Generell Topologi — Exercise Sheet 2

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Guide

The topic of the questions on this exercise sheet is that of a basis, or more generally a sub-basis, of a topological space.

(1) The two facts which are asked to be proven in Question 3 are important, and are used in later questions. They will give you practise in working with a basis for theoretical purposes.

(2) Question 5 introduces the notion of a sub-basis of a topological space, and will check your understanding of Proposition 2.2 in Lecture 2.

(3) Questions 1, 2, 4, 6, 7, and 9 will all help you to gain familiarity with bases and sub-bases in different settings. Question 7 (b) and Question 9 (b) are probably the most difficult of these, and I encourage you to give them a go.

(4) Question 8 has a somewhat different feel, introducing second-countable topological spaces. Second-countability is an important technical notion — it crops up, for example, in the theory of manifolds. Part (b) especially may be quite challenging.

(5) Question 10 continues our investigation of Alexandroff spaces and pre-orders from Exercise Sheet 1. The question essentially asks to show, making use of our new tool of a basis, that Alexandroff topologies on a set X correspond exactly to pre-orders on X, by means of the constructions we became acquainted with on Exercise Sheet 1.

Questions

1

Question. Let $(X, \mathcal{O})$ be a topological space. Prove that $\mathcal{O}' := \big\{\{x\} \mid x \in X\big\}$ is a basis for $(X, \mathcal{O})$ if and only if $\mathcal{O}$ is the discrete topology on X.

2

Question. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\mathcal{O}'_X$ be a basis for $(X, \mathcal{O}_X)$. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology upon A. Prove that

$$\mathcal{O}'_A := \{A \cap U' \mid U' \in \mathcal{O}'_X\}$$

defines a basis for $(A, \mathcal{O}_A)$.

3

Question.

(a) Let $(X, \mathcal{O})$ be a topological space, and let $\mathcal{O}'$ be a subset of $\mathcal{O}$. Then $\mathcal{O}'$ defines a basis for $\mathcal{O}$ if and only if for every $U \subset X$ which belongs to $\mathcal{O}$ and every $x \in U$ there is a $U' \in \mathcal{O}'$ such that $x \in U'$ and $U' \subset U$.

(b) Let $(X, \mathcal{O})$ be a topological space, and let $\mathcal{O}'$ be a basis for $(X, \mathcal{O})$. Then $U \subset X$ belongs to $\mathcal{O}$ if and only if for every $x \in U$ there is a $U' \in \mathcal{O}'$ such that $x \in U'$ and $U' \subset U$.

4

Question.

(a) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that

$$\mathcal{O}' := \{U \times U' \mid U \in \mathcal{O}_X \text{ and } U' \in \mathcal{O}_Y\}$$

defines a basis for the product topology $\mathcal{O}_{X \times Y}$ upon $X \times Y$.

(b) Find a pair of topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ for which it is not true that $\mathcal{O}'$ as defined in (a) itself defines a topology on $X \times Y$. Find a pair of topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ for which it is true.

5

Definition. Let $(X, \mathcal{O})$ be a topological space. A *sub-basis* for $(X, \mathcal{O})$ is a set $\mathcal{O}'$ of subsets of X belonging to $\mathcal{O}$ such that every subset of X belonging to $\mathcal{O}$ can be obtained as a (possibly infinite) union of finite intersections of subsets of X belonging to $\mathcal{O}'$.

Question. Let X be a set, and let $\mathcal{O}'$ be a set of subsets of X. Let $\mathcal{O}$ be the set of subsets of X which can be obtained as a (possibly infinite) union of finite intersections of subsets of X belonging to $\mathcal{O}'$. Suppose that $X \in \mathcal{O}$. Prove that $\mathcal{O}$ defines a topology on X with sub-basis $\mathcal{O}'$.

We refer to $\mathcal{O}$ as the topology *generated* by $\mathcal{O}'$.

6

Question.

(a) Let $X = \{a, b, c\}$, and let $\mathcal{O}$ denote the topology

$$\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$$

upon X. Which sets of subsets of X define bases of $(X, \mathcal{O})$? Find a sub-basis of $(X, \mathcal{O}_X)$ which consists of three subsets of X, and to which $\{a, b\}$ and $\{b, c\}$ both belong. Does this sub-basis define a basis?

(b) Let $X = \{a, b, c, d\}$. List the subsets of X belonging to the topology $\mathcal{O}_1$ on X generated by $\mathcal{O}'_1 := \big\{\{a\}, \{d\}, \{b, d\}, \{c, d\}\big\}$. Do the same for the topology $\mathcal{O}_2$ on X generated by $\mathcal{O}'_2 := \big\{\{a\}, \{b, c\}, \{c, d\}\big\}$.

We have two ways to generate a topology on a set X from a set $\mathcal{O}'$ of subsets of X, namely as in Proposition 2.2 of the lecture notes and as in Question 5. We can apply Question 5 to an arbitrary $\mathcal{O}'$, but to apply Proposition 2.2 of the lecture notes to $\mathcal{O}'$, we must have that conditions (1) and (2) of Proposition 2.2 are satisfied.

Note that when both Question 5 and Proposition 2.2 can be applied, we obtain the same topology in both cases. The key thing is not to try to apply Proposition 2.2 when the required conditions are not satisfied!

7

Question.

(a) Let $\mathcal{O}' := \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$. Prove that $\mathcal{O}'$ defines a sub-basis for the standard topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$.

(b) Let $\mathcal{O}' := \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$. Let $\mathcal{O}$ denote the topology on $\mathbb{R}$ generated by $\mathcal{O}'$. Prove that $\mathcal{O}'' := \{[a, b) \mid a, b \in \mathbb{R}\}$ defines a basis for $(\mathbb{R}, \mathcal{O})$. Prove that $\mathcal{O}_{\mathbb{R}} \subset \mathcal{O}$. Is it true that $\mathcal{O} = \mathcal{O}_{\mathbb{R}}$? Prove or disprove it!

Remark. The topology $\mathcal{O}$ on $\mathbb{R}$ is known as the *lower limit topology.* The topological space $(\mathbb{R}, \mathcal{O})$ is sometimes known as the *Sorgenfrey line.*

8

Terminology. We will say that a set X is *countable* if there exists an injection

$$X \longrightarrow \mathbb{N}.$$

Otherwise we say that X is *uncountable.*

Definition. A topological space $(X, \mathcal{O})$ is *second-countable* if it admits a basis $\mathcal{O}'$ which is a countable set.

Remark. In particular, any topological space $(X, \mathcal{O})$ such that X is finite is second-countable. Indeed, $\mathcal{O}$ is then finite, and we may take all of $\mathcal{O}$ as a basis for $(X, \mathcal{O})$.

Remark. There is also a notion of a *first-countable* topological space. We will meet it later in the course.

Remark. Recall that $\mathbb{Z}$ is countable, since there is a bijection

$$\mathbb{Z} \longrightarrow \mathbb{N},$$

given for example by

$$z \mapsto \begin{cases} 2z+1 & \text{if } z \geq 0, \\ -z & \text{if } z < 0. \end{cases}$$

Explicitly, the image of this bijection may be described as: $0, 1, -1, 2, -2, \ldots$.

Recall also that $\mathbb{Q}$ is countable. One way to prove this is as follows.

(i) There is a bijection

$$\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

given by $(n, n') \mapsto 2^n 3^{n'}$.

(ii) Since $\mathbb{Z}$ is countable, we deduce from (i) that $\mathbb{Z} \times \mathbb{Z}$ is countable.

(iii) Let

$$\mathcal{P} := \big\{ (z, z') \in \mathbb{Z} \times \mathbb{Z} \mid z \neq 0 \text{ and } \mathrm{hcf}(z, z') = 1 \big\},$$

where $\mathrm{hcf}(z, z')$ denotes the highest common factor of z and z'.

Since the inclusion map

$$\mathcal{P} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$$

is injective, we deduce from (ii) that $\mathcal{P}$ is countable.

(iv) The map

$$\mathbb{Q} \longrightarrow \mathcal{P}$$

which sends $q \in \mathbb{Q}$ to the unique $(z, z') \in \mathcal{P}$ such that $q = \frac{z}{z'}$ is bijective. Thus $\mathbb{Q}$ is second-countable.

Remark. Let us prove that $\mathbb{R}$ is uncountable. Let I denote the unit interval. We will rely crucially on the fact that if we have a set $\{A_n\}_{n \in \mathbb{N}}$ of closed subsets of I such that

$$A_0 \supset A_1 \supset A_2 \supset \dots$$

then $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty. We will prove this in a later Exercise Sheet, after we discussed the notion of a compact topological space and proven that $(I, \mathcal{O}_I)$ is compact.

Let us here assume it. Suppose that

$$\mathbb{R} \xrightarrow{f} \mathbb{N}$$

is an injective map. Let

$$I \xrightarrow{i} \mathbb{R}$$

denote the inclusion map. Since both i and f are injective, we have that

$$I \xrightarrow{f \circ i} \mathbb{N}$$

is injective. Let us denote this map by g.

Inductively, we construct for any $n \in \mathbb{N}$ a subset A_n of I with the following properties.

(1) A_n is a closed subset of $(I, \mathcal{O}_I)$.

(2) $A_n \cap g^{-1}\big(\{0, \dots, n\}\big) = \emptyset$.

(3) $A_{n+1} \subset A_n$.

We may construct a subset A_0 of I which satisfies conditions (1) – (3) by taking the complement in I of any neighbourhood of $g^{-1}(0)$. Suppose that we have constructed A_{n-1}.

Since g is injective, the $g^{-1}(i) \in I$ for $0 \leq i \leq n$ are distinct. Thus we may construct a subset U_n of I which is open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and which has the property that

$$U_n = U_n^0 \sqcup U_n^1 \sqcup \dots \sqcup U_n^n,$$

where U_n^i is a neighbourhood of $g^{-1}(i)$ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Check that you agree that the construction of such a U_n can be carried out!

Let $A = I \setminus U_n$. Then A is closed in $(I, \mathcal{O}_I)$, and $A \cap g^{-1}\big(\{0, \dots, n\}\big) = \emptyset$. Define A_n to be $A \cap A_{n-1}$. Then A_n satisfies conditions (1) – (3) above.

Since A_n is closed in $(I, \mathcal{O}_I)$ for every $n \in \mathbb{N}$, we have by Question 4 (i) of Exercise Sheet 1 that $\bigcap_{n \in \mathbb{N}} A_n$ is closed in $(I, \mathcal{O}_I)$. Since A_n satisfies property (2) for every $n \in \mathbb{N}$, we also have that

$$\begin{aligned} \bigcap_{n \in \mathbb{N}} A_n &= \Big(\bigcap_{n \in \mathbb{N}} A_n \Big) \cap I \\ &= \Big(\bigcap_{n \in \mathbb{N}} A_n \Big) \cap g^{-1}(\mathbb{N}) \\ &= \emptyset. \end{aligned}$$

But since A_n satisfies property (3) for every $n \in \mathbb{N}$, we have that $\emptyset \neq \bigcap_{n \in \mathbb{N}} A_n$, as discussed at the beginning of this proof. Thus we have a contradiction.

Question.

(a) Prove that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is second-countable.

(b) Prove that the topology $\mathcal{O}$ on $\mathbb{R}$ defined in Question 7 (b) is not second-countable.

9

Question.

(a) Which topology on $\mathbb{R}^2$ is generated by straight lines of infinite length? Does restricting to or allowing straight lines of finite length make any difference?

(b) Prove that there are topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\mathbb{R}$ such that the product topology on $\mathbb{R}^2$ with respect to $(\mathbb{R}, \mathcal{O}_1)$ and $(\mathbb{R}, \mathcal{O}_2)$ is the topology generated by straight lines of infinite length in $\mathbb{R}^2$ parallel to the y-axis.

10

This question builds upon Question 8 on Exercise Sheet 1.

Question.

(a) Let $(X, <)$ be a pre-order. Let $\mathcal{O}$ denote the corresponding topology upon X. For every $x \in X$, let $U^x := \{x' \in X \mid x < x'\}$. Prove that $\{U^x \mid x \in X\}$ defines a basis for $(X, \mathcal{O})$.

(b) Let $(X, \mathcal{O})$ be an Alexandroff space. As in Question 8 on Exercise Sheet 1, let U_x denote the intersection of all open subsets of X containing x. Prove that $\{U_x \mid x \in X\}$ defines a basis for $(X, \mathcal{O})$.

(c) Let $(X, \mathcal{O})$ be an Alexandroff space. Define $x << x'$ if $U_x \supset U_{x'}$. Prove that $<<$ defines a pre-ordering on X. This pre-ordering is the 'other way around' from the pre-ordering that was defined in Question 9 (f) of Exercise Sheet 1.

(d) Let $(X, <)$ be a pre-ordering, and let $\mathcal{O}$ denote the corresponding topology on X. Let $<<$ denote the pre-order on X of (c) corresponding to $(X, \mathcal{O})$. Prove that $<<$ coincides with $<$.

(e) Let $(X, \mathcal{O})$ be an Alexandroff space, and let $<$ denote the pre-order on X of (c) corresponding to $(X, \mathcal{O})$. Let $\mathcal{O}^<$ denote the topology on X corresponding to $<$. Prove that $\mathcal{O} = \mathcal{O}^<$.