Generell Topologi — Exercise Sheet 3

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Guide

This Exercise Sheet explores continuous maps.

(1) Question 1 contains several equivalent characterisations of continuity. The first two are vital from a theoretical point of view — we will make use of both in the lectures. The other three are useful for checking that a map is continuous in practise.

(2) Questions 3 – 9 lay the foundations for constructing all the continuous maps that we will see in the lectures. Following our philosophy throughout the course, our hard work will be to construct continuous maps

$$\mathbb{R} \longrightarrow \mathbb{R}$$

with respect to the standard topology $\mathcal{O}_{\mathbb{R}}$. We will then investigate tools to 'canonically' construct continuous maps between other spaces from these.

Our continuous maps

$$\mathbb{R} \longrightarrow \mathbb{R}$$

will be of two kinds.

(i) Polynomial maps, and quotients of polynomial maps — the continuity of these is the topic of Question 3.

(ii) Paths around a circle — in Question 9 you are given a definition of a map of this kind, and asked to prove that it is continuous.

Questions 4, 6, and 7 concern tools for 'canonical building up maps'. Question 4 discusses this in the setting of a product of topological spaces. Question 6 and Question 7 discuss this in the setting of a subspace of a topological space. Question 8 is important — you are asked to show that continuous maps can be 'glued'.

(3) Question 5 asks you to use the continuity of the projection maps to prove that a product of closed sets is closed in the product topology.

(4) Question 2 checks your understanding of the definition of a continuous map by asking you to work through an example involving topological spaces whose underlying set is finite, where everything can be seen explicitly.

(5) Question 10 helps to develop your ability to switch between the intuitive geometric idea of a continuous map and the rigorous definition.

(6) Question 11 asks you to prove that continuity for maps

$$\mathbb{R} \longrightarrow \mathbb{R}$$

with respect to the standard topology corresponds exactly to the $\epsilon - \delta$ sense which you have met before. We will not make use of this anywhere in the lectures, but it will allow you to practise working with the characterisation of continuity in terms of neighbourhoods of Question 1 (b).

(7) Question 12 continues our exploration of Alexandroff topological spaces from the previous two Exercise Sheets. You are asked to prove that not only do Alexandroff topological spaces corresponds exactly to pre-orderings, but moreover continuity in the setting of Alexandroff topological spaces can be understood entirely in the world of pre-orderings.

The Exercise Sheet should not hopefully take as long as it may look at first glance! Several of the questions with many parts should be able to be done quite quickly once you have managed one or two of the parts.

If you do not have time for all parts of all the questions at first, you might consider something like the following route:

(1) Question 1. Parts (c) and (d) are almost the same, so you could omit writing up one of these if you think you see how to do it.

(2) Question 11.

(3) Question 2.

(4) Question 5

(5) Question 3 (c). If you can do (c), you could probably manage parts (d) and (e), so you may prefer to skip them at first. In order to understand (c), you may find it helpful to warm up with one or both of (a) or (b). If you can do one of (a) or (b), you will probably be able to do the other quite quickly. If you do decide to skip (d) and (e), move on to (f) after (c) — just assume (d) and (e).

(6) Question 4 (a). For the rest of the question, try (b), but you could at first omit (c), one of (d) or (e), and one of (f) or (g).

(7) Questions 6, 7, and 8 are of a somewhat similar flavour to Question 4. You may prefer therefore to try something a little different, such as Question 9, next.

    All of Question 9 fits together, so you should do all parts here. Anything which you need from previous questions which you have not yet attempted you could just assume.

(8) After Question 9, you might like to go back to either Question 6 or Question 8. If you can do Question 6, you should find Question 7 easy, so you could at first omit Question 7.

(9) Next, you could try Question 10. This question could also be tackled much earlier, if you assume Question 9 and anything else that you need from previous questions.

(10) Then you could return to whichever of Questions 6 and 7 that you have not yet attempted.

(11) You could have a go at Question 12.

Of course, this is just a suggestion, and you can proceed however you wish! Different routes may be more beneficial for some of you. If you do follow something like the above route, you should of course return to those parts which you have omitted when you have time, and make sure that you understand my solutions!

Questions

1

Question.

(a) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(A)$ is closed in X for every closed subset A of Y.

(b) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if for every $x \in X$ and every neighbourhood U of $f(x)$ in $(Y, \mathcal{O}_Y)$, there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that $f(U') \subset U$.

(c) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let $\mathcal{O}'_Y$ be a basis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(U) \in \mathcal{O}_X$ for every $U \in \mathcal{O}'_Y$.

(d) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let $\mathcal{O}'_Y$ be a sub-basis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(U) \in \mathcal{O}_X$ for every $U \in \mathcal{O}'_Y$.

(e) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\{U_j\}_{j \in J}$ be a basis for $(X, \mathcal{O}_X)$, and let $\{U'_{j'}\}_{j' \in J'}$ be a basis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if for each $x \in X$ and each $j' \in J'$ such that $f(x) \in U'_{j'}$ there is a $j \in J$ such that $x \in U_j$ and $f(U_j) \subset U'_{j'}$.

2

Question. Let $X = \{a, b, c\}$ be equipped with the topology

$$\mathcal{O}_X := \big\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\big\},$$

and let $Y = \{a', b', c', d', e'\}$ be equipped with the topology

$$\mathcal{O}_Y := \big\{\emptyset, \{a'\}, \{e\}, \{a', e'\}, \{b', c'\}, \{a', b', c'\}, \{b', c', e'\}, \{a', b', c', e'\}, \{b', c', d', e'\}, Y\big\}.$$

Which of the following maps

$$X \xrightarrow{f} Y$$

are continuous?

(1) $a \mapsto d'$, $b \mapsto e'$, $c \mapsto d'$.

(2) $a \mapsto e'$, $b \mapsto e'$, $c \mapsto c'$.

(3) $a \mapsto c'$, $b \mapsto a'$, $c \mapsto d'$.

(4) $a \mapsto b'$, $b \mapsto c'$, $c \mapsto d'$.

3

Question. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\mathbb{R}$ be equipped with its standard topology $\mathcal{O}_{\mathbb{R}}$.

(a) Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Prove that the map

$$X \xrightarrow{|f|} \mathbb{R}$$

given by $x \mapsto |f(x)|$ is continuous.

(b) Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Prove that for any $k \in \mathbb{R}$, the map

$$X \xrightarrow{kf} \mathbb{R}$$

given by $x \mapsto k \cdot f(x)$ is continuous.

(c) Let

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R} \end{tikzcd}$$

be continuous maps. Prove that the map

$$X \xrightarrow{f+g} \mathbb{R}$$

given by $x \mapsto f(x) + g(x)$ is continuous.

Hint:

(i) Appeal to Question 7 (a) of Exercise Sheet 2.

(ii) For any $b \in \mathbb{R}$, appeal to the fact that

$$\{x \in X \mid f(x)+g(x) < b\} = \bigcup_{y \in \mathbb{R}} \Big(\{x \in X \mid f(x) < b-y\} \cap \{x \in X \mid g(x) < y\} \Big).$$

(iii) For any $a \in \mathbb{R}$, appeal to an analogous expression of

$$\{x \in X \mid f(x) + g(x) > a\}$$

as a union of intersections.

(d) Let

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R} \end{tikzcd}$$

be continuous maps. Prove that the map

$$X \xrightarrow{fg} \mathbb{R}$$

given by $x \mapsto f(x) \cdot g(x)$ is continuous.

Hint:

(i) Prove that if $f(x) \geq 0$ for all $x \in X$, then the map

$$X \xrightarrow{f^2} \mathbb{R}$$

given by $x \mapsto f(x) \cdot f(x)$ is continuous.

(ii) Find an expression for fg which allows you to deduce the continuity of fg from (i) and parts (a)–(c).

(e) Let

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R} \end{tikzcd}$$

be continuous maps, and suppose that $g(x) \neq 0$ for all $x \in X$. Prove that

$$X \xrightarrow{\frac{f}{g}} \mathbb{R}$$

given by $x \mapsto \frac{f(x)}{g(x)}$ is continuous.

Hint:

For any $a, b \in \mathbb{R}$, find an expression for $\left\{x \in X \mid \frac{1}{f(x)} < b\right\}$ and an expression for $\left\{x \in X \mid \frac{1}{f(x)} > a\right\}$ which allows you to deduce that $f^{-1}\big((-\infty, b)\big)$ and $f^{-1}\big((a, \infty)\big)$ are open in $(X, \mathcal{O}_X)$ from the continuity of f and the continuity of bf and af.

(f) Deduce that a map

$$\mathbb{R} \longrightarrow \mathbb{R}$$

which is a quotient of polynomials, namely a map of the form

$$x \mapsto \frac{k_0 + k_1 x + k_2 x^2 + \ldots + k_m x^m}{l_0 + l_1 x + l_2 x^2 + \ldots + l_n x^n}$$

where $m, n \in \mathbb{N}$, $k_i \in \mathbb{R}$ for all $0 \leq i \leq m$, and $l_j \in \mathbb{R}$ for all $0 \leq j \leq n$, is continuous.

Here we assume that

$$0 \neq l_0 + l_1 x + l_2 x^2 + \ldots + l_n x^n$$

for all $x \in \mathbb{R}$.

(g) Prove that the map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{\times} \mathbb{R}$$

given by $(x, y) \mapsto xy$ is continuous.

(h) Prove that the map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$$

given by $(x, y) \mapsto x + y$ is continuous.

4

Question. (a) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces, and let

$$Z \xrightarrow{f} X$$

and

$$Z \xrightarrow{g} Y$$

be continuous maps. Prove that the map

$$Z \xrightarrow{f \times g} X \times Y$$

given by $z \mapsto \big(f(z), g(z)\big)$ is continuous.

(b) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Prove that a map

$$Z \xrightarrow{f} X \times Y$$

is continuous if and only if the maps

$$Z \xrightarrow{p_1 \circ f} X$$

and

$$Z \xrightarrow{p_2 \circ f} Y$$

are continuous.

(c) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $(X', \mathcal{O}_{X'})$, and $(Y', \mathcal{O}_{Y'})$ be topological spaces, and let

$$X \xrightarrow{f} X'$$

and

$$Y \xrightarrow{g} Y'$$

be continuous maps. Prove that the map

$$X \times Y \xrightarrow{f \times g} X' \times Y'$$

given by $(x, y) \mapsto \big(f(x), g(y)\big)$ is continuous.

(d) Let $(X, \mathcal{O}_X)$ be a topological space. Prove that the map

$$X \xrightarrow{\Delta} X \times X$$

given by $x \mapsto (x, x)$ is continuous.

(e) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that the map

$$X \times Y \xrightarrow{\tau} Y \times X$$

given by $(x, y) \mapsto (y, x)$ is continuous.

5

Question. Let $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$ be topological spaces. Let

$$X \times Y \xrightarrow{p_1} X$$

and

$$X \times Y \xrightarrow{p_2} Y$$

denote the projection maps.

Let A be a closed subset of $(X, \mathcal{O}_X)$, and let A' be a closed subset of $(X', \mathcal{O}_{X'})$. By Proposition 3.2 in the Lecture Notes we have that p_1 and p_2 are continuous.

Use this to prove that $A \times A'$ is a closed subset of $(X \times X', \mathcal{O}_{X \times X'})$.

6

Question. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let A be a subset of X equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$.

(a) Let

$$X \xrightarrow{f} Y$$

be a continuous map. Prove that the restriction of f to A defines a continuous map

$$A \longrightarrow Y.$$

(b) Let

$$A \xhookrightarrow{i} X$$

denote the inclusion map. Prove that a map

$$Y \xrightarrow{f} A$$

is continuous if and only if the map

$$Y \xrightarrow{i \circ f} X$$

is continuous.

(c) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let A be a subset of X equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Give an example to show that a continuous map

$$A \xrightarrow{f} Y$$

need not extend to a continuous map

$$X \longrightarrow Y.$$

In other words, find topological spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(A, \mathcal{O}_A)$ and a continuous map

$$A \xrightarrow{f} Y$$

which cannot be the restriction of any continuous map

$$X \longrightarrow Y.$$

7

Question. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let A be a subset of Y. Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(Y, \mathcal{O}_Y)$.

(a) Prove that if

$$X \xrightarrow{f} Y$$

is a continuous map such that $f(X) \subset A$, then the map

$$X \longrightarrow A$$

given by $x \mapsto f(x)$ is continuous.

(b) Prove that if

$$X \xrightarrow{f} A$$

is a continuous map then the map

$$X \longrightarrow Y$$

given by $x \mapsto f(x)$ is continuous.

8

Let X and Y be sets, and let $\{A_j\}_{j \in J}$ be a set of subsets of $(X, \mathcal{O}_X)$ such that $X = \bigcup_{j \in J} A_j$. Let $A = \bigcap_{j \in J} A_j$.

Suppose that for every $j \in J$ we have a map

$$A_j \xrightarrow{f_j} Y$$

such that the restriction of f_j to A' is equal to the restriction of $f_{j'}$ to A for all $(j, j') \in J \times J$. Then the map

$$X \xrightarrow{g} Y$$

given by $x \mapsto f_j(x)$ if $x \in A_j$ is well-defined.

Now let $\mathcal{O}_X$ be a topology upon X, and let $\mathcal{O}_Y$ be a topology upon Y. Equip every A_j for $j \in J$ with the subspace topology with respect $(X, \mathcal{O}_X)$. Suppose that f_j is continuous for every $j \in J$.

Question.

(a) Prove that if A_j is open in $(X, \mathcal{O}_X)$ for every $j \in J$, then g is continuous.

(b) Prove that if J is finite and A_j is closed in $(X, \mathcal{O}_X)$ for every $j \in J$, then g is continuous.

(c) Find an example to show that for an arbitrary finite set $\{A_j\}$, it need not be the case that g is continuous.

(d) Find an example to show that when J is infinite, then g need not be continuous even if A_j is closed in $(X, \mathcal{O}_X)$ for every $j \in J$.

Remark. The result of (a) and (b) is known as the *glueing lemma* or *pasting lemma*.

9

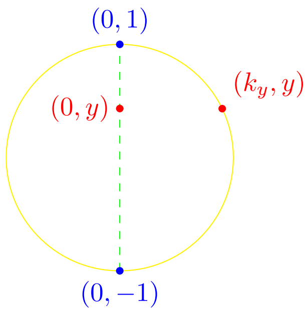
In this question, we will construct step-by-step a continuous map

$$\mathbb{R} \xrightarrow{\phi} S^1.$$

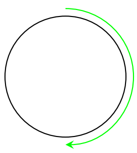
For any $y \in [-1, 1]$, there is a unique $k_y \in \mathbb{R}$ with $k_y \geq 0$ such that $\|(k_y, y)\| = 1$. We have that

$$k_y = \sqrt{1 - y^2},$$

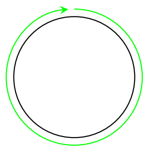
where we take the positive square root.



Given $x \in [0, \frac{1}{2}]$, let $y = 1 - 4x$, and define $\phi(x)$ to be (k_y, y). We may picture ϕ on $[0, 1]$ as follows.



Given $x \in \mathbb{R}$ such that $x \in [\frac{1}{2}, 1]$, let $y = 4x - 3$, and define $\phi(x)$ to be $(-k_y, y)$. We may picture ϕ on $[0, 1]$ as follows.



Given $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $x \in [n, n+1]$, we define $\phi(x)$ to be $\phi(x-n)$.

Remark. The map ϕ allows us to construct paths around a circle without using trigonometric maps. Sine and cosine define continuous maps, but the proof of this is quite involved. One has two choices.

(1) Appeal to a notion of angle, which requires a rigorous definition of arc length.

(2) Appeal to analytic methods such as power series.

Both of these approaches are quite far removed from our intuitive geometric understanding of paths around a circle! Thus we will not go into this. The map ϕ is simpler, and we can construct any path around a circle that we are interested in using it!

Question.

(a) Prove that the map

$$[0, \tfrac{1}{2}] \longrightarrow \mathbb{R}$$

given by $y \mapsto k_y$ is continuous.

(b) Deduce that the maps

$$[0, \tfrac{1}{2}] \xrightarrow{\phi} S^1$$

and

$$[\tfrac{1}{2}, 1] \xrightarrow{\phi} S^1$$

are continuous.

(c) Deduce that the map

$$[0, 1] \xrightarrow{\phi} S^1$$

is continuous.

(d) Conclude that the map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is continuous.

10

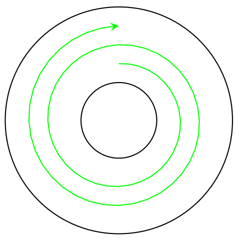
Question. (a) Prove that the map of Example 2.13 (2) in the Lecture Notes is continuous.

(b) Prove that the map of Example 2.13 (3) in the Lecture Notes is continuous.

(c) Find a continuous map

$$I \longrightarrow A_k$$

for a fixed $0 < k < \frac{1}{2}$ which describes a 'spiral' as roughly depicted below, starting at $(0, \frac{1}{2})$, passing through $(0, \frac{5}{8})$, and ending at $(0, \frac{3}{4})$.



(d) Prove that the map

$$I^2 \longrightarrow I$$

given by $(x, y) \mapsto \min\{x, y\}$ is continuous. Also, prove that the map

$$I^2 \longrightarrow I$$

given by $(x, y) \mapsto \max\{x, y\}$ is continuous. Draw a picture of each of these maps! You may find it helpful to think of the copy of I in the target as a diagonal in I^2.

11

Question. Let $\mathbb{R}$ be equipped with its standard topology $\mathcal{O}_{\mathbb{R}}$. Prove that a map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous in the topological sense if and only if it is continuous in the $\epsilon - \delta$ sense that you have met in real analysis/calculus, namely for all $x, c, \epsilon \in \mathbb{R}$ with $\epsilon > 0$ there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

Hint:

(1) Appeal to Examples 2.9 (1).

(2) Appeal to Question 1 (e).

12

Let $(X, <)$ and $(Y, <)$ be pre-orderings. A *morphism* from $(X, <)$ to $(Y, <)$ is a map

$$X \xrightarrow{f} Y$$

such that if $x < x'$ then $f(x) < f(x')$.

Question.

(a) What does this requirement correspond to if we picture $(X, <)$ and $(Y, <)$ via arrows as in Question 8 of Exercise Sheet 1?

Recall that by Question 10 of Exercise Sheet 2, Alexandroff topologies on a set X correspond bijectively to pre-orderings on X, in the following way.

(i) Let $(X, \mathcal{O}_X)$ be an Alexandroff topological space. Given $x \in X$, define U_x to be the intersection of all neighbourhoods of x in $(X, \mathcal{O}_X)$. To $(X, \mathcal{O}_X)$ we associate the pre-ordering $<$ defined by $x < x'$ if $U_x \supset U_{x'}$.

(ii) Let $(X, <)$ be a pre-ordering. We define a topology $\mathcal{O}_X$ on X by stipulating that $U \subset X$ belongs to $\mathcal{O}_X$ if for any $x \in U$ and any $x' \in X$ such that $x < x'$ we have that $x' \in U$. We have that $(X, \mathcal{O}_X)$ is an Alexandroff space.

Question.

(b) Let $(X, \mathcal{O}_X)$ be an Alexandroff topological space, and let $<_X$ denote the corresponding pre-ordering of X. Let $(Y, \mathcal{O}_Y)$ be another Alexandroff topological space, and let $<_Y$ denote the corresponding pre-ordering.

Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if f defines a morphsm from $(X, <_X)$ to $(Y, <_Y)$.