul}(a_1)"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

in $\mathcal{A}$, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(a_1)"] \arrow[dr, "\Gamma_{ul}(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\mathsf{Cyl}(i_1(a_1))"] \\
& a
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_1(a_1))"] \arrow[dr, "j_0 \circ k"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "j_0 \circ k \circ \Gamma_{ul}(a_1) \circ \mathsf{Cyl}(v(a_1))"] \\
& a
\end{tikzcd}$$

Putting everything together, we have now shown that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[d, "\mathsf{Cyl}(i_1(a_1))"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "j_1 \circ h \circ \Gamma_{ul}(a_1)"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "j_0 \circ k \circ \Gamma_{ul}(a_1) \circ \mathsf{Cyl}(v(a_1))"'] & a
\end{tikzcd}$$

Since Cyl preserves subdivision with respect to $\underline{\mathsf{Cyl}}$, the following diagram in $\mathcal{A}$ is co-cartesian.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[d, "\mathsf{Cyl}(i_1(a_1))"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\mathsf{Cyl}(r_0(a_1))"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(r_1(a_1))"'] & \mathsf{Cyl}(\mathsf{S}(a_1))
\end{tikzcd}$$

Thus there is an arrow

$$\mathsf{Cyl}(\mathsf{S}(a_1)) \xrightarrow{u} a$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[d, "\mathsf{Cyl}(i_1(a_1))"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\mathsf{Cyl}(r_0(a_1))"] \arrow[ddr, bend left, "j_1 \circ h \circ \Gamma_{ul}(a_1)"] & \\
\mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(r_1(a_1))"'] \arrow[drr, bend right, "j_0 \circ k \circ \Gamma_{ul}(a_1) \circ \mathsf{Cyl}(v(a_1))"'] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[dr, "u"'] & \\
 & & a
\end{tikzcd}$$

Let

$$\mathsf{Cyl}^2(a_1) \xrightarrow{\tau} a$$

denote the arrow $u \circ \mathsf{Cyl}(s(a_1))$ of $\mathcal{A}$. We claim that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[d, "l"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\tau"] \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "r_1(a_1)"] \arrow[d, "v(a_1)"'] \arrow[dr, "i_0(\mathsf{Cyl}(a_1))"] & & \mathsf{S}(a_1) \arrow[d, "i_0(\mathsf{S}(a_1))"] \\
\mathsf{Cyl}(a_1) \arrow[d, "i_0(\mathsf{Cyl}(a_1))"'] & \mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(r_1(a_1))"] \arrow[dl, "\mathsf{Cyl}(v(a_1))"] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}(a_1) \arrow[rr, "j_0 \circ k \circ \Gamma_{ul}(a_1)"'] & & a
\end{tikzcd}$$

Since the diagram

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[dr, "id"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ul}(a_1)"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

in $\mathcal{A}$ commutes, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"] \arrow[dr, "j_0 \circ k \circ v(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "u \circ i_0(\mathsf{S}(a_1))"] \\
& a
\end{tikzcd}
$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "r_0(a_1)"] \arrow[dr, "i_0(\mathsf{Cyl}(a_1))"] \arrow[dd, "id"'] & & \mathsf{S}(a_1) \arrow[d, "i_0(\mathsf{S}(a_1))"] \\
& \mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(r_0(a_1))"] \arrow[dl, "\Gamma_{ul}(a_1)"] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}(a_1) \arrow[rr, "j_1 \circ h"'] & & a
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "j_1 \circ h"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "j_0 \circ k \circ v(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "u \circ i_0(\mathsf{S}(a_1))"] & \\
& & a
\end{tikzcd}
$$

In the proof of Lemma XI.2, we showed that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\
\mathsf{Cyl}(a_1) \arrow[r, "f \circ k \circ v(a_1)"'] & a_1
\end{tikzcd}$$

Thus there is an arrow

$$\mathsf{S}(a_1) \xrightarrow{r} a_1$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "h"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "f \circ k \circ v(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "r"] & \\
& & a_1
\end{tikzcd}$$

In particular, since the diagram

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_0(a_1)"] \arrow[dr, "h"'] & \mathsf{S}(a_1) \arrow[d, "r"] \\
& a_1
\end{tikzcd}$$

in $\mathcal{A}$ commutes, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_0(a_1)"] \arrow[dr, "j_1 \circ h"'] & \mathsf{S}(a_1) \arrow[d, "j_1 \circ r"] \\
& a
\end{tikzcd}$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"] \arrow[d, "k \circ v(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "r"] \\
a_0 \arrow[r, "f"] \arrow[dr, "j_0"'] & a_1 \arrow[d, "j_1"] \\
& a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "j_1 \circ h"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "j_0 \circ k \circ v(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "j_1 \circ r"'] & \\
& & a
\end{tikzcd}$$

Appealing to the universal property of $\mathsf{S}(a_1)$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{S}(a_1) \arrow[r, "i_0(\mathsf{S}(a_1))"] \arrow[d, "r"'] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "i_0(\mathsf{S}(a_1))"] \arrow[d, "s(a_1)"'] & & \mathsf{Cyl}^2(a_1) \arrow[dl, "\mathsf{Cyl}(s(a_1))"'] \arrow[dd, "\tau"] \\
\mathsf{S}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"'] \arrow[d, "r"'] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[dr, "u"] & \\
a_1 \arrow[rr, "j_1"'] & & a
\end{tikzcd}
$$

By definition of the homotopy l with respect to $\underline{\mathsf{Cyl}}$ constructed in the proof of Lemma XI.2, the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "s(a_1)"] \arrow[dr, "l"'] & \mathsf{S}(a_1) \arrow[d, "r"] \\
& a_1
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[d, "l"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\tau"] \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}
$$

This completes the proof of the claim.

$\square$

Lemma XI.4. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v *compatible with* p*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*, and an upper left connection structure* Γ_{ul}*. Suppose that* Cyl *preserves subdivision with respect to* $\underline{\mathsf{Cyl}}$*.*

Let

$$a \xrightarrow{j_0} a_0$$

and

$$a \xrightarrow{j_1} a_1$$

be arrows of $\mathcal{A}$, which are fibrations with respect to $\underline{\mathsf{Cyl}}$.

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, such that the diagram

$$\begin{tikzcd} a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, and such that f is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$.

Let

$$a_1 \xrightarrow{g} a_0$$

and

$$\mathsf{Cyl}(a_1) \xrightarrow{l} a_1$$

denote the arrows of $\mathcal{A}$ constructed in Lemma XI.2, such that the diagram

$$\begin{tikzcd} a_1 \arrow[d, "g"'] \arrow[dr, "j_1"] & \\ a_0 \arrow[r, "j_0"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, and such that l defines a homotopy from fg to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$.

There is an arrow

$$\mathsf{Cyl}^2(a_1) \xrightarrow{\sigma} a$$

of $\mathcal{A}$ with the following properties.

(i) *The following diagram in $\mathcal{A}$ commutes.*

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[d, "l"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

(ii) *Let h_1, h_2, and h_3 denote the right, left, and bottom boundary homotopies of σ respectively, so that we may depict σ as follows, in the pictorial notation of Remark VII.27.*

$$\begin{tikzcd} \arrow[r, "l"] \arrow[d, "h_2"'] & \arrow[d, "h_1"] \\ \arrow[r, "h_3"'] \arrow[ur, phantom, "\sigma"] & \end{tikzcd}$$

Then the following diagrams in $\mathcal{A}$ commute.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "h_1"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "h_2"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "h_3"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd}$$

Proof. Let

$$\mathsf{Cyl}^2(a_1) \xrightarrow{\tau} a$$

denote the double homotopy with respect to $\underline{\mathsf{Cyl}}$ constructed in the proof of Lemma XI.3. In particular, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "l"] \arrow[d, "i_0(\mathsf{Cyl}(a_1))"'] & a_1 \arrow[d, "\tau"] \\ \mathsf{Cyl}^2(a_1) \arrow[r, "j_1"'] & a \end{tikzcd}$$

Since j_1 is a fibration with respect to $\underline{\mathsf{Cyl}}$, there is thus an arrow

$$\mathsf{Cyl}^2(a_1) \xrightarrow{\sigma} a_1$$

of $\mathcal{A}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "l"] \arrow[d, "i_0(\mathsf{Cyl}(a_1))"'] & a_1 \arrow[d, "\tau"] \\
\mathsf{Cyl}^2(a_1) \arrow[ur, "\sigma"'] \arrow[r, "j_1"'] & a
\end{tikzcd}$$

Let h_1, h_2, and h_3 denote the right, left, and bottom boundary homotopies of σ respectively, so that we may depict σ as follows, in the pictorial notation of Remark VII.27.

$$\begin{tikzcd}
\cdot \arrow[r, "l"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\sigma"] & \cdot \arrow[d, "h_1"] \\
\cdot \arrow[r, "h_3"'] & \cdot
\end{tikzcd}$$

Firstly, let us prove that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "h_1"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

By definition of Γ_{ul} as an upper left connection structure, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_1(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ul}(a_1)"] \\
a_1 \arrow[r, "i_1(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Let

$$\mathsf{Cyl}(\mathsf{S}(a_1)) \xrightarrow{u} a$$

denote the arrow of $\mathcal{A}$ constructed in the proof of Lemma XI.3. We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "\mathsf{Cyl}(i_1(a_1))"] \arrow[d, swap, "p(a_1)"] \arrow[dr, "\mathsf{Cyl}(i_1(a_1))"] & & \mathsf{Cyl}^2(a_1) \arrow[d, "\mathsf{Cyl}(s(a_1))"] \\
a_1 \arrow[d, swap, "i_1(a_1)"] & \mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(r_0(a_1))"] \arrow[dl, "\Gamma_{ul}(a_1)"] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}(a_1) \arrow[rr, swap, "j_1 \circ h"] & & a
\end{tikzcd}$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, swap, "id"] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\
& a_1
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_1(a_1))"] \arrow[d, swap, "p(a_1)"] & \mathsf{Cyl}^2(a_1) \arrow[d, "u \circ \mathsf{Cyl}(s(a_1))"] \\
a_1 \arrow[r, swap, "j_1"] & a
\end{tikzcd}$$

Moreover, by definition of τ, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(s(a_1))"] \arrow[dr, swap, "\tau"] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
& a
\end{tikzcd}$$

Thus we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_1(a_1))"] \arrow[d, swap, "p(a_1)"] & \mathsf{Cyl}^2(a_1) \arrow[d, "\tau"] \\
a_1 \arrow[r, swap, "j_1"] & a
\end{tikzcd}$$

By definition of h_1, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_1(a_1))"] \arrow[dr, "h_1"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\
& a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "h_1"] \arrow[dd, "p(a_1)"'] \arrow[dr, "\mathsf{Cyl}(i_1(a_1))"] & & a \arrow[dd, "j_1"] \\
& \mathsf{Cyl}^2(a_1) \arrow[ur, "\sigma"] \arrow[dr, "\tau"'] & \\
a_1 \arrow[rr, "j_1"'] & & a
\end{tikzcd}$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "p(a_1)"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "h_1"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

Secondly, let us prove that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "h_2"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

Let

$$\mathsf{Cyl}(a_1) \xrightarrow{k} a_0$$

denote the arrow of $\mathcal{A}$ of the proof of Lemma XI.2. We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "\mathsf{Cyl}(i_0(a_1))"] \arrow[dr, "\mathsf{Cyl}(i_0(a_1))"] \arrow[dd, "\mathsf{Cyl}(i_1(a_1))"'] & & \mathsf{Cyl}^2(a_1) \arrow[d, "\mathsf{Cyl}(s(a_1))"] \\
& \mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(r_1(a_1))"] \arrow[dl, "\mathsf{Cyl}(v(a_1))"] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}^2(a_1) \arrow[rr, "j_0 \circ k \circ \Gamma_{ul}(a_1)"'] & & a
\end{tikzcd}$$

By definition of Γ_{ul} as an upper left connection structurei, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_1(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ul}(a_1)"] \\
a_1 \arrow[r, "i_1(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Again, we also have that the following diagram in $\mathcal{A}$ commutes, by definition of τ.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(s(a_1))"] \arrow[dr, "\tau"'] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
& a
\end{tikzcd}$$

Putting the last three observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\tau"] \\
a_1 \arrow[r, "j_0 \circ k \circ i_1(a_1)"'] & a
\end{tikzcd}$$

By definition of g, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "g"'] & \mathsf{Cyl}(a_1) \arrow[d, "k"] \\
& a_0
\end{tikzcd}$$

Moreover, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "g"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "j_0"] \\
& a
\end{tikzcd}$$

Putting the last three observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\tau"] \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

By definition of h_2, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[dr, "h_2"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\
& a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "h_2"] \arrow[dr, "\mathsf{Cyl}(i_0(a_1))"] \arrow[dd, "p(a_1)"'] & & a \arrow[dd, "j_1"] \\
& \mathsf{Cyl}^2(a_1) \arrow[ur, "\sigma"'] \arrow[dr, "\tau"'] & \\
a_1 \arrow[rr, "j_1"'] & & a
\end{tikzcd}
$$

Appealing to the commutativity of the diagram

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "p(a_1)"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}
$$

in $\mathcal{A}$, we deduce that the following diagram in $\mathcal{A}$ commutes, as required.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "h_2"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}
$$

Thirdly, let us prove that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "h_3"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}
$$

The following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "r_1(a_1)"] \arrow[d, "v(a_1)"'] \arrow[dr, "i_1(\mathsf{Cyl}(a_1))"] & & \mathsf{S}(a_1) \arrow[d, "i_1(\mathsf{S}(a_1))"] \\
\mathsf{Cyl}(a_1) \arrow[d, "i_1(\mathsf{Cyl}(a_1))"'] & \mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(r_1(a_1))"] \arrow[dl, "\mathsf{Cyl}(v(a_1))"] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}^2(a_1) \arrow[rr, "j_0 \circ k \circ \Gamma_{ul}(a_1)"'] & & a
\end{tikzcd}$$

By definition of Γ_{ul} as an upper left connection structure, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ul}(a_1)"] \\
a_1 \arrow[r, "i_1(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"] \arrow[d, "p(a_1) \circ v(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "u \circ i_1(\mathsf{S}(a_1))"] \\
a_1 \arrow[r, "j_0 \circ k \circ i_1(a_1)"'] & a
\end{tikzcd}$$

Moreover, as earlier in the proof, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "j_1"'] & \mathsf{Cyl}(a_1) \arrow[d, "k \circ j_0"] \\
& a
\end{tikzcd}$$

In addition, since v is compatible with p, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "v(a_1)"] \arrow[dr, "p(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "p(a_1)"] \\
& a_1
\end{tikzcd}$$

Putting the last three observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"] \arrow[d, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "u \circ i_1(\mathsf{S}(a_1))"] \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

Let

$$\mathsf{Cyl}(a_1) \xrightarrow{h} a_1$$

denote the homotopy from ff^{-1} to $id(a_1)$ of the proof of Lemma XI.2. In particular, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\
& a_1
\end{tikzcd}$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "r_0(a_1)"] \arrow[d, "p(a_1)"'] \arrow[dr, "i_1(\mathsf{Cyl}(a_1))"] & & \mathsf{S}(a_1) \arrow[d, "i_1(\mathsf{S}(a_1))"] \\
a_1 \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(r_0(a_1))"] \arrow[dl, "\Gamma_{ul}(a_1)"] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}(a_1) \arrow[rr, "j_1 \circ h"'] & & a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_0(a_1)"] \arrow[d, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "u \circ i_1(\mathsf{S}(a_1))"] \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}
$$

Putting everything together, we have now shown that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "j_1 \circ p(a_1)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "j_1 \circ p(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "u \circ i_1(\mathsf{S}(a_1))"'] & \\
& & a
\end{tikzcd}
$$

Let

$$
\mathsf{S} \xrightarrow{\overline{p}} \mathsf{id}_{\mathcal{A}}
$$

denote the 2-arrow of $\mathcal{C}$ of Definition III.13. By definition, the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "p(a_1)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "\overline{p}(a_1)"'] & \\
& & a_1
\end{tikzcd}
$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "j_1 \circ p(a_1)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "j_1 \circ p(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "j_1 \circ \overline{p}(a_1)"'] & \\
& & a
\end{tikzcd}
$$

Appealing to the universal property of $\mathsf{S}(a_1)$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{S}(a_1) \arrow[r, "i_1(\mathsf{S}(a_1))"] \arrow[d, "\overline{p}(a_1)"'] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}
$$

Since the subdivision structure $(\mathsf{S}, r_0, r_1, s)$ is compatible with p, we also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "s(a_1)"] \arrow[dr, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "\overline{p}(a_1)"] \\
& a_1
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "i_1(\mathsf{Cyl}(a_1))"] \arrow[dr, "s(a_1)"] \arrow[dd, "p(a_1)"'] & & \mathsf{S}(a_1) \arrow[d, "\mathsf{Cyl}(s(a_1))"] \\
& \mathsf{S}(a_1) \arrow[r, "i_1(\mathsf{S}(a_1))"] \arrow[dl, "\overline{p}(a_1)"] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
a_1 \arrow[rr, "j_1"'] & & a
\end{tikzcd}
$$

Once more, we also have that the following diagram in $\mathcal{A}$ commutes, by definition of τ.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_1) \arrow[r, "\mathsf{Cyl}(s(a_1))"] \arrow[dr, "\tau"'] & \mathsf{Cyl}(\mathsf{S}(a_1)) \arrow[d, "u"] \\
& a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "\tau"] \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

By definition of h_3i, the following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[dr, "h_3"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\
& a
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "h_3"] \arrow[dr, "i_0(\mathsf{Cyl}(a_1))"] \arrow[dd, "p(a_1)"'] & & a \arrow[dd, "j_1"] \\
& \mathsf{Cyl}^2(a_1) \arrow[ur, "\sigma"'] \arrow[dr, "\tau"] & \\
a_1 \arrow[rr, "j_1"'] & & a
\end{tikzcd}$$

Appealing to the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "p(a_1)"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\
\mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a
\end{tikzcd}$$

in $\mathcal{A}$, we deduce that the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "h_3"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd}$$

□

Lemma XI.5. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v *compatible with* p*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*, and an upper left connection structure* Γ_{ul}*. Suppose that* Cyl *preserves subdivision with respect to* $\underline{\mathsf{Cyl}}$*.*

Let

$$a \xrightarrow{j_0} a_0$$

and

$$a \xrightarrow{j_1} a_1$$

be arrows of $\mathcal{A}$ *which are fibrations with respect to* $\underline{\mathsf{Cyl}}$*.*

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$*, such that the diagram*

$$\begin{tikzcd} a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

in $\mathcal{A}$ *commutes, and such that* f *is a homotopy equivalence with respect to* $\underline{\mathsf{Cyl}}$*.*

There is an arrow

$$a_1 \xrightarrow{g} a_0$$

of $\mathcal{A}$*, such that the diagram*

$$\begin{tikzcd} a_1 \arrow[d, "g"'] \arrow[dr, "j_1"] & \\ a_0 \arrow[r, "j_0"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, and such that there is a homotopy over a from fg to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_1, j_1).

Proof. Let

$$a_1 \xrightarrow{g} a_0$$

and

$$\mathsf{Cyl}(a_1) \xrightarrow{l} a_1$$

denote the arrows of $\mathcal{A}$ constructed in Lemma XI.2. Let

$$\mathsf{Cyl}^2(a_1) \xrightarrow{\sigma} a_1$$

denote the arrow of $\mathcal{A}$ constructed in Lemma XI.4. In the pictorial notation of Remark VII.27, the boundary of σ is as follows, where h_1, h_2, and h_3 are all homotopies over a with respect to $\underline{\mathsf{Cyl}}$ and (j_1, j_1).

$$\begin{tikzcd} \cdot \arrow[r, "l"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\sigma"] & \cdot \arrow[d, "h_1"] \\ \cdot \arrow[r, "h_3"'] & \cdot \end{tikzcd}$$

By Proposition VII.42 and Proposition VII.44, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_1) \arrow[r, "(h_2 + h_3) + h_1^{-1}"] \arrow[d, "\mathsf{Cyl}(j_1)"'] & a_1 \arrow[d, "j_1"] \\ \mathsf{Cyl}(a) \arrow[r, "p(a)"'] & a \end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_0(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "(h_2) + h_3) + h_1^{-1}"] \\
\mathsf{Cyl}(a_1) \arrow[r, "l"'] & a_1
\end{tikzcd}$$

Since the diagram

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[dr, "gf"'] & \mathsf{Cyl}(a_1) \arrow[d, "l"] \\
& a_1
\end{tikzcd}$$

in $\mathcal{A}$ commutes, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[dr, "gf"'] & \mathsf{Cyl}(a_1) \arrow[d, "(h_2 + h_3) + h_1^{-1}"] \\
& a_1
\end{tikzcd}$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_1(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "(h_2) + h_3) + h_1^{-1}"] \\
\mathsf{Cyl}(a_1) \arrow[r, "l"'] & a_1
\end{tikzcd}$$

Since the diagram

$$\begin{tikzcd}
a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_1) \arrow[d, "l"] \\
& a_1
\end{tikzcd}$$

in $\mathcal{A}$ commutes, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_1(a_1)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_1) \arrow[d, "(h_2 + h_3) + h_1^{-1}"] \\
& a_1
\end{tikzcd}$$

Putting everything together, we have that $(h_2 + h_3) + h_1^{-1}$ defines a homotopy over a from gf to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_1, j_1). $\square$

Proposition XI.6. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v *compatible with* p*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*, and an upper left connection structure* Γ_{ul}*. Suppose that* Cyl *preserves subdivision with respect to* $\underline{\mathsf{Cyl}}$*.*

Let

$$a \xrightarrow{j_0} a_0$$

and

$$a \xrightarrow{j_1} a_1$$

be arrows of $\mathcal{A}$ *which are fibrations with respect to* $\underline{\mathsf{Cyl}}$*.*

Let

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ *be an arrow of* $\mathcal{A}$*, such that the diagram*

$$\begin{tikzcd}
a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\
a_1 \arrow[r, "j_1"'] & a
\end{tikzcd}$$

in $\mathcal{A}$ *commutes, and such that* f *is a homotopy equivalence with respect to* $\underline{\mathsf{Cyl}}$*.*

Then f *is, moreover, a homotopy equivalence over* a *with respect to* $\underline{\mathsf{Cyl}}$ *and* (j_0, j_1)*.*

Proof. By Lemma XI.5, there is an arrow

$$a_1 \xrightarrow{g} a_0$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_1 \arrow[d, "g"'] \arrow[dr, "j_1"] & \\ a_0 \arrow[r, "j_0"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, together with a homotopy

$$\mathsf{Cyl}(a_1) \xrightarrow{h} a_1$$

over a from fg to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_1, j_1). It remains to construct a homotopy over a from gf to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_0).

By Lemma VII.23, we have that g is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$. Thus g satisfies the hypotheses of Lemma XI.5. We deduce that there is an arrow

$$a_0 \xrightarrow{g'} a_1$$

of $\mathcal{A}$, such that the diagram

$$\begin{tikzcd} a_0 \arrow[d, "g'"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, and such that there is a homotopy

$$\mathsf{Cyl}(a_0) \xrightarrow{h'} a_0$$

over a from gg' to $id(a_0)$ with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_0).

By Corollary VII.43, we have that $(h')^{-1}$ defines a homotopy over a from $id(a_0)$ to gg' with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_0). Thus, by Lemma VII.62, the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{(h')^{-1} \circ \mathsf{Cyl}(gf)} a_1$$

of $\mathcal{A}$ defines a homotopy over a from gf to $gfgg'$ with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_0).

Appealing again to Lemma VII.62, we also have that the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{g \circ h \circ \mathsf{Cyl}(g')} a_1$$

of $\mathcal{A}$ defines a homotopy over a from $gfgg'$ to gg' with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_0).

By virtue of Corollary VII.45, we have that the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{\left((h')^{-1} \circ \mathsf{Cyl}(gf)\right) + \left(g \circ h \circ \mathsf{Cyl}(g')\right)} a_1$$

of $\mathcal{A}$ defines a homotopy over a from gf to gg' with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_0). Let us denote it by k for brevity. Appealing to Corollary VII.45 once more, we have that the arrow

$$\mathsf{Cyl}(a_1) \xrightarrow{k + h'} a_1$$

of $\mathcal{A}$ defines a homotopy over a from gf to $id(a_0)$. □

Corollary XI.7. *Let* $\underline{\text{co-}\mathsf{Cyl}} = (\text{co-}\mathsf{Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*, an involution structure* v *compatible with* c*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* c*, and an upper left connection structure* Γ_{ul}*. Suppose that* co-Cyl *preserves subdivision with respect to* $\underline{\text{co-}\mathsf{Cyl}}$*.*

Let

$$a \xrightarrow{j_0} a_0$$

and

$$a \xrightarrow{j_1} a_1$$

be arrows of $\mathcal{A}$ *which are cofibrations with respect to* $\underline{\text{co-}\mathsf{Cyl}}$*.*

Let

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ *be an arrow of* $\mathcal{A}$*, such that the diagram*

$$\begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f"] \\ & a_1 \end{tikzcd}$$

in $\mathcal{A}$ *commutes, and such that* f *is a homotopy equivalence with respect to* $\underline{\text{co-}\mathsf{Cyl}}$*.*

Then f *is, moreover, a homotopy equivalence under* a *with respect to* $\underline{\text{co-}\mathsf{Cyl}}$ *and* (j_0, j_1)*.*

Proof. Follows immediately from Proposition XI.6 by duality. □

Corollary XI.8. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*. Let* $\underline{\text{co-}\mathsf{Cyl}} = (\text{co-}\mathsf{Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*, an involution structure* v *compatible with* c*, a* *subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* c*, and an upper left connection structure* Γ_{ul}*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Let

$$a \xrightarrow{j_0} a_0$$

and

$$a \xrightarrow{j_1} a_1$$

be arrows of $\mathcal{A}$ *which are cofibrations with respect to* $\underline{\mathsf{Cyl}}$*.*

Let

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ *be an arrow of* $\mathcal{A}$*, such that the diagram*

$$\begin{tikzcd} a \arrow[r, "j_0"] \arrow[dr, "j_1"'] & a_0 \arrow[d, "f"] \\ & a_1 \end{tikzcd}$$

in $\mathcal{A}$ *commutes, and such that* f *is a homotopy equivalence with respect to* $\underline{\mathsf{Cyl}}$*. Then* f *is, moreover, a homotopy equivalence under* a *with respect to* $\underline{\mathsf{Cyl}}$ *and* (j_0, j_1)*.*

Proof. Since $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, we have that $\underline{\mathsf{co\text{-}Cyl}}$ preserves subdivision. By Proposition VIII.10, we have that j_0 and j_1 are cofibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$. By Proposition VII.17, we have that f is a homotopy equivalence with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

We deduce, by Corollary XI.7, that f is, moreover, a homotopy equivalence under a with respect to $\underline{\mathsf{co\text{-}Cyl}}$ and (j_0, j_1). Hence, by Corollary VII.40, f is a homotopy equivalence under a with respect to $\underline{\mathsf{Cyl}}$ and (j_0, j_1). $\square$

Remark XI.9. Assuming that $\underline{\mathsf{Cyl}}$ preserves subdivision, but not necessarily that it is left adjoint to a co-cylinder, it is possible to prove Corollary XI.8 directly for a cylinder $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ equipped with a contraction structure p, an involution structure v compatible with p, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$ compatible with p, and an upper left connection structure Γ_{ul}.

For this, we need that if an arrow j of $\mathcal{A}$ is a cofibration, then so is $\mathsf{Cyl}(j)$. This can be proven, and is the approach taken in the book [25] of May and the book [23] of Kamps and Porter.

However, the proof relies upon the assumption that $\underline{\mathsf{Cyl}}$ admits a *transposition* structure, namely a 2-arrow

$$\mathsf{Cyl}^2 \xrightarrow{t} \mathsf{Cyl}^2$$

of $\mathcal{C}$ such that the following diagrams in $\underline{\mathsf{Hom}}_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ commute.

$$\begin{tikzcd} \mathsf{Cyl} \arrow[r, "i_0 \cdot \mathsf{Cyl}"] \arrow[dr, "\mathsf{Cyl} \cdot i_0"'] & \mathsf{Cyl}^2 \arrow[d, "t"] \\ & \mathsf{Cyl}^2 \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Cyl} \arrow[r, "\mathsf{Cyl} \cdot i_0"] \arrow[dr, "i_0 \cdot \mathsf{Cyl}"'] & \mathsf{Cyl}^2 \arrow[d, "t"] \\ & \mathsf{Cyl}^2 \end{tikzcd}$$

We might also require that the two analogous diagrams involving i_1 commute, but this is not necessary for the proof of Dold's theorem.

For example, suppose that we are working in the 2-category of categories, and that $\underline{\mathsf{Cyl}}$ arises from an interval $\widehat{I}$ in a braided monoidal category $\mathcal{A}$. Then the arrow

$$I^2 \longrightarrow I^2$$

of $\mathcal{A}$ which defines the braiding gives a transposition structure with respect to $\underline{\mathsf{Cyl}}$.

We shall not need a transposition structure anywhere else in this work. Moreover, it will later be indispensable for us to assume that we have an adjoint cylinder and co-cylinder. For these reasons, we have chosen to give a different proof.

Corollary XI.10. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v *compatible with* p*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s,)$ *compatible with* p*, and an upper left connection structure* Γ_{ul}*. Let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Let

$$a \xrightarrow{j_0} a_0$$

and

$$a \xrightarrow{j_1} a_1$$

be arrows of $\mathcal{A}$ *which are fibrations with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

Let

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ *be an arrow of* $\mathcal{A}$*, such that the diagram*

$$\begin{tikzcd} a_0 \arrow[d, "f"'] \arrow[dr, "j_0"] & \\ a_1 \arrow[r, "j_1"'] & a \end{tikzcd}$$

in $\mathcal{A}$ commutes, and such that f is a homotopy equivalence with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Then f is, moreover, a homotopy equivalence over a with respect to $\underline{\mathsf{co\text{-}Cyl}}$ and (j_0, j_1).

Proof. Follows immediately from Corollary XI.8 by duality. □

Corollary XI.11. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, an involution structure v compatible with c, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$ compatible with p, and an upper left connection structure Γ_{ul}. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.*

Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ which is a trivial cofibration with respect to $\underline{\mathsf{Cyl}}$. Then j admits a strong deformation retraction with respect to $\underline{\mathsf{Cyl}}$.

Proof. By Proposition VIII.11, $id(a_0)$ is a cofibration with respect to $\underline{\mathsf{Cyl}}$. Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "id"] \arrow[dr, "j"'] & a_0 \arrow[d, "j"] \\ & a_1 \end{tikzcd}$$

Thus, by Corollary XI.8, j defines a homotopy equivalence under a_0 with respect to $\underline{\mathsf{Cyl}}$ and $(id(a_0), j)$. This means exactly that there is an arrow

$$a_1 \xrightarrow{j'} a_0$$

of $\mathcal{A}$ which is a strong deformation retraction of j with respect to $\underline{\mathsf{Cyl}}$. □

Corollary XI.12. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, an involution structure v compatible with p, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$ compatible with p, and an upper left connection structure Γ_{ul}. Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.*

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ which is a trivial fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$. Then there is an arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ such that f is a strong deformation retraction of j with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Corollary XI.11 by duality. $\square$

XII. Lifting axioms

Let $\underline{\mathsf{Cyl}}$ be a cylinder in a formal category $\mathcal{A}$, and let $\underline{\mathsf{co\text{-}Cyl}}$ be a co-cylinder in $\mathcal{A}$. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$. We prove that if an arrow j of $\mathcal{A}$ is a cofibration, then the canonical arrow $m_j^{\underline{\mathsf{Cyl}}}$ of $\mathcal{A}$ defined in IX admits a strong deformation retraction. For this we assume, for the first time, that our cylinder is equipped with upper and lower right connection structures, which we require to be compatible with subdivision.

For topological spaces, the fact that $m_j^{\underline{\mathsf{Cyl}}}$ admits a strong deformation retraction if j is a cofibration is due to Strøm, proven in §2 of the paper [32]. Strøm's argument is, however, quite different to ours. It relies on the fact that the homotopy theory of topological spaces is defined with respect to a cartesian monoidal structure.

Next, we prove that a normally cloven fibration has the right lifting property with respect to arrows admitting a strong deformation retraction. We deduce that normally cloven fibrations have the covering homotopy extension property with respect to cofibrations.

In a similar way, we prove that trivial normally cloven fibrations have the right lifting property with respect to cofibrations. Dualising, we deduce that fibrations have the right lifting property with respect to trivial normally cloven cofibrations, and that trivial fibrations have the right lifting property with respect to normally cloven cofibrations.

Assumption XII.1. Let $\mathcal{C}$ be a 2-category with a final object. Suppose that pushouts and pullbacks of 2-arrows of $\mathcal{C}$ give rise to pushouts and pullbacks in formal categories, in the sense of Definition II.14. Let $\mathcal{A}$ be an object of $\mathcal{C}$. As before, we view $\mathcal{A}$ as a formal category, writing of objects and arrows of $\mathcal{A}$.

Proposition XII.2. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr}, \Gamma_{ur})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*, a lower right connection structure* Γ_{lr}*, and an upper right connection structure* Γ_{ur}*. Suppose that* Γ_{lr} *and* Γ_{ur} *are compatible with the subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and that* Cyl *preserves mapping cylinders with respect to* $\underline{\mathsf{Cyl}}$.

Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ *which is a cofibration with respect to* $\underline{\mathsf{Cyl}}$*, and suppose that* $(a_j^{\underline{\mathsf{Cyl}}}, d_j^0, d_j^1)$ *defines a mapping cylinder of* j *with respect to* $\underline{\mathsf{Cyl}}$*. Let*

$$a_j^{\underline{\mathsf{Cyl}}} \xrightarrow{m_j^{\underline{\mathsf{Cyl}}}} \mathsf{Cyl}(a_1)$$

denote the corresponding canonical arrow of $\mathcal{A}$ of Notation IX.9. Let

$$\mathsf{Cyl}(a_1) \xrightarrow{r_{\overline{j}}^{\mathsf{Cyl}}} a_{\overline{j}}^{\mathsf{Cyl}}$$

be the arrow of $\mathcal{A}$ of Proposition IX.11. Then $r_{\overline{j}}^{\mathsf{Cyl}}$ is a strong deformation retraction of $m_{\overline{j}}^{\mathsf{Cyl}}$ with respect to $\underline{\mathsf{Cyl}}$.

Proof. By Proposition IX.11, we have that $r_{\overline{j}}^{\mathsf{Cyl}}$ is a retraction of $m_{\overline{j}}^{\mathsf{Cyl}}$. It remains to prove that there is a homotopy

$$\mathsf{Cyl}^2(a_1) \xrightarrow{\sigma} \mathsf{Cyl}(a_1)$$

from $m_{\overline{j}}^{\mathsf{Cyl}} \circ r_{\overline{j}}^{\mathsf{Cyl}}$ to $id(\mathsf{Cyl}(a_1))$ with respect to $\underline{\mathsf{Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_{\overline{j}}^{\mathsf{Cyl}}) \arrow[r, "p(a_{\overline{j}}^{\mathsf{Cyl}})"] \arrow[d, "\mathsf{Cyl}(m_{\overline{j}}^{\mathsf{Cyl}})"'] & a_{\overline{j}}^{\mathsf{Cyl}} \arrow[d, "m_{\overline{j}}^{\mathsf{Cyl}}"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "\sigma"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Let us construct σ. We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_0(a_1)"'] \arrow[dr, "d_j^1"] & \mathsf{Cyl}(a_1) \arrow[d, "r_{\overline{j}}^{\mathsf{Cyl}}"] \\
\mathsf{Cyl}(a_1) & a_{\overline{j}}^{\mathsf{Cyl}} \arrow[l, "m_{\overline{j}}^{\mathsf{Cyl}}"]
\end{tikzcd}$$

By definition of Γ_{ur} as an upper right connection structure, we also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ur}(a_1)"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}} \circ \Gamma_{ur}(a_1)"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

By definition of Γ_{lr} as a lower right connection structure, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ur}(a_1)"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[d, "i_1(\mathsf{Cyl}(a_1))"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{lr}(a_1)"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}} \circ \Gamma_{ur}(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

We define σ to be the arrow

$$\mathsf{Cyl}^2(a_1) \xrightarrow{(m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}} \circ \Gamma_{ur}(a_1)) + \Gamma_{lr}(a_1)} \mathsf{Cyl}(a_1)$$

of $\mathcal{A}$.

Let us first prove that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[dr, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}}"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\
 & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

By definition of Γ_{ur} as an upper right connection structure, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[dr, "id"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ur}(a_1)"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[dr, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}}"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}} \circ \Gamma_{ur}(a_1)"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[d, "i_0(\mathsf{Cyl}(a_1))"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}} \circ \Gamma_{ur}(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[dr, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}}"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Next, let us prove that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[dr, "id"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

By definition of Γ_{lr} as a lower right connection structure, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[dr, "id"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{lr}(a_1)"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[d, "i_1(\mathsf{Cyl}(a_1))"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\sigma"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "\Gamma_{lr}(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes, as required.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_1(\mathsf{Cyl}(a_1))"] \arrow[dr, "id"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{lr}(a_1)"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Let us now prove that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_{\overline{j}}^{\mathsf{Cyl}}) \arrow[r, "p(a_{\overline{j}}^{\mathsf{Cyl}})"] \arrow[d, "\mathsf{Cyl}(m_{\overline{j}}^{\mathsf{Cyl}})"'] & a_{\overline{j}}^{\mathsf{Cyl}} \arrow[d, "m_{\overline{j}}^{\mathsf{Cyl}}"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "\sigma"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Let

$$
\mathsf{S}(\mathsf{Cyl}(a_1)) \xrightarrow{u} \mathsf{Cyl}(a_1)
$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "i_0(\mathsf{Cyl}(a_1))"] \arrow[d, swap, "i_1(\mathsf{Cyl}(a_1))"] & \mathsf{Cyl}^2(a_1) \arrow[d, "r_0(\mathsf{Cyl}(a_1))"] \arrow[rdd, bend left, "\Gamma_{lr}(a_1)"] & \\
\mathsf{Cyl}^2(a_1) \arrow[r, swap, "r_1(\mathsf{Cyl}(a_1))"] \arrow[rrd, bend right, swap, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}} \circ \Gamma_{ur}(a_1)"] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[rd, "u"] & \\
 & & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Let

$$
\mathsf{S} \xrightarrow{\overline{p}} \mathsf{id}
$$

denote the canonical 2-arrow of $\mathcal{C}$ of Definition III.13. We claim that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "s(a_1)"] \arrow[d, swap, "p(a_1)"] & \mathsf{S}(a_1) \arrow[d, "u \circ \mathsf{S}(i_0(a_1))"] \\
a_1 \arrow[r, swap, "i_0(a_1)"] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

By definition of $\Gamma_{lr}(a_1)$ as a lower right connection structure, the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[d, swap, "p(a_1)"] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ur}(a_1)"] \\
a_1 \arrow[r, swap, "i_0(a_1)"] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_0(a_1)"] \arrow[d, "\mathsf{Cyl}(i_0(a_1))"'] & \mathsf{S}(a_1) \arrow[d, "\mathsf{S}(i_0(a_1))"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "r_0(\mathsf{Cyl}(a_1))"] \arrow[dr, "\Gamma_{lr}(a_1)"'] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[d, "u"] \\
 & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_0(a_1)"] \arrow[d, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "u \circ \mathsf{S}(i_0(a_1))"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

By definition of $\Gamma_{ur}(a_1)$ as an upper right connection structure, the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{ur}(a_1)"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Thus we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[rr, "r_1(a_1)"] \arrow[d, "p(a_1)"'] \arrow[dr, "\mathsf{Cyl}(i_0(a_1))"] & & \mathsf{S}(a_1) \arrow[d, "\mathsf{S}(i_0(a_1))"] \\
a_1 \arrow[d, "i_0(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[r, "r_1(\mathsf{Cyl}(a_1))"] \arrow[dl, "\Gamma_{ur}(a_1)"] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}(a_1) \arrow[rr, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}}"'] & & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Earlier in the proof, we observed that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[dr, "i_0(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "m_j^{\mathsf{Cyl}} \circ r_j^{\mathsf{Cyl}}"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"] \arrow[d, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "u \circ \mathsf{S}(i_0(a_1))"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

We have now shown that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "i_0(a_1) \circ p(a_1)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "i_0(a_1) \circ p(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "u \circ \mathsf{S}(i_0(a_1))"'] & \\
& & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

By definition of $\overline{p}$, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "p(a_1)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "\overline{p}(a_1)"'] & \\
& & a_1
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[d, "i_1(a_1)"'] & \mathsf{Cyl}(a_1) \arrow[d, "r_0(a_1)"] \arrow[ddr, bend left, "i_0(a_1) \circ p(a_1)"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "r_1(a_1)"'] \arrow[drr, bend right, "i_0(a_1) \circ p(a_1)"'] & \mathsf{S}(a_1) \arrow[dr, "i_0(a_1) \circ \overline{p}(a_1)"'] & \\
 & & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Appealing to the universal property of $\mathsf{S}(a_1)$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{S}(a_1) \arrow[r, "\mathsf{S}(i_0(a_1))"] \arrow[d, "\overline{p}(a_1)"'] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[d, "u"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Since the subdivision structure $(\mathsf{S}, r_0, r_1, s)$ is compatible with p, we also have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "s(a_1)"] \arrow[dr, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "\overline{p}(a_1)"] \\
 & a_1
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes, completing the proof of the claim.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "s(a_1)"] \arrow[d, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "u \circ \mathsf{S}(i_0(a_1))"] \arrow[dl, "\overline{p}(a_1)"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

We also claim that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "s(\mathsf{Cyl}(a_0))"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[d, "u \circ \mathsf{S}(\mathsf{Cyl}(j))"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[rr, "r_1(\mathsf{Cyl}(a_0))"] \arrow[d, "\Gamma_{ur}(a_0)"'] \arrow[dr, "\mathsf{Cyl}^2(j)"] & & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[d, "\mathsf{S}(\mathsf{Cyl}(j))"] \\
\mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}^2(a_1) \arrow[r, "r_1(\mathsf{Cyl}(a_1))"] \arrow[dl, "\Gamma_{ur}(a_1)"] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}(a_1) \arrow[rr, "m^{\underline{\mathsf{Cyl}}}_{j} \circ r^{\underline{\mathsf{Cyl}}}_{j}"'] & & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"] \arrow[d, "\mathsf{Cyl}(j)"'] \arrow[dr, "d^0_j"'] & \mathsf{Cyl}(a_1) \arrow[d, "r^{\underline{\mathsf{Cyl}}}_{j}"] \\
\mathsf{Cyl}(a_1) & a^{\underline{\mathsf{Cyl}}}_{j} \arrow[l, "m^{\underline{\mathsf{Cyl}}}_{j}"]
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "r_1(\mathsf{Cyl}(a_0))"] \arrow[d, "\Gamma_{ur}(a_0)"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[d, "u \circ \mathsf{S}(\mathsf{Cyl}(j))"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "r_0(\mathsf{Cyl}(a_0))"] \arrow[d, "\mathsf{Cyl}^2(j)"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[d, "\mathsf{S}(\mathsf{Cyl}(j))"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "r_0(\mathsf{Cyl}(a_1))"] \arrow[dr, "\Gamma_{lr}(a_1)"'] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[d, "u"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

The following diagram in $\mathcal{A}$ also commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}^2(j)"] \arrow[d, "\Gamma_{lr}(a_0)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "\Gamma_{lr}(a_1)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "r_0(\mathsf{Cyl}(a_0))"] \arrow[d, "\Gamma_{lr}(a_0)"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[d, "u \circ \mathsf{S}(\mathsf{Cyl}(j))"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

We have now shown that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "i_0(\mathsf{Cyl}(a_0))"] \arrow[d, "i_1(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "r_0(\mathsf{Cyl}(a_0))"] \arrow[ddr, bend left, "\mathsf{Cyl}(j) \circ \Gamma_{lr}(a_0)"] & \\
\mathsf{Cyl}^2(a_0) \arrow[r, "r_1(\mathsf{Cyl}(a_0))"'] \arrow[drr, bend right, "\mathsf{Cyl}(j) \circ \Gamma_{ur}(a_0)"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[dr, "u \circ \mathsf{S}(\mathsf{Cyl}(j))"'] & \\
& & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Let

$$\mathsf{S} \circ \mathsf{Cyl} \xrightarrow{x} \mathsf{Cyl}$$

denote the canonical 2-arrow of $\mathcal{C}$ of Definition III.27. By definition of x, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "i_0(\mathsf{Cyl}(a_0))"] \arrow[d, "i_1(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "r_0(\mathsf{Cyl}(a_0))"] \arrow[ddr, bend left, "\Gamma_{lr}(a_0)"] & \\
\mathsf{Cyl}^2(a_0) \arrow[r, "r_1(\mathsf{Cyl}(a_0))"'] \arrow[drr, bend right, "\Gamma_{ur}(a_0)"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[dr, "x(a_0)"'] & \\
& & \mathsf{Cyl}(a_0)
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "i_0(\mathsf{Cyl}(a_0))"] \arrow[d, "i_1(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "r_0(\mathsf{Cyl}(a_0))"] \arrow[ddr, bend left, "\mathsf{Cyl}(j) \circ \Gamma_{lr}(a_0)"] & \\
\mathsf{Cyl}^2(a_0) \arrow[r, "r_1(\mathsf{Cyl}(a_0))"'] \arrow[drr, bend right, "\mathsf{Cyl}(j) \circ \Gamma_{ur}(a_0)"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[dr, "\mathsf{Cyl}(j) \circ x(a_0)"'] & \\
& & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Appealing to the universal property of $\mathsf{S}(\mathsf{Cyl}(a_0))$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[r, "\mathsf{S}(\mathsf{Cyl}(j))"] \arrow[d, "x(a_0)"'] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[d, "u"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Since Γ_{lr} and Γ_{ur} are compatible with the subdivision structure $(\mathsf{S}, r_0, r_1, s)$, we also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "s(\mathsf{Cyl}(a_0))"] \arrow[dr, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[d, "x(a_0)"] \\
& \mathsf{Cyl}(a_0)
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes, completing the proof of the claim.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "s(\mathsf{Cyl}(a_0))"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[d, "u \circ \mathsf{S}(\mathsf{Cyl}(j))"] \arrow[dl, "x(a_0)"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Next, we claim that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d_j^0)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}(a_j^{\mathsf{Cyl}}) \arrow[d, "\sigma \circ \mathsf{Cyl}(m_j^{\mathsf{Cyl}})"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "d_j^0"] \arrow[dr, "\mathsf{Cyl}(j)"'] & a_j^{\mathsf{Cyl}} \arrow[d, "m_j^{\mathsf{Cyl}}"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d_j^0)"] \arrow[dr, "\mathsf{Cyl}^2(j)"'] & \mathsf{Cyl}(a_j^{\mathsf{Cyl}}) \arrow[d, "\mathsf{Cyl}(m_j^{\mathsf{Cyl}})"] \\
& \mathsf{Cyl}^2(a_1)
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}^2(j)"] \arrow[d, "s(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "s(\mathsf{Cyl}(a_1))"] \\
\mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[r, "\mathsf{S}(\mathsf{Cyl}(j))"'] & \mathsf{S}(\mathsf{Cyl}(a_1))
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d_j^0)"] \arrow[d, "s(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}(a_j^{\mathsf{Cyl}}) \arrow[d, "s(\mathsf{Cyl}(a_1)) \circ \mathsf{Cyl}(m_j^{\mathsf{Cyl}})"] \\
\mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[r, "\mathsf{S}(\mathsf{Cyl}(j))"'] & \mathsf{S}(\mathsf{Cyl}(a_1))
\end{tikzcd}
$$

Earlier in the proof, we established that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "s(\mathsf{Cyl}(a_0))"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{S}(\mathsf{Cyl}(a_0)) \arrow[d, "u \circ \mathsf{S}(\mathsf{Cyl}(j))"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

In addition, by definition of σ, the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_1) \arrow[r, "s(\mathsf{Cyl}(a_1))"] \arrow[dr, "\sigma"'] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[d, "u"] \\
 & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Putting the last three observations together, we have that the following diagram in $\mathcal{A}$ commutes, completing our proof of the claim.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d_j^0)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}(a_j^{\mathsf{Cyl}}) \arrow[d, "\sigma \circ \mathsf{Cyl}(m_j^{\mathsf{Cyl}})"] \\
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Next, we claim that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d^1_j)"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_j) \arrow[d, "\sigma \circ \mathsf{Cyl}(m^{\mathsf{Cyl}}_j)"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "d^1_j"] \arrow[dr, "i_0(a_1)"'] & a^{\mathsf{Cyl}}_j \arrow[d, "m^{\mathsf{Cyl}}_j"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d^1_j)"] \arrow[dr, "\mathsf{Cyl}(i_0(a_1))"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_j) \arrow[d, "\mathsf{Cyl}(m^{\mathsf{Cyl}}_j)"] \\
& \mathsf{Cyl}^2(a_1)
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(i_0(a_1))"] \arrow[d, "s(a_1)"'] & \mathsf{Cyl}^2(a_1) \arrow[d, "s(\mathsf{Cyl}(a_1))"] \\
\mathsf{S}(a_1) \arrow[r, "\mathsf{S}(i_0(a_1))"'] & \mathsf{S}(\mathsf{Cyl}(a_1))
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d^1_j)"] \arrow[d, "s(a_1)"'] & \mathsf{Cyl}(a^{\mathsf{Cyl}}_j) \arrow[d, "s(\mathsf{Cyl}(a_1)) \circ \mathsf{Cyl}(m^{\mathsf{Cyl}}_j)"] \\
a_1 \arrow[r, "\mathsf{S}(i_0(a_1))"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Earlier in the proof, we established that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "s(a_1)"] \arrow[d, "p(a_1)"'] & \mathsf{S}(a_1) \arrow[d, "u \circ \mathsf{S}(i_0(a_1))"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

In addition, by definition of σ, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}^2(a_1) \arrow[r, "s(\mathsf{Cyl}(a_1))"] \arrow[dr, "\sigma"'] & \mathsf{S}(\mathsf{Cyl}(a_1)) \arrow[d, "u"] \\
 & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

Putting the last three observations together, we have that the following diagram in $\mathcal{A}$ commutes, completing the proof of the claim.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_j^1)"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}(a_j^{\mathsf{Cyl}}) \arrow[d, "\sigma \circ \mathsf{Cyl}(m_j^{\mathsf{Cyl}})"] \\
a_1 \arrow[r, "i_0(a_1)"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

We have now shown that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_j^0)"] \arrow[ddr, bend left, "\mathsf{Cyl}(j) \circ p(\mathsf{Cyl}(a_0))"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_j^1)"'] \arrow[drr, bend right, "i_0(a_1) \circ p(a_1)"'] & \mathsf{Cyl}(a_j^{\mathsf{Cyl}}) \arrow[dr, "\sigma \circ \mathsf{Cyl}(m_j^{\mathsf{Cyl}})"'] & \\
 & & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(d_j^0)"] \arrow[d, "p(\mathsf{Cyl}(a_0))"'] & \mathsf{Cyl}(a_j^{\underline{\mathsf{Cyl}}}) \arrow[d, "p(a_j^{\underline{\mathsf{Cyl}}})"] \\
\mathsf{Cyl}(a_0) \arrow[r, "d_j^0"] \arrow[dr, "\mathsf{Cyl}(j)"'] & a_j^{\underline{\mathsf{Cyl}}} \arrow[d, "m_j^{\underline{\mathsf{Cyl}}}"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

The following diagram in $\mathcal{A}$ also commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_j^1)"] \arrow[d, "p(a_1)"'] & \mathsf{Cyl}(a_j^{\underline{\mathsf{Cyl}}}) \arrow[d, "p(a_j^{\underline{\mathsf{Cyl}}})"] \\
a_1 \arrow[r, "d_j^1"] \arrow[dr, "i_0(a_1)"'] & a_j^{\underline{\mathsf{Cyl}}} \arrow[d, "m_j^{\underline{\mathsf{Cyl}}}"] \\
& \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$
\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_j^0)"] \arrow[ddr, bend left, "\mathsf{Cyl}(j) \circ p(\mathsf{Cyl}(a_0))"] & \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_j^1)"'] \arrow[drr, bend right, "i_0(a_1) \circ p(a_1)"'] & \mathsf{Cyl}(a_j^{\underline{\mathsf{Cyl}}}) \arrow[dr, "m_j^{\underline{\mathsf{Cyl}}} \circ p(a_j^{\underline{\mathsf{Cyl}}})"'] & \\
& & \mathsf{Cyl}(a_1)
\end{tikzcd}
$$

Since Cyl preserves mapping cylinders with respect to $\underline{\mathsf{Cyl}}$, the following diagram in $\mathcal{A}$ is co-cartesian.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(i_0(a_0))"] \arrow[d, "\mathsf{Cyl}(j)"'] & \mathsf{Cyl}^2(a_0) \arrow[d, "\mathsf{Cyl}(d_j^0)"] \\
\mathsf{Cyl}(a_1) \arrow[r, "\mathsf{Cyl}(d_j^1)"'] & \mathsf{Cyl}(a_j^{\mathsf{Cyl}})
\end{tikzcd}$$

Appealing to the universal property of $\mathsf{Cyl}(a_j^{\mathsf{Cyl}})$, we deduce that the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd}
\mathsf{Cyl}(a_j^{\mathsf{Cyl}}) \arrow[r, "p(a_j^{\mathsf{Cyl}})"] \arrow[d, "\mathsf{Cyl}(m_j^{\mathsf{Cyl}})"'] & a_j^{\mathsf{Cyl}} \arrow[d, "m_j^{\mathsf{Cyl}}"] \\
\mathsf{Cyl}^2(a_1) \arrow[r, "\sigma"'] & \mathsf{Cyl}(a_1)
\end{tikzcd}$$

$\square$

Proposition XII.3. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*. Let*

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ *which admits a strong deformation retraction*

$$a_1 \xrightarrow{r} a_0$$

with respect to $\underline{\mathsf{Cyl}}$*, and let*

$$a_2 \xrightarrow{f} a_3$$

be an arrow of $\mathcal{A}$ *which is a normally cloven fibration with respect to* $\underline{\mathsf{Cyl}}$*.*

For any arrows

$$a_0 \xrightarrow{g_0} a_2$$

and

$$a_1 \xrightarrow{g_1} a_3$$

of $\mathcal{A}$ *such that the diagram*

$$\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\
a_1 \arrow[r, "g_1"'] & a_3
\end{tikzcd}$$

in $\mathcal{A}$ commutes, there is an arrow

$$a_1 \xrightarrow{l} a_2$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\
a_1 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3
\end{tikzcd}$$

Proof. Since r is a strong deformation retraction of j with respect to $\underline{\mathsf{Cyl}}$, there is a homotopy

$$\mathsf{Cyl}(a_1) \xrightarrow{h} a_1$$

under a_0 from jr to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$. In particular, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "i_0(a_1)"] \arrow[dr, "j \circ r"'] & \mathsf{Cyl}(a_1) \arrow[d, "h"] \\
 & a_1
\end{tikzcd}$$

By assumption, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\
a_1 \arrow[r, "g_1"'] & a_3
\end{tikzcd}$$

Together, the commutativity of these two diagrams implies that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "g_0 \circ r"] \arrow[d, "i_0(a_1)"'] & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_1) \arrow[r, "g_1 \circ h"'] & a_3
\end{tikzcd}$$

For any object a of $\mathcal{A}$, let

$$\Delta^{\mathsf{Cyl}}_{f,a} \xrightarrow{k_a} \Omega^{\mathsf{Cyl}}_{f,a}$$

denote the map of the cleavage with which f is equipped. Let

$$\mathsf{Cyl}(a_1) \xrightarrow{k} a_2$$

denote the arrow $k_{a_1}(g_0 \circ r, g_1 \circ h)$ of $\mathcal{A}$. We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[r, "g_0 \circ r"] \arrow[d, "i_0(a_1)"'] & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_1) \arrow[r, "g_1 \circ h"'] \arrow[ur, "k"'] & a_3
\end{tikzcd}$$

Let

$$a_1 \xrightarrow{l} a_2$$

denote the arrow $k \circ i_1(a_1)$ of $\mathcal{A}$. We claim that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\
a_1 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3
\end{tikzcd}$$

Firstly, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_1 \arrow[rr, "l"] \arrow[dr, "i_1(a_1)"] \arrow[dd, "id"'] & & a_2 \arrow[dd, "f"] \\
& \mathsf{Cyl}(a_1) \arrow[ur, "k"'] \arrow[dl, "h"] & \\
a_1 \arrow[rr, "g_1"'] & & a_3
\end{tikzcd}$$

Thus the triangle

$$\begin{tikzcd}
a_1 \arrow[r, "l"] \arrow[dr, "g_1"'] & a_2 \arrow[d, "f"] \\
& a_3
\end{tikzcd}$$

in $\mathcal{A}$ commutes. It remains to prove the commutativity of the triangle

$$\begin{tikzcd}
a_0 \arrow[r, "j"] \arrow[dr, "g_0"'] & a_1 \arrow[d, "l"] \\
& a_2
\end{tikzcd}$$

in $\mathcal{A}$.

Let

$$\mathsf{Cyl}(a_0) \xrightarrow{k'} a_2$$

denote the arrow $k_{a_0}(g_0 \circ r \circ j, g_1 \circ h \circ \mathsf{Cyl}(j))$ of $\mathcal{A}$. Since f is a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$, its cleavage satisfies property (ii) of Definition VIII.33. Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"] \arrow[dr, "k'"'] & \mathsf{Cyl}(a_1) \arrow[d, "k"] \\
& a_3
\end{tikzcd}$$

By definition of k', we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g_0 \circ r \circ j"] \arrow[d, "i_0(a_1)"'] & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[ur, "k'"'] \arrow[r, "g_1 \circ h \circ \mathsf{Cyl}(j)"'] & a_3
\end{tikzcd}$$

By the commutativity of the diagram

$$\begin{tikzcd}
a_0 \arrow[r, "j"] \arrow[dr, "id"'] & a_1 \arrow[d, "r"] \\
& a_0
\end{tikzcd}$$

in $\mathcal{A}$, we thus have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "i_0(a_0)"'] & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[ur, "k'"'] \arrow[r, "g_1 \circ h \circ \mathsf{Cyl}(j)"'] & a_3
\end{tikzcd}$$

The following diagram in $\mathcal{A}$ also commutes, by definition of h.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[d, "\mathsf{Cyl}(j)"'] & a_0 \arrow[d, "j"] \\
\mathsf{Cyl}(a_1) \arrow[r, "h"'] & a_1
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, "g_1 \circ h \circ \mathsf{Cyl}(j)"'] & a_0 \arrow[d, "g_1 \circ j"] \\
& a_3
\end{tikzcd}$$

We now have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[rr, "g_0"] \arrow[d, "i_0(a_0)"'] & & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[rru, "k'"] \arrow[rr, "g_1 \circ j \circ p(a_0)"'] & & a_3
\end{tikzcd}$$

Since f is a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$, its cleavage satisfies property (i) of Definition VIII.33. Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "p(a_0)"] \arrow[dr, "k'"'] & a_0 \arrow[d, "g_0"] \\
& a_2
\end{tikzcd}$$

Putting everything together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"] \arrow[d, "p(a_0)"'] \arrow[dr, "k'"] & \mathsf{Cyl}(a_1) \arrow[d, "k"] \\
a_0 \arrow[r, "g_0"'] & a_2
\end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd}
a_0 \arrow[rrr, "j"] \arrow[dr, "i_1(a_0)"] \arrow[dd, "id"'] & & & a_1 \arrow[dl, "i_1(a_1)"'] \arrow[dd, "l"] \\
& \mathsf{Cyl}(a_0) \arrow[r, "\mathsf{Cyl}(j)"] \arrow[dl, "p(a_0)"] & \mathsf{Cyl}(a_1) \arrow[dr, "k"'] & \\
a_0 \arrow[rrr, "g_0"'] & & & a_2
\end{tikzcd}$$

□

Corollary XII.4. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr}, \Gamma_{ur})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*, a lower right connection structure* Γ_{lr}*, and an upper right connection structure* Γ_{ur}*. Suppose that* Γ_{lr} *and* Γ_{ur} *are compatible with the subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and that* Cyl *preserves mapping cylinders with respect to* $\underline{\mathsf{Cyl}}$*.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v', \mathsf{S}', r_0', r_1', s', \Gamma_{ul}')$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*, an involution structure* v' *compatible with* c*, a subdivision* *structure* $(\mathsf{S}', r_0', r_1', s')$ *compatible with* c*, and an upper left connection structure* Γ'_{ul}*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ *which is a trivial cofibration with respect to* $\underline{\mathsf{Cyl}}$*, and let*

$$a_2 \xrightarrow{f} a_3$$

be an arrow of $\mathcal{A}$ *which is a normally cloven fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

For any commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, there is an arrow*

$$a_1 \xrightarrow{l} a_2$$

of $\mathcal{A}$ *such that the following diagram in* $\mathcal{A}$ *commutes.*

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3 \end{tikzcd}$$

Proof. By Corollary XI.11, we have that j admits a strong deformation retraction with respect to $\underline{\mathsf{Cyl}}$. By Proposition VIII.40, we have that f is a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$. Thus we may appeal to Proposition XII.3 for a construction of l. $\square$

Corollary XII.5. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v *compatible with* p*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*, and an upper left connection structure* Γ_{ul}*.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v', \mathsf{S}', r_0', r_1', s', \Gamma'_{lr}, \Gamma'_{ur})$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*, an involution structure* v' *compatible with* c*, a subdivision structure* $(\mathsf{S}', r_0', r_1', s')$ *compatible with* c*, a lower right connection structure* Γ'_{lr}*, and an upper right connection structure* Γ'_{ur}*.*

Suppose that Γ'_{lr} *and* Γ'_{ur} *are compatible with the subdivision structure* $(\mathsf{S}', r'_0, r'_1, s')$*, and that* $\mathsf{co\text{-}Cyl}$ *preserves mapping co-cylinders with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ *which is a normally cloven cofibration with respect to* $\underline{\mathsf{Cyl}}$*, and let*

$$a_2 \xrightarrow{f} a_3$$

be an arrow of $\mathcal{A}$ *which is a trivial fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

For any commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, there is an arrow*

$$a_1 \xrightarrow{l} a_2$$

of $\mathcal{A}$ *such that the following diagram in* $\mathcal{A}$ *commutes.*

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3 \end{tikzcd}$$

Proof. Follows immediately from Corollary XII.4 by duality. ☐

Corollary XII.6. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr}, \Gamma_{ur})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*, a lower right connection structure* Γ_{lr}*, and an upper right connection structure* Γ_{ur}*. Suppose that* Γ_{lr} *and* Γ_{ur} *are compatible with the subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and that* Cyl *preserves mapping cylinders with respect to* $\underline{\mathsf{Cyl}}$*.*

Let f *be an arrow of* $\mathcal{A}$ *which is a normally cloven fibration with respect to* $\underline{\mathsf{Cyl}}$*, and let* j *be an arrow of* $\mathcal{A}$ *which is a cofibration with respect to* $\underline{\mathsf{Cyl}}$*. Then* f *has the covering homotopy extension property with respect to* j *and* $\underline{\mathsf{Cyl}}$*.*

Proof. Follows immediately from Proposition XII.2 and Proposition XII.3. ☐

Corollary XII.7. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul}, \Gamma_{lr}, \Gamma_{ur})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* v *compatible with* p*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*, an upper left connection structure* Γ_{ul}*, a lower right connection structure* Γ_{lr}*, and an upper right connection structure* Γ_{ur}*. Suppose that* Γ_{lr} *and* Γ_{ur} *are compatible with the subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and that* Cyl *preserves mapping cylinders with respect to* $\underline{\mathsf{Cyl}}$*.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ *which is a cofibration with respect to* $\underline{\mathsf{Cyl}}$*, and let*

$$a_2 \xrightarrow{f} a_3$$

be an arrow of $\mathcal{A}$ *which is a trivial normally cloven fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

For any commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, there is an arrow*

$$a_1 \xrightarrow{l} a_2$$

of $\mathcal{A}$ *such that the following diagram in* $\mathcal{A}$ *commutes.*

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "g_1"'] \arrow[ur, "l"] & a_3 \end{tikzcd}$$

Proof. By Proposition VIII.40, we have that f is a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}$. Thus by Corollary XII.6, f has the covering homotopy extension property with respect to j and $\underline{\mathsf{Cyl}}$.

Moreover, by Corollary XI.12, there is an arrow

$$a_3 \xrightarrow{j'} a_2$$

of $\mathcal{A}$ such that f is a strong deformation retraction of j' with respect to $\underline{\mathsf{co\text{-}Cyl}}$. Thus we may appeal to Proposition X.4 for a construction of l. $\square$

Corollary XII.8. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*. Let* $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul}, \Gamma_{lr}, \Gamma_{ur})$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*, an involution structure* v *compatible with* c*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* c*, an upper left connection structure* Γ_{ul}*, a lower right connection structure* Γ_{lr}*, and an upper right connection structure* Γ_{ur}*.*

Suppose that Γ_{lr} *and* Γ_{ur} *are compatible with the subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and that* $\mathsf{co\text{-}Cyl}$ *preserves mapping co-cylinders with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*. Suppose that* $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Let

$$a_0 \xrightarrow{j} a_1$$

be an arrow of $\mathcal{A}$ *which is a trivial normally cloven cofibration with respect to* $\underline{\mathsf{Cyl}}$*, and let*

$$a_2 \xrightarrow{f} a_3$$

be an arrow of $\mathcal{A}$ *which is a fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

For any commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$*, there is an arrow*

$$a_1 \xrightarrow{l} a_2$$

of $\mathcal{A}$ *such that the following diagram in* $\mathcal{A}$ *commutes.*

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3 \end{tikzcd}$$

Proof. Follows immediately from Corollary XII.7 by duality. $\square$

XIII. Factorisation axioms

Let $\underline{\mathsf{Cyl}}$ be a cylinder in a formal category $\mathcal{A}$. In IX, we showed that if $\underline{\mathsf{Cyl}}$ is equipped with certain structures, and has strictness of right identities, then the mapping cylinder with respect to $\underline{\mathsf{Cyl}}$ of an arrow f gives rise to a factorisation into a normally cloven cofibration followed by a strong deformation retraction.

We now prove that, if $\underline{\mathsf{Cyl}}$ has strictness of left identities, then a strong deformation retraction with respect to $\underline{\mathsf{Cyl}}$ is a trivial fibration with respect to $\underline{\mathsf{Cyl}}$. Thus, if $\underline{\mathsf{Cyl}}$ has strictness of both left and right identities, then the mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$ yields a factorisation of f into a normally cloven cofibration followed by a trivial fibration.

Dually, if a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}$ in $\mathcal{A}$ is equipped with sufficient structures and has strictness of identities, the mapping co-cylinder of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$ yields a factorisation of f into a trivial cofibration followed by a normally cloven fibration.

Morever, building upon this, we construct a factorisation of f into a cofibration followed by a trivial normally cloven fibration, and into a trivial normally cloven cofibration followed by a fibration.

Neither the strictness of left identities hypothesis nor the strictness of right identities hypothesis holds with respect to the usual cylinder and co-cylinder in the category of topological spaces. Essentially, this is the observation that by glueing a path g to a constant path, we obtain a path homotopic to g, but not g itself.

Whilst the mapping cylinder of a map between topological spaces gives a factorisation into a cofibration followed by a homotopy equivalence, this homotopy equivalence is not necessarily a fibration. An example is given by the mapping cylinder factorisation of the inclusion of a circle into a disc. Dually, whilst the mapping co-cylinder of a map between topological spaces gives a factorisation into a homotopy equivalence followed by a fibration, this homotopy equivalence is not necessarily a cofibration. An example is given by the mapping co-cylinder factorisation of any inclusion of spaces which is not closed. We refer the reader to I for a discussion of a way around this.

We shall explore in XVI a guiding example in which strictness of identities does hold, namely the homotopy theory of categories or groupoids.

Assumption XIII.1. Let $\mathcal{C}$ be a 2-category with a final object. Suppose that pushouts and pullbacks of 2-arrows of $\mathcal{C}$ give rise to pushouts and pullbacks in formal categories, in the sense of Definition II.14. Let $\mathcal{A}$ be an object of $\mathcal{C}$. As before, we view $\mathcal{A}$ as a formal category, writing of objects and arrows of $\mathcal{A}$.

Proposition XIII.2. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*. Suppose that* $\underline{\mathsf{Cyl}}$ *has strictness of left identities.*

Let

$$a_1 \xrightarrow{j} a_2$$

be an arrow of $\mathcal{A}$, and let

$$a_2 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$ which is a retraction of j. Suppose that

$$\mathsf{Cyl}(a_2) \xrightarrow{h} a_2$$

defines a homotopy over a_1 from $id(a_2)$ to jf with respect to $\underline{\mathsf{Cyl}}$ and (f, f). Then f is a fibration with respect to $\underline{\mathsf{Cyl}}$.

Proof. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows.

$$\begin{tikzcd}
a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_2 \arrow[d, "f"] \\
\mathsf{Cyl}(a_0) \arrow[r, "k"'] & a_1
\end{tikzcd}$$

By definition of h, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[r, "i_1(a_2)"] \arrow[dr, "j \circ f"'] & \mathsf{Cyl}(a_2) \arrow[d, "h"] \\
& a_2
\end{tikzcd}$$

Appealing to the commutativity of the diagram

$$\begin{tikzcd}
a_0 \arrow[r, "i_1(a_0)"] \arrow[d, "g"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(g)"] \\
a_2 \arrow[r, "i_1(a_2)"'] & \mathsf{Cyl}(a_2)
\end{tikzcd}$$

in $\mathcal{A}$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_1(a_0)"] \arrow[dr, "j \circ f \circ g"'] & \mathsf{Cyl}(a_0) \arrow[d, "h \circ \mathsf{Cyl}(g)"] \\ & a_2 \end{tikzcd}$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[dr, "j \circ f \circ g"'] & \mathsf{Cyl}(a_0) \arrow[d, "j \circ k"] \\ & a_2 \end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "j \circ k"] \\ \mathsf{Cyl}(a_0) \arrow[r, "h \circ \mathsf{Cyl}(g)"'] & a_2 \end{tikzcd}$$

Let

$$\mathsf{Cyl}(a_0) \xrightarrow{l} a_2$$

denote the homotopy $(h \circ \mathsf{Cyl}(g)) + (j \circ k)$ with respect to $\underline{\mathsf{Cyl}}$. We claim that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "g"] \arrow[d, "i_0(a_0)"'] & a_2 \arrow[d, "f"] \\ \mathsf{Cyl}(a_0) \arrow[ur, "l"] \arrow[r, "k"'] & a_1 \end{tikzcd}$$

Firstly, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "g"'] & \mathsf{Cyl}(a_0) \arrow[d, "\mathsf{Cyl}(g)"] \\
a_2 \arrow[r, "i_0(a_2)"] \arrow[dr, "id"'] & \mathsf{Cyl}(a_2) \arrow[d, "h"] \\
& a_2
\end{tikzcd}$$

By definition of l, we also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[dr, "h \circ \mathsf{Cyl}(g)"'] & \mathsf{Cyl}(a_0) \arrow[d, "l"] \\
& a_2
\end{tikzcd}$$

Hence the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[dr, "g"'] & \mathsf{Cyl}(a_0) \arrow[d, "l"] \\
& a_2
\end{tikzcd}$$

Secondly, let us prove that the diagram

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "l"] \arrow[dr, "k"'] & a_2 \arrow[d, "f"] \\
& a_1
\end{tikzcd}$$

in $\mathcal{A}$ commutes. Let

$$\mathsf{S}(a_0) \xrightarrow{u} a_2$$

denote the canonical arrow of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "j \circ k"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "h \circ \mathsf{Cyl}(g)"'] & \mathsf{S}(a_0) \arrow[dr, "u"] & \\
 & & a_2
\end{tikzcd}$$

By definition of h, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_1) \arrow[r, "h"] \arrow[d, "\mathsf{Cyl}(f)"'] & a_1 \arrow[d, "f"] \\
\mathsf{Cyl}(a_2) \arrow[r, "p(a_2)"'] & a_2
\end{tikzcd}$$

Appealing to the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"] \arrow[dr, "h \circ \mathsf{Cyl}(g)"'] & \mathsf{S}(a_0) \arrow[d, "u"] \\
 & a_2
\end{tikzcd}$$

in $\mathcal{A}$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "s(a_0)"] \arrow[dr, "p(a_1) \circ \mathsf{Cyl}(f \circ g)"'] & \mathsf{S}(a_0) \arrow[d, "f \circ u"] \\
 & a_1
\end{tikzcd}$$

We also have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[rr, "\mathsf{Cyl}(g)"] \arrow[d, "p(a_0)"'] \arrow[dr, "\mathsf{Cyl}(i_0(a_0))"] & & \mathsf{Cyl}(a_2) \arrow[d, "\mathsf{Cyl}(f)"] \\
a_0 \arrow[d, "i_0(a_0)"'] & \mathsf{Cyl}^2(a_0) \arrow[r, "\mathsf{Cyl}(k)"] \arrow[dl, "p(\mathsf{Cyl}(a_0))"] & \mathsf{Cyl}(a_1) \arrow[d, "p(a_1)"] \\
\mathsf{Cyl}(a_0) \arrow[rr, "k"'] & & a_1
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "s(a_0)"] \arrow[dr, "k \circ i_0(a_0) \circ p(a_0)"'] & \mathsf{S}(a_0) \arrow[d, "f \circ u"] \\
& a_1
\end{tikzcd}$$

Moreover, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{Cyl}(a_0) \arrow[r, "r_0(a_0)"] \arrow[d, "k"'] & \mathsf{S}(a_0) \arrow[d, "u"] \\
a_1 \arrow[r, "j"] \arrow[dr, "id"'] & a_2 \arrow[d, "f"] \\
& a_1
\end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "k"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "k \circ i_0(a_0) \circ p(a_0)"'] & \mathsf{S}(a_0) \arrow[dr, "f \circ u"] & \\
& & a_1
\end{tikzcd}$$

Let

$$\mathsf{S} \xrightarrow{q_l} \mathsf{Cyl}$$

denote the canonical 2-arrow of $\mathcal{C}$ of Definition III.31. We have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "id"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "i_0(a_0) \circ p(a_0)"'] & \mathsf{S}(a_0) \arrow[dr, "q_l(a_0)"'] & \\
& & \mathsf{Cyl}(a_0)
\end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_0 \arrow[r, "i_0(a_0)"] \arrow[d, "i_1(a_0)"'] & \mathsf{Cyl}(a_0) \arrow[d, "r_0(a_0)"] \arrow[ddr, bend left, "k"] & \\
\mathsf{Cyl}(a_0) \arrow[r, "r_1(a_0)"'] \arrow[drr, bend right, "k \circ i_0(a_0) \circ p(a_0)"'] & \mathsf{S}(a_0) \arrow[dr, "k \circ q_l(a_0)"'] & \\
& & a_1
\end{tikzcd}$$

Appealing to the universal property of $\mathsf{S}(a_0)$, we deduce that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
\mathsf{S}(a_0) \arrow[r, "q_l(a_0)"] \arrow[dr, "f \circ u"'] & \mathsf{Cyl}(a_0) \arrow[d, "k"] \\
& a_1
\end{tikzcd}$$

By definition of l, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "s(a_0)"] \arrow[dr, "l"'] & \mathsf{S}(a_0) \arrow[d, "u"] \\ & a_2 \end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "l"] \arrow[dr, "k \circ q_l(a_0) \circ s(a_0)"'] & a_2 \arrow[d, "f"] \\ & a_1 \end{tikzcd}$$

Since $\underline{\mathsf{Cyl}}$ has strictness of left identities, the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "s(a_0)"] \arrow[dr, "id"'] & \mathsf{S}(a_0) \arrow[d, "q_l(a_0)"] \\ & \mathsf{Cyl}(a_0) \end{tikzcd}$$

Putting the last two observations together, we have that the following diagram in $\mathcal{A}$ commutes, as required.

$$\begin{tikzcd} \mathsf{Cyl}(a_0) \arrow[r, "l"] \arrow[dr, "k"'] & a_2 \arrow[d, "f"] \\ & a_1 \end{tikzcd}$$

□

Corollary XIII.3. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, \mathsf{S}, r_0, r_1, s)$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, and a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*. Suppose that* $\underline{\mathsf{Cyl}}$ *has strictness of left identities.*

An arrow

$$a_1 \xrightarrow{f} a_0$$

of $\mathcal{A}$ *is a trivial fibration with respect to* $\underline{\mathsf{Cyl}}$ *if and only if there is an arrow*

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$, such that f is a retraction of j, and such that there exists a homotopy over a_0 from jf to $id(a_1)$ with respect to $\underline{\mathsf{Cyl}}$ and (f,f).

Proof. Follows immediately from Proposition XI.6 and Proposition XIII.2. $\square$

Corollary XIII.4. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, an involution structure v compatible with p, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Suppose that $\underline{\mathsf{Cyl}}$ has strictness of left identities.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

An arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is a trivial fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$ if and only if it is a strong deformation retraction with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition VII.61, Proposition VII.42, and Corollary XIII.3. $\square$

Corollary XIII.5. *Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, an involution structure v compatible with c, and a subdivision structure $(\mathsf{S}, r_0, r_1, s)$. Suppose that $\underline{\mathsf{co\text{-}Cyl}}$ has strictness of left identities.*

Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

An arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$ is a trivial cofibration with respect to $\underline{\mathsf{Cyl}}$ if and only if it admits a strong deformation retraction with respect to $\underline{\mathsf{Cyl}}$.

Proof. Follows immediately from Corollary XIII.4 by duality. $\square$

Corollary XIII.6. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, an involution structure p compatible with p, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, and a lower right connection structure Γ_{lr}. Suppose that Γ_{lr} is compatible with p, and that $\underline{\mathsf{Cyl}}$ has strictness of identities. Suppose moreover that Cyl preserves mapping cylinders with respect to $\underline{\mathsf{Cyl}}$.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c)$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and let $(a_f^{\mathsf{Cyl}}, d_f^0, d_f^1)$ be a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$. Let

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping cylinder factorisation of f. Then j is a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$, and g is a trivial fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Proposition IX.20, Corollary IX.17, and Corollary XIII.4. □

Corollary XIII.7. *Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, an involution structure compatible with c, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, and a lower right connection structure Γ_{lr}. Suppose that Γ_{lr} is compatible with c, and that $\underline{\mathsf{co\text{-}Cyl}}$ has strictness of identities. Suppose moreover that $\mathsf{co\text{-}Cyl}$ preserves mapping co-cylinders with respect to $\underline{\mathsf{co\text{-}Cyl}}$.*

Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p)$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p. Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$, and let $(a_f^{\mathsf{co\text{-}Cyl}}, d_f^0, d_f^1)$ be a mapping co-cylinder of f with respect to $\underline{\mathsf{co\text{-}Cyl}}$. Let

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_f^{\mathsf{co\text{-}Cyl}} \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping co-cylinder factorisation of f.

Then j is a trivial cofibration with respect to $\underline{\mathsf{Cyl}}$, and g is a normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Proof. Follows immediately from Corollary XIII.6 by duality. □

Corollary XIII.8. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ *be a cylinder in* $\mathcal{A}$ *equipped with a contraction structure* p*, an involution structure* p *compatible with* p*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and a lower right connection structure* Γ_{lr}*. Suppose that* Γ_{lr} *is compatible with* p*, and that* $\underline{\mathsf{Cyl}}$ *has strictness of identities. Suppose moreover that* Cyl *preserves mapping cylinders with respect to* $\underline{\mathsf{Cyl}}$*.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \overline{\Gamma_{lr}})$ *be a co-cylinder in* $\mathcal{A}$ *equipped with a contraction structure* c*, an involution structure compatible with* c*, a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$*, and a lower right connection structure* Γ_{lr}*. Suppose that* Γ_{lr} *is compatible with* c*, and that* $\underline{\mathsf{co\text{-}Cyl}}$ *has strictness of identities. Suppose moreover that* $\mathsf{co\text{-}Cyl}$ *preserves mapping co-cylinders with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*.*

Suppose that $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$*. There is an object* a *of* $\mathcal{A}$*, an arrow*

$$a_0 \xrightarrow{j} a$$

of $\mathcal{A}$ *which is a cofibration with respect to* $\underline{\mathsf{Cyl}}$*, and an arrow*

$$a \xrightarrow{g} a_1$$

of $\mathcal{A}$ *which is a trivial normally cloven fibration with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*, such that the following diagram in* $\mathcal{A}$ *commutes.*

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

Proof. Let $(a_f^{\mathsf{Cyl}}, d_f^0, d_f^1)$ be a mapping cylinder of f with respect to $\underline{\mathsf{Cyl}}$. Let

$$\begin{tikzcd} a_0 \arrow[r, "j'"] \arrow[dr, "f"'] & a_f^{\mathsf{Cyl}} \arrow[d, "g'"] \\ & a_1 \end{tikzcd}$$

denote the corresponding mapping cylinder factorisation of f. By Proposition IX.20, we have that j' is a cofibration with respect to $\underline{\mathsf{Cyl}}$.

Let $(a_{g'}^{\mathsf{co\text{-}Cyl}}, d_{g'}^0, d_{g'}^1)$ be a mapping co-cylinder of g' with respect to $\underline{\mathsf{co\text{-}Cyl}}$. Let

$$\begin{tikzcd}
a_f^{\mathsf{Cyl}} \arrow[r, "j''"] \arrow[dr, "g'"'] & a_{g'}^{\mathsf{co\text{-}Cyl}} \arrow[d, "g"] \\
& a_1
\end{tikzcd}$$

denote the corresponding mapping co-cylinder factorisation of g'. Let us take a to be $a_{g'}^{\mathsf{co\text{-}Cyl}}$. By Corollary XIII.7, we have that j'' is a trivial cofibration with respect to $\underline{\mathsf{Cyl}}$, and that g is a normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Since g' and j'' are homotopy equivalences with respect to $\underline{\mathsf{Cyl}}$, it follows from Proposition VII.21 that g is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}$. Hence, by Proposition VII.17, g is a homotopy equivalence with respect to $\underline{\mathsf{co\text{-}Cyl}}$. Thus g is a trivial normally cloven fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Let

$$a_0 \xrightarrow{j} a_{g'}^{\mathsf{co\text{-}Cyl}}$$

denote the arrow $j'' \circ j'$ of $\mathcal{A}$. Since both j' and j'' are cofibrations with respect to $\underline{\mathsf{Cyl}}$, we conclude that j is a cofibration with respect to $\underline{\mathsf{Cyl}}$ by Proposition VIII.12.

□

Corollary XIII.9. *Let $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ be a cylinder in $\mathcal{A}$ equipped with a contraction structure p, an involution structure p compatible with p, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, and a lower right connection structure Γ_{lr}. Suppose that Γ_{lr} is compatible with p, and that $\underline{\mathsf{Cyl}}$ has strictness of identities. Suppose moreover that Cyl preserves mapping cylinders with respect to $\underline{\mathsf{Cyl}}$.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v, \mathsf{S}, r_0, r_1, s, \Gamma_{lr})$ be a co-cylinder in $\mathcal{A}$ equipped with a contraction structure c, an involution structure compatible with c, a subdivision structure $(\mathsf{S}, r_0, r_1, s)$, and a lower right connection structure Γ_{lr}. Suppose that Γ_{lr} is compatible with c, and that $\underline{\mathsf{co\text{-}Cyl}}$ has strictness of identities. Suppose moreover that $\mathsf{co\text{-}Cyl}$ preserves mapping co-cylinders with respect to $\underline{\mathsf{co\text{-}Cyl}}$.

Suppose that $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and that the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with p and c.

Let

$$a_0 \xrightarrow{f} a_1$$

be an arrow of $\mathcal{A}$. There is an object a of $\mathcal{A}$, an arrow

$$a_0 \xrightarrow{j} a$$

of $\mathcal{A}$ which is a trivial normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}$, and an arrow

$$a \xrightarrow{g} a_1$$

of $\mathcal{A}$ which is a fibration with respect to $\underline{\mathsf{co\text{-}Cyl}}$, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

Proof. Follows immediately from Corollary XIII.9 by duality. □

XIV. Model category recollections

In XV, we shall bring together all the theory we have developed so far. In order to do so, we now present a few recollections on model categories.

The notion of a model category was introduced by Quillen in [28]. We recall two definitions, and prove that they are equivalent. Our arguments comprise part of the proof of Proposition 2 of §5 of [28]. Our definitions are equivalent to the definition of a closed model category given in §5 of [28].

Definition XIV.1. Let $\mathcal{A}$ be a category with finite limits and colimits. A *model structure* upon $\mathcal{A}$ consists of three sets W, F, and C of arrows of $\mathcal{A}$, such that the following conditions are satisfied.

(i) If any two of the arrows in a commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[dr, "g_2"'] & a_1 \arrow[d, "g_1"] \\ & a_2 \end{tikzcd}$$

in $\mathcal{A}$ belong to W, so does the third.

(ii) An arrow

$$a_2 \xrightarrow{f} a_3$$

of $\mathcal{A}$ belongs to F if and only if, for every commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ such that j belongs to both W and C, there is an arrow

$$a_2 \xrightarrow{l} a_1$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[ur, "l"'] \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ commutes.

(iii) An arrow

$$a_2 \xrightarrow{f} a_3$$

of $\mathcal{A}$ belongs to both F and W if and only if, for every commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ such that j belongs to C, there is an arrow

$$a_2 \xrightarrow{l} a_1$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[ur, "l"'] \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ commutes.

(iv) An arrow

$$a_0 \xrightarrow{j} a_1$$

of $\mathcal{A}$ belongs to C if and only if, for every commutative diagram

$$
\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\
a_2 \arrow[r, "g_1"'] & a_3
\end{tikzcd}
$$

in $\mathcal{A}$ such that f belongs to both F and W, there is an arrow

$$a_2 \xrightarrow{l} a_1$$

of $\mathcal{A}$ such that the diagram

$$
\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\
a_1 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3
\end{tikzcd}
$$

in $\mathcal{A}$ commutes.

(v) An arrow

$$a_0 \xrightarrow{j} a_1$$

belongs to both C and W if and only if, for every commutative diagram

$$
\begin{tikzcd}
a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\
a_2 \arrow[r, "g_1"'] & a_3
\end{tikzcd}
$$

in $\mathcal{A}$ such that f belongs to F, there is an arrow

$$a_2 \xrightarrow{l} a_1$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[ur, "l"'] \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ commutes.

(vi) For every arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$, there is an arrow

$$a_0 \xrightarrow{j} a_2$$

of $\mathcal{A}$ which belongs to C, and an arrow

$$a_2 \xrightarrow{g} a_1$$

of $\mathcal{A}$ which belongs to W and F, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_2 \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

(vii) For every arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$, there is an arrow

$$a_0 \xrightarrow{j} a_2$$

of $\mathcal{A}$ which belongs to W and C, and an arrow

$$a_2 \xrightarrow{g} a_1$$

of $\mathcal{A}$ which belongs to F, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_2 \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

Definition XIV.2. Let $\mathcal{A}$ be a category with finite limits and colimits. Let W, F, and C be sets of arrows of $\mathcal{A}$ which equip $\mathcal{A}$ with a model structure. We refer to an arrow of $\mathcal{A}$ which belongs to W as a *weak equivalence*, to an arrow of $\mathcal{A}$ which belongs to F as a *fibration*, and to an arrow of $\mathcal{A}$ which belongs to C as a *cofibration*. We refer to an arrow of $\mathcal{A}$ which belongs to both W and C as a *trivial cofibration*, and to an arrow of $\mathcal{A}$ which belongs to both W and F as a *trivial fibration*.

Definition XIV.3. A *model category* is a category $\mathcal{A}$ which has finite limits and colimits, together with a model structure upon $\mathcal{A}$.

Proposition XIV.4. *Let $\mathcal{A}$ be a category with finite limits and colimits. Let W, F, C be sets of arrows of $\mathcal{A}$. Then (W, F, C) equips $\mathcal{A}$ with a model structure if and only if the following conditions are satisfied.*

(i) If any two of the arrows in a commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[dr, "g_2"'] & a_1 \arrow[d, "g_1"] \\ & a_2 \end{tikzcd}$$

in $\mathcal{A}$ belong to W, so does the third.

(ii) Suppose that we have commutative diagrams

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "j'"'] & a_0 \arrow[d, "j"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "j"'] & a_2 \arrow[d, "j'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$, such that r_0 is a retraction of g_0, and such that r_1 is a retraction of g_1. If j belongs to C, then j' belongs to C. If j belongs to both C and W, then j' belongs to both C and W.

(iii) *Suppose that we have commutative diagrams*

$$\begin{tikzcd} a_2 \arrow[r, "g_0"] \arrow[d, "f'"'] & a_0 \arrow[d, "f"] \\ a_3 \arrow[r, "g_1"'] & a_1 \end{tikzcd} \qquad \begin{tikzcd} a_0 \arrow[r, "r_0"] \arrow[d, "f"'] & a_2 \arrow[d, "f'"] \\ a_1 \arrow[r, "r_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ such that r_0 is a retraction of g_0, and such that r_1 is a retraction of g_1. If f belongs to F then f' belongs to F. If f belongs to both F and W then f' belongs to both F and W.

(iv) *For every diagram*

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$, such that j belongs to W and C, and f belongs to F, there is an arrow

$$a_2 \xrightarrow{l} a_1$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ commutes.

(v) *For every diagram*

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$, such that j belongs to C, and f belongs to W and F, there is an arrow

$$a_2 \xrightarrow{l} a_1$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ commutes.

(vi) *For every arrow*

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$, there is an arrow

$$a_0 \xrightarrow{j} a_2$$

of $\mathcal{A}$ which belongs to C, and an arrow

$$a_2 \xrightarrow{g} a_1$$

of $\mathcal{A}$ which belongs to W and F, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_2 \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

(vii) For every arrow

$$a_0 \xrightarrow{f} a_1$$

of $\mathcal{A}$, there is an arrow

$$a_0 \xrightarrow{j} a_2$$

of $\mathcal{A}$ which belongs to W and C, and an arrow

$$a_2 \xrightarrow{g} a_1$$

of $\mathcal{A}$ which belongs to F, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0 \arrow[r, "j"] \arrow[dr, "f"'] & a_2 \arrow[d, "g"] \\ & a_1 \end{tikzcd}$$

Proof. We first prove that if the conditions of Proposition XIV.4 are satisfied, then (W, F, C) equips $\mathcal{A}$ with a model structure. Let us demonstrate that condition (ii) of Definition XIV.1 holds.

Given that condition (iv) of Proposition XIV.4 holds, it suffices to show that if

$$a_2 \xrightarrow{f} a_3$$

is an arrow of $\mathcal{A}$ with the property that, for every commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ such that j belongs to both W and C, there is an arrow

$$a_2 \xrightarrow{l} a_1$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ commutes, then f belongs to F.

To this end, by condition (vii) of Proposition XIV.4, there is an arrow

$$a_2 \xrightarrow{j'} a_4$$

of $\mathcal{A}$ which belongs to both W and C, and an arrow

$$a_4 \xrightarrow{f'} a_3$$

of $\mathcal{A}$ which belongs to F, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_2 \arrow[r, "j'"] \arrow[dr, "f"'] & a_4 \arrow[d, "f'"] \\ & a_3 \end{tikzcd}$$

By assumption, there is an arrow

$$a_4 \xrightarrow{l'} a_2$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_2 \arrow[r, "id"] \arrow[d, "j'"'] & a_2 \arrow[d, "f"] \\ a_4 \arrow[r, "f'"'] \arrow[ur, "l'"'] & a_3 \end{tikzcd}$$

In other words, we have a pair of commutative diagrams in $\mathcal{A}$ as follows such that l' is a retraction of j'.

$$\begin{tikzcd} a_2 \arrow[r, "j'"] \arrow[d, "f"'] & a_4 \arrow[d, "f'"] \\ a_3 \arrow[r, "id"'] & a_3 \end{tikzcd} \qquad \begin{tikzcd} a_4 \arrow[r, "l'"] \arrow[d, "f'"'] & a_2 \arrow[d, "f"] \\ a_3 \arrow[r, "id"'] & a_3 \end{tikzcd}$$

Appealing to condition (iii) of Proposition XIV.4, we deduce that f belongs to F.

Next, let us demonstrate that condition (iii) of Definition XIV.1 holds. Given that condition (v) of Proposition XIV.4 holds, it suffices to show that if

$$a_2 \xrightarrow{f} a_3$$

is now an arrow of $\mathcal{A}$ with the property that, for every commutative diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_1 \arrow[d, "f"] \\ a_2 \arrow[r, "g_1"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ such that j belongs to C, there is an arrow

$$a_2 \xrightarrow{l} a_1$$

of $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} a_0 \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1 \arrow[r, "g_1"'] \arrow[ur, "l"'] & a_3 \end{tikzcd}$$

in $\mathcal{A}$ commutes, then f belongs to both W and F.

To this end, by condition (vi) of Proposition XIV.4, there is an arrow

$$a_2 \xrightarrow{j'} a_4$$

of $\mathcal{A}$ which belongs to C, and an arrow

$$a_4 \xrightarrow{f'} a_3$$

of $\mathcal{A}$ which belongs to both F and W, such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_2 \arrow[r, "j'"] \arrow[dr, "f"'] & a_4 \arrow[d, "f'"] \\ & a_3 \end{tikzcd}$$

By assumption, there is an arrow

$$a_4 \xrightarrow{l'} a_2$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd}
a_2 \arrow[r, "id"] \arrow[d, "j'"'] & a_2 \arrow[d, "f"] \\
a_4 \arrow[r, "f'"'] \arrow[ur, "l'"'] & a_3
\end{tikzcd}$$

In other words, we have a pair of commutative diagrams in $\mathcal{A}$ as follows such that l' is a retraction of j'.

$$\begin{tikzcd}
a_2 \arrow[r, "j'"] \arrow[d, "f"'] & a_4 \arrow[d, "f'"] \\
a_3 \arrow[r, "id"'] & a_3
\end{tikzcd}
\qquad
\begin{tikzcd}
a_4 \arrow[r, "l'"] \arrow[d, "f'"'] & a_2 \arrow[d, "f"] \\
a_3 \arrow[r, "id"'] & a_3
\end{tikzcd}$$

Appealing to condition (iii) of Proposition XIV.4, we deduce that f belongs to both F and W.

That conditions (iv) and (v) of Definition XIV.1 hold, given that conditions (ii), (iv), (v), (vi), and (vii) of Proposition XIV.4 hold, follows formally, by duality, from the two arguments we have already given in this proof.

Conversely, suppose that (W, F, C) equips $\mathcal{A}$ with a model structure. We must demonstrate that conditions (ii) and (iii) of Proposition XIV.4 are satisfied.

Suppose that we have commutative diagrams

$$\begin{tikzcd}
a_2 \arrow[r, "g_0'"] \arrow[d, "f'"'] & a_0 \arrow[d, "f"] \\
a_3 \arrow[r, "g_1'"'] & a_1
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[r, "r_0"] \arrow[d, "f"'] & a_2 \arrow[d, "f'"] \\
a_1 \arrow[r, "r_1"'] & a_3
\end{tikzcd}$$

in $\mathcal{A}$, such that r_0 is a retraction of g_0, such that r_1 is a retraction of g_1, and such that f belongs to F. Suppose that we have a commutative diagram in $\mathcal{A}$ as follows, in which j belongs to both C and W.

$$\begin{tikzcd}
a_0' \arrow[r, "g_0"] \arrow[d, "j"'] & a_2 \arrow[d, "f'"] \\
a_1' \arrow[r, "g_1"'] & a_3'
\end{tikzcd}$$

Then the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0' \arrow[r, "g_0 \circ g_0'"] \arrow[d, "j"'] & a_0 \arrow[d, "f"] \\ a_1' \arrow[r, "g_1 \circ g_1'"'] & a_1 \end{tikzcd}$$

Since f belongs to F, by condition (ii) of Definition XIV.1 there is an arrow

$$a_1' \xrightarrow{l} a_0$$

of $\mathcal{A}$ such that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0' \arrow[r, "g_0 \circ g_0'"] \arrow[d, "j"'] & a_0 \arrow[d, "f"] \\ a_1' \arrow[r, "g_1 \circ g_1'"'] \arrow[ur, "l"'] & a_1 \end{tikzcd}$$

Thus the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0' \arrow[r, "r_0 \circ g_0 \circ g_0'"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1' \arrow[r, "r_1 \circ g_1 \circ g_1'"'] \arrow[ur, "r_0 \circ l"'] & a_3 \end{tikzcd}$$

Since r_0 is a retraction of g_0, and since r_1 is a retraction of g_1, we thus have that the following diagram in $\mathcal{A}$ commutes.

$$\begin{tikzcd} a_0' \arrow[r, "g_0'"] \arrow[d, "j"'] & a_2 \arrow[d, "f"] \\ a_1' \arrow[r, "g_1'"'] \arrow[ur, "r_0 \circ l"'] & a_3 \end{tikzcd}$$

An entirely similar argument, appealing to condition (iii) rather than condition (ii) of Definition XIV.1, proves that if f belongs to both F and W, then f' belongs to both F and W. This completes our proof that condition (ii) of Proposition XIV.4 is satisfied.

That condition (iii) of Proposition XIV.4 is satisfied, given that conditions (iv) and (v) of Definition XIV.1 hold, follows formally, by duality, from the proof we have just given that condition (ii) of Proposition XIV.4 holds.

□

XV. Model structure

Suppose that we have a cylinder $\underline{\mathsf{Cyl}}$ and a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}$ in a category $\mathcal{A}$, such that the following hold:

(i) both $\underline{\mathsf{Cyl}}$ and $\underline{\mathsf{co\text{-}Cyl}}$ are equipped with all the structures we have considered in this work, and have strictness of identities;

(ii) $\underline{\mathsf{Cyl}}$ is left adjoint to $\underline{\mathsf{co\text{-}Cyl}}$, and the adjunction between Cyl and $\mathsf{co\text{-}Cyl}$ is compatible with their respective contraction structures.

We bring all our theory together, to prove that we obtain a model structure upon $\mathcal{A}$ by taking:

(i) weak equivalences to be homotopy equivalences with respect to $\underline{\mathsf{Cyl}}$, or equivalently with respect to $\underline{\mathsf{co\text{-}Cyl}}$,

(ii) fibrations to be fibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$,

(iii) cofibrations to be normally cloven cofibrations with respect to $\underline{\mathsf{Cyl}}$.

Equally, we prove that we obtain a model structure upon $\mathcal{A}$ by taking:

(i) weak equivalences to be homotopy equivalences with respect to $\underline{\mathsf{Cyl}}$, or equivalently with respect to $\underline{\mathsf{co\text{-}Cyl}}$;

(ii) fibrations to be normally cloven fibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$;

(iii) cofibrations to be cofibrations with respect to $\underline{\mathsf{Cyl}}$.

An interval $\widehat{\mathsf{I}}$ with respect to a monoidal structure upon $\mathcal{A}$ gives rise, as in VI, to a cylinder $\underline{\mathsf{Cyl}}(\mathsf{I})$ and a co-cylinder $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$ in $\mathcal{A}$, under certain conditions. In this way, we also obtain two model structures upon $\mathcal{A}$ from an interval $\widehat{\mathsf{I}}$ in a monoidal category, equipped with all the structures we have considered in this work, and satisfying strictness of identities.

Assumption XV.1. Let $\mathcal{A}$ be a category with finite limits and colimits.

Remark XV.2. We make this assumption to be consistent with the definition of a model category which was recalled in XIV. Our work in fact relies only upon the existence of mapping cylinders and mapping co-cylinders in $\mathcal{A}$.

This is a significant difference. Mapping cylinders and mapping co-cylinders exist in the category of chain complexes in any additive category, for example, whereas arbitrary finite limits and colimits do not.

Theorem XV.3. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul}, \Gamma_{lr}, \Gamma_{ur})$ *be a cylinder in* $\mathcal{A}$ *equipped with:*

(i) a contraction structure p*,*

(ii) an involution structure v *compatible with* p*,*

(iii) a subdivision structure $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*,*

(iv) an upper left connection structure Γ_{ul}*,*

(v) a lower right connection structure Γ_{lr} *compatible with* p*,*

(vi) an upper right connection structure Γ_{ur}*.*

Suppose that:

(i) Γ_{lr} *and* Γ_{ur} *are compatible with* $(\mathsf{S}, r_0, r_1, s)$*,*

(ii) Cyl *preserves mapping cylinders with respect to* $\underline{\mathsf{Cyl}}$*,*

(iii) $\underline{\mathsf{Cyl}}$ *has strictness of identities.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v', \mathsf{S}', r'_0, r'_1, s', \Gamma'_{ul}, \Gamma'_{lr})$ *be a co-cylinder in* $\mathcal{A}$ *equipped with:*

(i) a contraction structure c*,*

(ii) an involution structure v' *compatible with* c*,*

(iii) a subdivision structure $(\mathsf{S}', r'_0, r'_1, s')$ *compatible with* c*,*

(iv) an upper left connection structure Γ'_{ul}*,*

(v) a lower right connection structure Γ'_{lr} *compatible with* c*.*

Suppose that:

(i) $\mathsf{co\text{-}Cyl}$ *preserves mapping co-cylinders with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*,*

(ii) $\underline{\mathsf{co\text{-}Cyl}}$ *has strictness of identities.*

Suppose moreover that $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

We obtain a model structure upon $\mathcal{A}$ *by taking:*

(i) weak equivalences to be the homotopy equivalences with respect to $\underline{\mathsf{Cyl}}$*, or equivalently, by Proposition VII.17, to be the homotopy equivalences with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*;*

(ii) fibrations to be the normally cloven fibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}$*;*

(iii) cofibrations to be the cofibrations with respect to $\underline{\mathsf{Cyl}}$*.*

Proof. That the conditions of Proposition XIV.4 hold has been established as follows:

(i) Proposition VII.21,

(ii) Proposition VIII.14 and Corollary VIII.15,

(iii) Corollary VIII.44 and Corollary VIII.45,

(iv) Corollary XII.4,

(v) Corollary XII.7,

(vi) Corollary XIII.8,

(vii) Corollary XIII.7.

□

Theorem XV.4. *Let* $\underline{\mathsf{Cyl}} = (\mathsf{Cyl}, i_0, i_1, p, v, \mathsf{S}, r_0, r_1, s, \Gamma_{ul}, \Gamma_{lr})$ *be a cylinder in* $\mathcal{A}$ *equipped with:*

(i) a contraction structure p*,*

(ii) an involution structure v *compatible with* p*,*

(iii) *a subdivision structure* $(\mathsf{S}, r_0, r_1, s)$ *compatible with* p*,*

(iv) *an upper left connection structure* Γ_{ul}*,*

(v) *a lower right connection structure* Γ_{lr} *compatible with* p*.*

Suppose that:

(i) Cyl *preserves mapping cylinders with respect to* $\underline{\mathsf{Cyl}}$*,*

(ii) $\underline{\mathsf{Cyl}}$ *has strictness of identities.*

Let $\underline{\mathsf{co\text{-}Cyl}} = (\mathsf{co\text{-}Cyl}, e_0, e_1, c, v', \mathsf{S}', r'_0, r'_1, s', \Gamma'_{ul}, \Gamma'_{lr}, \Gamma'_{ur})$ *be a co-cylinder in* $\mathcal{A}$ *equipped with:*

(i) *a contraction structure* c*,*

(ii) *an involution structure* v' *compatible with* c*,*

(iii) *a subdivision structure* $(\mathsf{S}', r'_0, r'_1, s')$ *compatible with* c*,*

(iv) *an upper left connection structure* Γ'_{ul}*,*

(v) *a lower right connection structure* Γ'_{lr} *compatible with* c*,*

(vi) *an upper right connection structure* Γ'_{ur}*.*

Suppose that:

(i) Γ'_{lr} *and* Γ'_{ur} *are compatible with* $(\mathsf{S}', r'_0, r'_1, s')$*.*

(ii) $\mathsf{co\text{-}Cyl}$ *preserves mapping co-cylinders with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*,*

(iii) $\underline{\mathsf{co\text{-}Cyl}}$ *has strictness of identities.*

Suppose moreover that $\underline{\mathsf{Cyl}}$ *is left adjoint to* $\underline{\mathsf{co\text{-}Cyl}}$*, and that the adjunction between* Cyl *and* $\mathsf{co\text{-}Cyl}$ *is compatible with* p *and* c*.*

We obtain a model structure upon $\mathcal{A}$ *by taking:*

(i) *weak equivalences to be the homotopy equivalences with respect to* $\underline{\mathsf{Cyl}}$*, or equivalently, by Proposition VII.17, the homotopy equivalences with respect to* $\underline{\mathsf{co\text{-}Cyl}}$*;*

(ii) fibrations to be the fibrations with respect to $\underline{\textsf{co-Cyl}}$*;*

(iii) cofibrations to be the normally cloven cofibrations with respect to $\underline{\textsf{Cyl}}$.

Proof. That the conditions of Proposition XIV.4 hold has been established as follows:

(i) Proposition VII.21,

(ii) Proposition VIII.42 and Corollary VIII.43,

(iii) Corollary VIII.16 and Corollary VIII.17,

(iv) Corollary XII.8,

(v) Corollary XII.5,

(vi) Corollary XIII.6,

(vii) Corollary XIII.9.

$\square$

Assumption XV.5. Let $\otimes$ be a monoidal structure upon $\mathcal{A}$.

Corollary XV.6. *Let* $\widehat{\mathsf{I}} = (I, i_0, i_1, p, v, S, r_0, r_1, s, \Gamma_{ul}, \Gamma_{lr}, \Gamma_{ur})$ *be an interval in* $\mathcal{A}$ *equipped with:*

(i) a contraction structure p*,*

(ii) an involution structure v *compatible with* p*,*

(iii) a subdivision structure (S, r_0, r_1, s) *compatible with* p*,*

(iv) an upper left connection structure Γ_{ul}*,*

(v) a lower right connection structure Γ_{lr} *compatible with* p*,*

(vi) an upper right connection structure Γ_{ur}*.*

Suppose that:

(i) Γ_{lr} *and* Γ_{ur} *are compatible with* (S, r_0, r_1, s)*,*

(ii) I *and* S *are exponentiable with respect to* $\otimes$*,*

(iii) Requirement VI.15 holds,

(iv) $\widehat{\mathsf{I}}$ *has strictness of identities.*

We obtain a model structure upon $\mathcal{A}$ *by taking:*

(i) weak equivalences to be the homotopy equivalences with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$*, or equivalently, by Proposition VII.17, the homotopy equivalences with respect to* $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$*;*

(ii) fibrations to be the normally cloven fibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$*;*

(iii) cofibrations to be the cofibrations with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$*.*

Proof. Follows immediately from Theorem XV.3 by the observations of VI. □

Corollary XV.7. *Let* $\widehat{\mathsf{I}} = (I, i_0, i_1, p, v, S, r_0, r_1, s, \Gamma_{ul}, \Gamma_{lr}, \Gamma_{ur})$ *be an interval in* $\mathcal{A}$ *equipped with:*

(i) a contraction structure p*,*

(ii) an involution structure v *compatible with* p*,*

(iii) a subdivision structure (S, r_0, r_1, s) *compatible with* p*,*

(iv) an upper left connection structure Γ_{ul}*,*

(v) a lower right connection structure Γ_{lr} *compatible with* p*,*

(vi) an upper right connection structure Γ_{ur}*.*

Suppose that:

(i) Γ_{lr} *and* Γ_{ur} *are compatible with* (S, r_0, r_1, s)*,*

(ii) I *and* S *are exponentiable with respect to* $\otimes$*,*

(iii) Requirement VI.15 holds,

(iv) $\widehat{\mathsf{I}}$ *has strictness of identities.*

We obtain a model structure upon $\mathcal{A}$ by taking:

(i) weak equivalences to be the homotopy equivalences with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$, or equivalently, by Proposition VII.17 the homotopy equivalences with respect to $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$;

(ii) fibrations to be the fibrations with respect to $\underline{\mathsf{co\text{-}Cyl}}(\mathsf{I})$;

(iii) cofibrations to be the normally cloven cofibrations with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$.

Proof. Follows immediately from Theorem XV.4 by the observations of VI. $\square$

XVI. Example — categories and groupoids

We define an interval in the category Cat of categories, equipped with its cartesian monoidal structure. It admits all the structures of VI, and has strictness of identities. By XV, we thus obtain two model structures upon Cat. In a non-constructive setting, both model structures can be proven to coincide with folk model structure.

In the same way, we obtain two model structures upon the category Grpd of groupoids. Again both may be demonstrated, by a non-constructive argument, to coincide with the folk model structure.

The folk model structure on Cat was constructed by Joyal and Tierney in [20]. Independently, a construction was given by Rezk in [29].

The folk model structure on Grpd appeared in the literature earlier. It was first described by Anderson in §5 of [1], and is also discussed in §14.1 of the paper [3] of Bousfield. A detailed construction is given in §6.1 of the article [31] of Strickland, built upon in §3 of the thesis [19] of Hollander.

The folk model structure on groupoids can also be seen to arise as the restriction to groupoids of the model structure on Cat constructed by Thomason in [35]. This is observed, for example, in §1 of the paper [8] of Casacuberta, Golasiński, and Tonks.

In all these works, the non-constructive characterisation of equivalences of categories as functors which are fully faithful and essentially surjective is essential. From this point of view, the folk model structure on Cat or Grpd is akin to the Serre model structure on topological spaces, which was first constructed in §II.3 of [28].

The conceptual approach we have taken is significantly different. Our two model structures are instead akin to the model structure on topological spaces, which was constructed by Strøm in [34]. The fact that we may non-constructively identify the two model structures on categories or groupoids which we construct with the folk model structure might reasonably, we think, be viewed as something of a coincidence.

Notation XVI.1. Let Cat denote the category of categories, and let Grpd denote the category of groupoids. We denote by 1 the final object of Cat and Grpd, the category with a unique object $\bullet$ and a unique arrow.

Remark XVI.2. We shall regard Cat and Grpd as equipped with their cartesian monoidal structures. These monoidal structures are closed, and thus Requirement VI.15 is satisfied.

Notation XVI.3. Let $\mathcal{I}$ denote the free groupoid on the following directed graph.

$$0 \longrightarrow 1$$

Notation XVI.4. Let

$$1 \xrightarrow{i_0} \mathcal{I}$$

denote the unique functor which maps $\bullet$ to 0.

Notation XVI.5. Let

$$1 \xrightarrow{i_1} \mathcal{I}$$

denote the unique functor which maps $\bullet$ to 1.

Notation XVI.6. Let $\widehat{\mathsf{I}}$ denote the interval $(\mathcal{I}, i_0, i_1)$ in Cat or Grpd.

Observation XVI.7. The canonical functor

$$\mathcal{I} \xrightarrow{p} 1$$

equips $\widehat{\mathsf{I}}$ with a contraction structure.

Notation XVI.8. Let

$$\mathcal{I} \xrightarrow{v} \mathcal{I}$$

denote the unique functor which maps the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$ to the arrow

$$1 \longrightarrow 0$$

of $\mathcal{I}$.

Observation XVI.9. The functor v equips $\widehat{\mathsf{I}}$ with an involution structure which is compatible with p.

Notation XVI.10. Let $\mathcal{S}$ denote the free groupoid on the following directed graph.

$$0 \longrightarrow 1 \longrightarrow 2$$

Notation XVI.11. Let

$$\mathcal{I} \xrightarrow{r_0} \mathcal{S}$$

denote the unique functor which maps the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$ to the arrow

$$1 \longrightarrow 2$$

of $\mathcal{S}$.

Notation XVI.12. Let

$$\mathcal{I} \xrightarrow{r_1} \mathcal{S}$$

denote the unique functor which maps the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$ to the arrow

$$0 \longrightarrow 1$$

of $\mathcal{S}$.

Notation XVI.13. Let

$$\mathcal{I} \xrightarrow{s} \mathcal{S}$$

denote the unique functor which maps the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$ to the arrow

$$0 \longrightarrow 2$$

of $\mathcal{S}$.

Observation XVI.14. We have that $(\mathcal{S}, r_0, r_1, s)$ equips $\widehat{\mathsf{I}}$ with a subdivision structure, which is compatible with p.

Observation XVI.15. With respect to the involution structure v and the subdivision structure (S, r_0, r_1, s), the interval $\widehat{\mathsf{I}}$ has strictness of identities and strictness of left inverses.

Observation XVI.16. The groupoid $\mathcal{I}^2 = \mathcal{I} \times \mathcal{I}$ is the unique groupoid with objects and arrows as follows, excluding the four identity arrows.

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[dr] & (1,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1)
\end{tikzcd}
\qquad
\begin{tikzcd}
(0,0) & (1,0) \arrow[l] \\
(0,1) \arrow[u] & (1,1) \arrow[l] \arrow[u] \arrow[ul]
\end{tikzcd}$$

Notation XVI.17. Let

$$\mathcal{I}^2 \xrightarrow{\Gamma_{ul}} \mathcal{I}$$

denote the unique functor with the following properties:

(i) the arrow

$$(0,0) \longrightarrow (1,0)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$.

(ii) the arrow

$$(0,0) \longrightarrow (0,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$.

(iii) the arrow

$$(1,0) \longrightarrow (1,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$1 \longrightarrow 1$$

of $\mathcal{I}$.

(iv) the arrow

$$(0,1) \longrightarrow (1,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$1 \longrightarrow 1$$

of $\mathcal{I}$.

Observation XVI.18. The functor Γ_{ul} equips $\widehat{\mathsf{I}}$ with an upper left connection structure.

Notation XVI.19. Let

$$\mathcal{I}^2 \xrightarrow{\Gamma_{lr}} \mathcal{I}$$

denote the unique functor with the following properties:

(i) the arrow

$$(0,0) \longrightarrow (1,0)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 0$$

of $\mathcal{I}$.

(ii) the arrow

$$(0,0) \longrightarrow (0,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 0$$

of $\mathcal{I}$.

(iii) the arrow

$$(1,0) \longrightarrow (1,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$.

(iv) the arrow

$$(0,1) \longrightarrow (1,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$.

Observation XVI.20. The functor Γ_{lr} equips $\widehat{\mathsf{I}}$ with a lower right connection structure, which is compatible with p.

Notation XVI.21. Let

$$\mathcal{I}^2 \xrightarrow{\Gamma_{ur}} \mathcal{I}$$

denote the unique functor with the following properties:

(i) the arrow

$$(0,0) \longrightarrow (1,0)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 1$$

of $\mathcal{I}$.

(ii) the arrow

$$(0,0) \longrightarrow (0,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 0$$

of $\mathcal{I}$.

(iii) the arrow

$$(1,0) \longrightarrow (1,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$1 \longrightarrow 0$$

of $\mathcal{I}$.

(iv) the arrow

$$(0,1) \longrightarrow (1,1)$$

of $\mathcal{I}^2$ maps to the arrow

$$0 \longrightarrow 0$$

of $\mathcal{I}$.

Observation XVI.22. The functor Γ_{ur} equips $\widehat{\mathsf{I}}$ with an upper right connection structure. We have that Γ_{lr} and Γ_{ur} are compatible with $(\mathcal{S}, r_0, r_1, s)$.

Observation XVI.23. A functor is a homotopy equivalence with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$ if and only if it is an equivalence of categories.

Recollection XVI.24. An *iso-fibration* is a functor

$$\mathcal{A}_0 \xrightarrow{F} \mathcal{A}_1$$

with the property that, for every commutative diagram

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "i_0"'] & \mathcal{A}_0 \arrow[d, "F"] \\ \mathcal{I} \arrow[r, "g"'] & \mathcal{A}_1 \end{tikzcd}$$

in Cat, there is a functor

$$\mathcal{I} \xrightarrow{l} \mathcal{A}_0$$

such that the following diagram in Cat commutes.

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "i_0"'] & \mathcal{A}_0 \arrow[d, "F"] \\ \mathcal{I} \arrow[r, "g"'] \arrow[ur, "l"'] & \mathcal{A}_1 \end{tikzcd}$$

Recollection XVI.25. A *normally cloven iso-fibration* is a functor

$$\mathcal{A}_0 \xrightarrow{F} \mathcal{A}_1$$

with the property that, to every commutative diagram

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "i_0"'] & \mathcal{A}_0 \arrow[d, "F"] \\ \mathcal{I} \arrow[r, "g"'] & \mathcal{A}_1 \end{tikzcd}$$

in Cat, we can associate a functor

$$\mathcal{I} \xrightarrow{l} \mathcal{A}_0$$

such that the following hold.

(i) The diagram

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "i_0"'] & \mathcal{A}_0 \arrow[d, "F"] \\ \mathcal{I} \arrow[r, "g"'] \arrow[ur, "l"'] & \mathcal{A}_1 \end{tikzcd}$$

in Cat commutes.

(ii) If the diagram

$$\begin{tikzcd} \mathcal{I} \arrow[r, "p"] \arrow[dr, "g"'] & 1 \arrow[d, "F(a)"] \\ & \mathcal{A}_1 \end{tikzcd}$$

in Cat commutes, then the diagram

$$\begin{tikzcd} \mathcal{I} \arrow[r, "p"] \arrow[dr, "l"'] & 1 \arrow[d, "a"] \\ & \mathcal{A}_0 \end{tikzcd}$$

in Cat commutes.

Observation XVI.26. A functor

$$\mathcal{A}_0 \xrightarrow{F} \mathcal{A}_1$$

is a fibration with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$ if and only if it is an iso-fibration. This goes back to Proposition 2.1 of the paper [4] of Brown.

Moreover, F is a normally cloven fibration with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$ if and only if it is a normally cloven iso-fibration.

Definition XVI.27. An *iso-cofibration* is a functor

$$\mathcal{A}_0 \xrightarrow{j} \mathcal{A}_1$$

such that j is a cofibration with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$.

Remark XVI.28. Non-constructively, it is possible to characterise an iso-cofibration as a functor which is injective on objects.

Definition XVI.29. A *normally cloven iso-cofibration* is a functor

$$\mathcal{A}_0 \xrightarrow{j} \mathcal{A}_1$$

such that j is a normally cloven cofibration with respect to $\underline{\mathsf{Cyl}}(\mathsf{I})$.

Theorem XVI.30. *We obtain a model structure on* Cat *and* Grpd *by taking:*

(i) weak equivalences to be equivalences of categories,

(ii) fibrations to be iso-fibrations,

(iii) cofibrations to be normally cloven iso-cofibrations.

Proof. Follows immediately from Corollary XV.7. □

Theorem XVI.31. *We obtain a model structure on* Cat *and* Grpd *by taking:*

(i) weak equivalences to be equivalences of categories,

(ii) fibrations to be normally cloven iso-fibrations,

(iii) cofibrations to be iso-cofibrations.

Proof. Follows immediately from Corollary XV.6. □

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