Generell Topologi — Exercise Sheet 4

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Guide

The questions are organised into three overlapping themes.

Have a go at as many as you have time for — try to tackle questions from each of the three themes. For the rest, check that you understand my solutions — let me know if not!

(1) Questions 1 – 6 cover limit points, closure, and boundary.

  (i) Questions 1 allows you to practise working with closure theoretically.

  (ii) Question 2 asks you to calculate boundaries and closures in a finite example and two geometric examples.

  (iii) Question 3 allows you to practise working theoretically with boundaries. It also introduces the notion of interior.

  (iv) Question 4 explores the boundary of a product of topological spaces, and introduces some examples where this can be used.

  (v) In Question 5 you are asked to work with limit points in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

  (v) Question 6 investigates the behaviour of boundary under a homeomorphism.

(2) Questions 7 – 9 will help to develop your understanding of the notion of homeomorphism.

  (i) Question 7 explores two examples of homeomorphisms that were introduced in the lectures. You may particularly learn a lot from trying part (b).

  (ii) Question 8 is a question about restricting homeomorphisms which we will use frequently in the lectures.

(iii) Question 9 is very important in geometric topology. It is part of a construction known as *Alexander's trick*. You are asked to use it to deduce that any 'blob' in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is homeomorphic to a disc.

(3) Questions 10 – 17 concern quotient spaces. I hope that you will find the geometric questions fun! All of the geometric examples are important in algebraic and geometric topology.

They will help you to become familiar with and visualise glueing topological spaces. This is the most important tool in a topologist's armoury!

(i) Question 10 allows you to practise working with the definition of a quotient topology in a finite example.

(ii) Question 11 introduces the *universal property* of a quotient space which will make use of when we discuss locally compact topological spaces in the lectures. It is an important theoretical question, but is quite abstract — don't worry if you find it a little difficult.

(iii) In Question 12 you are introduced to two examples of *projective spaces*. The projective plane $\mathbb{P}^2(\mathbb{R})$ cannot be visualised in $\mathbb{R}^3$, like the Klein bottle.

(iv) In Question 13 you are introduced to the cone and suspension of a topological space.

(v) Question 14 allows you to practise working with the Möbius band and the Klein bottle. The different parts of the question are independent of each other — if you cannot do a particular part, move on to the next.

(vi) In Question 15 you are introduced to the wedge sum of a pair of topological spaces.

(vii) I particularly recommend that you try Question 16, which will allow you to practise working with a torus. Part (c) concerns knot theory.

(viii) In Question 17 you will explore glueing sides of polygons other than the square. The topological spaces that we obtain, known as handlebodies, will be very important when we discuss the classification of surfaces in the last few lectures.

(4) Question 18 is included mostly for fun. It will help you to develop your feeling for the notion of homeomorphism.

(5) Question 19 is included mostly as a curious puzzle. Don't worry if you cannot do it, it will not be examined!

Questions

1

Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$.

Question.

(a) Prove Proposition 5.9 in the Lecture Notes.

(b) Let A' be a closed subset of A. Prove that $A' = A \cap A''$ for a closed subset A'' of $(X, \mathcal{O}_X)$.

(c) Let A' be a subset of A which is closed in $(X, \mathcal{O}_X)$. Prove that A' is closed in $(A, \mathcal{O}_A)$.

(d) Let A' be a closed subset of A. Prove that if A is closed in $(X, \mathcal{O}_X)$ then A' is closed in $(X, \mathcal{O}_X)$.

(e) Let A' be a subset of A. Let $\overline{A'}$ denote the closure of A' in $(A, \mathcal{O}_A)$, and let $\widehat{A'}$ denote the closure of A' in $(X, \mathcal{O}_X)$. Prove that $\overline{A'} = \widehat{A'} \cap A$.

(f) Let A' be a subset of A such that $A \cap A' = \emptyset$. Prove that if A is open in $(X, \mathcal{O}_X)$ then $A \cap \overline{A'} = \emptyset$.

2

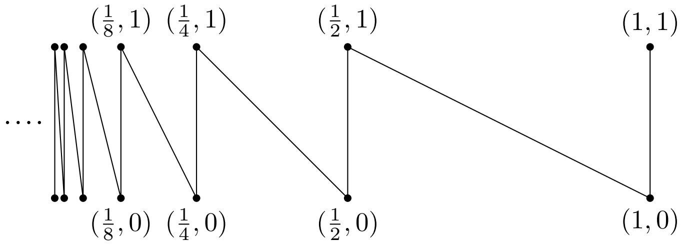
Question.

(a) Let $X = \{a, b, c, d\}$, and let $\mathcal{O}$ be the topology on X defined by

$$\big\{\emptyset, \{a\}, \{b, c\}, \{b, c, d\}, \{a, b, c\}, X\big\}.$$

What is the closure of $\{b\}$ in X? Find a dense subset of X consisting of two elements. What is the boundary of $\{b\}$ in X? What is the boundary of $\{b, c\}$ in $(X, \mathcal{O}_X)$?

(b) Let $X \subset \mathbb{R}^2$ be the set depicted below.



Explicitly, let X be the union of the sets

$$\bigcup_{n\geq 0} \{(\frac{1}{2^n}, y) \mid y \in [0,1]\}$$

and

$$\bigcup_{n\geq 0} \{(x, -2^{n+1}x + 2) \mid x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}]\}.$$

We equip X with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

What is the closure of X in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$?

(c) What is the boundary of the vertical line $\{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$ in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$? What is the boundary of $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y > 0\}$ in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$? What is the boundary of the union of these two sets in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$?

3

Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X equipped with the subspace topology $(A, \mathcal{O}_A)$ with respect to $(X, \mathcal{O}_X)$.

The *interior* of A in X is the union of all the subsets of A which are open in X. We shall denote it by A°.

Let $\overline{A}$ denote the closure of A in X.

Question.

(a) Prove that $\partial_X A = \overline{A} \cap \overline{X \setminus A}$, where $\overline{X \setminus A}$ is the closure of $X \setminus A$ in X.

(b) Deduce that $\partial_X A$ is a closed subset of X.

(c) Prove that $\partial_X A = \overline{A} \setminus A^\circ$.

(d) Prove that $A^\circ = A \setminus \partial_X A$.

(e) Find the interior of $[0, 1]$ regarded as a subset of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ equipped with the subspace topology. Do the same for $[0, 1)$, $(0, 1]$, and $(0, 1)$.

(f) Find the interior of D^2 regarded as a subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

(g) Prove that A is open in X if and only if $A^\circ = A$.

4

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of X, and let A' be a subset of Y. Let $\overline{A}$ denote the closure of A in X, and let $\overline{A'}$ denote the closure of A' in Y.

Question.

(a) Let $\overline{A \times A'}$ denote the closure of $A \times A'$ in $X \times Y$. Prove that $\overline{A \times A'} = \overline{A} \times \overline{A'}$.

(b) Prove that $\partial_{X \times Y} A \times A' = (\partial_X A \times \overline{A'}) \cup (\overline{A} \times \partial_Y A')$.

(c) Deduce that $\partial_{\mathbb{R}^2}(I^2)$ is as claimed in Examples 5.16 (2) of the Lecture Notes.

(d) What is the boundary of the solid cylinder $D^2 \times I$ in $(\mathbb{R}^2 \times I, \mathcal{O}_{\mathbb{R}^2 \times I})$? What is its boundary in $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$?

5

Question.

(a) Let $X \subset \mathbb{R}$ be bounded above. Prove that $\sup X$ is a limit point of X in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

(b) Let $X \subset \mathbb{R}$ be bounded below. Prove that $\inf X$ is a limit point of X in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

(c) Let $a, b \in \mathbb{R}$. Prove that $\overline{(a,b)} = \overline{[a,b)} = \overline{(a,b]} = \overline{[a,b]} = [a,b]$. Prove that $\partial_{\mathbb{R}}(a,b) = \partial_{\mathbb{R}}[a,b) = \partial_{\mathbb{R}}(a,b] = \partial_{\mathbb{R}}[a,b] = \{a,b\}$.

(d) Let (a,b) be an open interval in $\mathbb{R}$. Prove that

$$\overline{\mathbb{Q} \cap (a,b)} = \overline{\mathbb{Q} \cap [a,b)} = \overline{\mathbb{Q} \cap (a,b]} = \overline{\mathbb{Q} \cap [a,b]} = [a,b],$$

where the closures are in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. What is the boundary of these four sets in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$?

6

Question.

(a) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Let A be a subset of X equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Let $f(A)$ be equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$.

Prove that $f(\partial_X A) = \partial_Y f(A)$.

(b) Find an example of a topological space $(X, \mathcal{O}_X)$ and subsets A and A' such that the following conditions are satisfied:

(1) $\partial_X A = \partial_X A'$,

(2) $(A, \mathcal{O}_A)$ is not homeomorphic to $(A', \mathcal{O}_{A'})$. Here $\mathcal{O}_A$ denotes the subspace topology of A with respect to $(X, \mathcal{O}_X)$, and $\mathcal{O}_{A'}$ denotes the subspace topology of A' with respect to $(X, \mathcal{O}_X)$.

(c) Find an example of a topological space $(X, \mathcal{O}_X)$ and subsets A and A' of X such that the following conditions are satisfied:

(1) $(A, \mathcal{O}_A)$ is homeomorphic to $(A', \mathcal{O}_{A'})$. Here $\mathcal{O}_A$ denotes the subspace topology of A with respect to $(X, \mathcal{O}_X)$, and $\mathcal{O}_{A'}$ denotes the subspace topology of A' with respect to $(X, \mathcal{O}_X)$.

(2) $\partial_X A$ is not homeomorphic to $\partial_X A'$, where each is equipped with its subspace topology with respect to $(X, \mathcal{O}_X)$.

7

Question.

(a) Define a homeomorphism

$$I^2 \xrightarrow{f} D^2$$

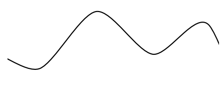
such that $f(\partial_{\mathbb{R}^2} I^2) = S^1$.

(b) Let us define a 'squiggle' in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ to be a bounded, closed, connected subset X of $\mathbb{R}^2$ which has the following properties.

(1) For every point $x \in X$ there is a neighbourhood U of x in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ such that $U \cap X$ equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is homeomorphic to either an open interval or a half-open interval.

(2) There is at least one point $x \in X$ which admits a neighbourhood U in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ such that $U \cap X$ equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is homeomorphic to a half-open interval.

The squiggle of Examples 4.10 (5) in the Lecture Notes satisfies these properties.



So does the letter S regarded as a subset of $\mathbb{R}^2$, for example. However, the letters K and T are not squiggles — why? Also the circle is not a squiggle — why?

Prove that any squiggle is homeomorphic to $(I, \mathcal{O}_I)$.

Prove that if $(X, \mathcal{O}_X)$ satisfies property (1) but not property (2) then $(X, \mathcal{O}_X)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$. Is boundedness a necessary assumption for this?

Prove that if $(X, \mathcal{O}_X)$ satisfies properties (1) and (2) but is not closed then $(X, \mathcal{O}_X)$ is homeomorphic to a half-open interval.

Prove that if $(X, \mathcal{O}_X)$ satisfies property (1) but not property (2) and is not closed then $(X, \mathcal{O}_X)$ is homeomorphic to an open interval.

8

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Let A be a subset of X.

Question.

(a) Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$, and let $f(A)$ be equipped with the subspace topology $\mathcal{O}_{f(A)}$ with respect to $(Y, \mathcal{O}_Y)$. Prove that f restricts to a homeomorphism

$$A \longrightarrow f(A).$$

(b) Appealing to Question 7 (a) deduce that $\partial_{\mathbb{R}^2} I^2$ equipped with its subspace topology with respect to I^2 is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

(c) Let $X \setminus A$ be equipped with the subspace topology $\mathcal{O}_{X \setminus A}$ with respect to $(X, \mathcal{O}_X)$, and let $Y \setminus f(A)$ be equipped with the subspace topology $\mathcal{O}_{Y \setminus f(A)}$ with respect to $(Y, \mathcal{O}_Y)$. Prove that f restricts to a homeomorphism

$$X \setminus A \longrightarrow Y \setminus f(A).$$

9

Question.

(a) Let

$$S^1 \xrightarrow{f} S^1$$

be a homeomorphism. Prove that there is a homeomorphism

$$D^2 \xrightarrow{g} D^2$$

such that the restriction of g to S^1 is f.

(b) Let X and Y be subsets of $\mathbb{R}^2$ equipped with their subspace topologies $\mathcal{O}_X$ and $\mathcal{O}_Y$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Suppose that

$$X \xrightarrow{f} D^2$$

and

$$Y \xrightarrow{g} D^2$$

are homeomorphisms such that $f(\partial_{\mathbb{R}^2} X) = S^1$ and $g(\partial_{\mathbb{R}^2} Y) = S^1$.

Suppose that $X \cap Y \subset \partial_{\mathbb{R}^2 X}$ and $X \cap Y \subset \partial_{\mathbb{R}^2 X}$. Suppose moreover that $X \cap Y$ equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is homeomorphic to $(I, \mathcal{O}_I)$.

Draw a couple of 'blobs' in $\mathbb{R}^2$ which have this property.

Prove that $(X \cup Y, \mathcal{O}_{X \cup Y})$ is homeomorphic to $(D^2, \mathcal{O}_{D^2})$, where $\mathcal{O}_{X \cup Y}$ is the subspace topology on $X \cup Y$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$,

(c) Fill in the details of Examples 4.10 (4) in the Lecture Notes.

To be more precise:

(1) Let X be a bounded star shaped subset of $\mathbb{R}^2$. Let X be equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that there is a homeomorphism

$$X \xrightarrow{f} D^2$$

such that $f(\partial_{\mathbb{R}^2} X) = S^1$.

Hint: consider a disc large enough to contain X.

(2) Let B be a subset of $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ with the property that $B = \cup_{j \in J} X_j$ for a finite set J where:

(i) X_j is star shaped for all $j \in J$.

(ii) $X_j \cap X_{j'}$ equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is homeomorphic to $(I, \mathcal{O}_I)$ for all $j, j' \in J$.

Let $\mathcal{O}_B$ denote the subspace topology on B with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

By (1) and part (b) prove by induction that $(B, \mathcal{O}_B)$ is homeomorphic to $(D^2, \mathcal{O}_{D^2})$.

(3) Deduce that $(B, \mathcal{O}_B) \cong (I^2, \mathcal{O}_{I^2})$.

10

Question. Let $(X, \mathcal{O}_X)$ be the pseudo-circle of Question 8 of Exercise Sheet 1. Recall that $X = \{a, b, c, d\}$, and that

$$\mathcal{O}_X = \big\{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\big\}.$$

Let $\sim$ be the equivalence relation on X defined by $b \sim c$. List every subset of $X/\sim$ which belongs to $\mathcal{O}_{X/\sim}$.

Draw the pre-order which corresponds to $(X/\sim, \mathcal{O}_{X/\sim})$.

11

Let $(X, \mathcal{O}_X)$ be a topological space. Let $\sim$ be an equivalence relation upon X, and let $\mathcal{O}_{X/\sim}$ denote the quotient topology upon $X/\sim$. Let

$$X \xrightarrow{\pi} X/\sim$$

denote the quotient map given by $x \mapsto [x]$.

Given a topological space $(Y, \mathcal{O}_Y)$ and a continuous map

$$X \xrightarrow{f} Y$$

let us write that f *respects* $\sim$ if for all $x, x' \in X$ such that $x \sim x'$ we have that $f(x) = f(x')$.

Question.

(a) Let $(Y, \mathcal{O}_Y)$ be a topological space, and let

$$X \xrightarrow{f} Y$$

be a continuous map such that f respects $\sim$.

Let

$$X/\sim \xrightarrow{g} Y$$

be the map defined by $[x] \mapsto f(x)$. Since f respects $\sim$ we have that g is well-defined.

Prove that g is continuous.

Note that g is the unique map with $g \circ \pi = f$. In particular g is the unique continuous map with $g \circ \pi = f$. This is known as the *universal property* of a quotient topology.

(b) Let $(Z, \mathcal{O}_Z)$ be a topological space, and let

$$X \xrightarrow{\pi'} Z$$

be a continuous map. Suppose that for any topological space $(Y, \mathcal{O}_Y)$ and any continuous map

$$X \xrightarrow{f} Y$$

which respects $\sim$ there is a unique continuous map

$$Z \xrightarrow{g} Y$$

such that $g \circ \pi' = f$.

Prove that $(Z, \mathcal{O}_Z)$ is homeomorphic to $(X/\sim, \mathcal{O}_{X/\sim})$.

(c) Let $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$ be topological spaces, and let

$$X \xrightarrow{f} X'$$

be a homeomorphism. Let $\sim$ be an equivalence relation on X, and let $\sim'$ be an equivalence relation on Y. Prove that if f respects $\sim$ and f^{-1} respect $\sim'$, then $(X/\sim, \mathcal{O}_{X/\sim})$ is homeomorphic to $(X', \mathcal{O}_{X'/\sim'})$.

(d) Prove that $(S^2, \mathcal{O}_{S^2})$ is homeomorphic to the topological space $(I^2/\sim, \mathcal{O}_{I^2/\sim})$, where $\sim$ is the equivalence relation on I^2 defined by $x \sim x'$ if $x, x' \in \partial_{\mathbb{R}^2} I^2$.

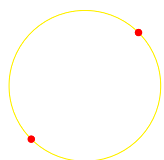
12

Let $\mathbb{R}^2 \setminus \{0\}$ be equipped with its subspace topology $\mathcal{O}_{\mathbb{R}^2 \setminus \{0\}}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $\mathbb{P}^1(\mathbb{R})$ be the quotient of $\mathbb{R}^2 \setminus \{0\}$ by the equivalence relation $\sim$ defined by $x \sim y$ if there is a line through the origin in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ on which x and y both lie. Then $\mathbb{P}^1(\mathbb{R})$ equipped with its quotient topology $\mathcal{O}_{\mathbb{R}^2/\sim}$ is known as the *real projective line*.

Let $\mathbb{R}^3 \setminus \{0\}$ be equipped with its subspace topology $\mathcal{O}_{\mathbb{R}^3 \setminus \{0\}}$ with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$. Let $\mathbb{P}^2(\mathbb{R})$ be the quotient of $\mathbb{R}^3 \setminus \{0\}$ by the equivalence relation $\sim$ defined by $x \sim y$ if there is a line through the origin in $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$ on which x and y both lie. Then $\mathbb{P}^2(\mathbb{R})$ equipped with its quotient topology $\mathcal{O}_{\mathbb{R}^3/\sim}$ is known as the *real projective plane*.

Question.

(a) Let us refer to two points on S^1 as *antipodal* if they both lie on a straight line through the origin. A pair of antipodal points are pictured below.



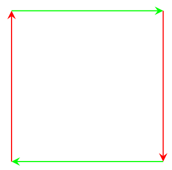
Define an equivalence relation $\sim$ on S^1 which identifies antipodal points.

Prove that $(S^1/\sim, \mathcal{O}_{S^1/\sim})$ is homeomorphic to $\big(\mathbb{P}^1(\mathbb{R}), \mathcal{O}_{\mathbb{P}^1(\mathbb{R})}\big)$.

Prove moreover that $(S^1/\sim, \mathcal{O}_{S^1/\sim})$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

Conclude that $(S^1, \mathcal{O}_{S^1})$ is homeomorphic $\big(\mathbb{P}^1(\mathbb{R}), \mathcal{O}_{\mathbb{P}^1(\mathbb{R})}\big)$.

(b) Define an equivalence relation $\sim$ on I^2 to capture glueing together the two vertical edges with a twist and glueing together the two horizontal edges with a twist.



Prove that the $\big(\mathbb{P}^2(\mathbb{R}), \mathcal{O}_{\mathbb{P}^2(\mathbb{R})}\big)$ is homeomorphic to $(I^2/\sim, \mathcal{O}_{I^2/\sim})$.

Hint: can you find a generalisation of your argument in the second step of part (a)?

13

Let $(X, \mathcal{O}_X)$ be a topological space. Let $X \times I$ be equipped with the product topology $\mathcal{O}_{X\times I}$.

The *cone* of X is the quotient of $X \times I$ by the equivalence relation $\sim$ defined by $(x, 1) \sim (x', 1)$ for all $x \in X$, equipped with the quotient topology $\mathcal{O}_{(X\times I)/\sim}$. Let us denote it by $(CX, \mathcal{O}_{CX})$.

The *suspension* of X is the quotient of $X \times I$ by the equivalence relation $\sim$ defined by $(x, 1) \sim (x', 1)$ and $(x, 0) \sim (x', 0)$ for all $x \in X$, equipped with the quotient topology $\mathcal{O}_{(X\times I)/\sim}$. Let us denote it by $(\Sigma X, \mathcal{O}_{\Sigma X})$.

Question.

(a) Draw the cone of S^1. Prove that it is homeomorphic to $(D^2, \mathcal{O}_{D^2})$.

(b) Let X be a subset of $\mathbb{R}^n$, equipped with the subspace topology with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Let $p \in \mathbb{R}^n \setminus X$.

For any $x \in X$, let L_x denote the straight line from x to p. Let $C_p(X) = \bigcup_{x\in X} L_x$. We equip C_p with the subspace topology $\mathcal{O}_{C_p(X)}$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

Prove that $\big(C_p(X), \mathcal{O}_{C_p(X)}\big)$ is homeomorphic to $\big(C(X), \mathcal{O}_{C(X)}\big)$.

(c) Let $(X, \mathcal{O}_X)$ be a topological space, and let $\big(C(X), \mathcal{O}_{C(X)}\big)$ be its cone. Let $c \in C(X)$.

Find a continuous map

$$C(X) \times I \xrightarrow{f} C(X)$$

with the property that $f(y, 0) = c$ for all $y \in C(X)$ and $f(y, 1) = y$ for all $y \in C(X)$.

A topological space with this property is said to be *contractible*.

(d) Draw the suspension of S^1.

(e) Prove that $(S^n, \mathcal{O}_{S^n})$ is homeomorphic to $(\Sigma S^{n-1}, \mathcal{O}_{\Sigma S^{n-1}})$, for any $n \geq 1$.

(f) Let $(X, \mathcal{O}_X)$ be a topological space. Prove that $(\Sigma X, \mathcal{O}_{\Sigma X})$ is homeomorphic to a topological space obtained by glueing together two copies of $(CX, \mathcal{O}_{CX})$.

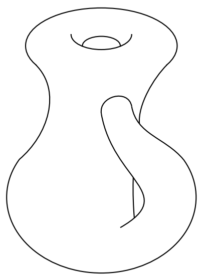
Looking at the pictures you drew in parts (a) and (d) should help!

Part of what you are required to do is to express this glueing rigorously.

14

Question.

(a) Find an equivalence relation $\sim$ on I^2 such that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ can truly be pictured as follows.



(b) Define a continuous surjective map

$$M^2 \xrightarrow{f} S^1$$

with the following properties.

(1) For every $x \in S^1$ we have that $f^{-1}(x)$ equipped with the subspace topology with respect to $(M^2, \mathcal{O}_{M^2})$ is homeomorphic to $(I, \mathcal{O}_I)$.

(2) For every $x \in S^1$ there is a neighbourhood U of x such that there exists a homeomorphism

$$f^{-1}(U) \xrightarrow{h_x} U \times I$$

with $f(y) = p_U \circ h_x(y)$ for all $y \in f^{-1}(U)$.

Here U is equipped with the subspace topology with respect to S^1, $U \times I$ is equipped with the product topology $\mathcal{O}_{U \times I}$, $f^{-1}(U)$ is equipped with the subspace topology with respect to $(M^2, \mathcal{O}_{M^2})$, and

$$U \times I \xrightarrow{p_U} U$$

is the projection map.

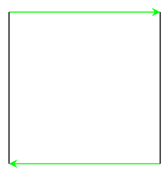
A map f with these properties is known as a *fibre bundle*.

(3) Make sense of the limerick at the end of Examples 3.9 (5) in the Lecture Notes!

More precisely, prove first that the image under the quotient map

$$I^2 \xrightarrow{\pi} M^2$$

of the two vertical black lines pictured below is homeomorphic to $(S^1, \mathcal{O}_{S^1})$. We refer to it as the *boundary circle* of M^2.



Then prove that $(K^2, \mathcal{O}_{K^2})$ is homeomorphic to the topological space obtained by glueing together two copies of $(M^2, \mathcal{O}_{M^2})$ at their boundary circles.

Part of what you are required to do is to make sense of this glueing precisely!

(4) Prove that the topological space obtained by glueing a Möbius band $(M^2, \mathcal{O}_{M^2})$ to a disc $(D^2, \mathcal{O}_{D^2})$ by identifying the boundary circle of M^2 defined in (3) to the boundary circle of D^2 is homeomorphic to $\big(\mathbb{P}^2(\mathbb{R}), \mathcal{O}_{\mathbb{P}^2(\mathbb{R})}\big)$.

Again, part of what you are required to do is to make sense of this glueing precisely.

15

Question.
Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $x \in X$ and $y \in Y$.

The *wedge sum* of X and Y is the quotient of $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$ be the equivalence relation defined by $(x', y) \sim (x, y')$ for all $x' \in X$ and $y' \in Y$.

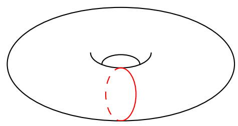
We denote it by $X \vee Y$.

(a) Draw the wedge sum $S^1 \vee S^1$, picking an arbitrary pair of points to work with. Make sense of and draw the wedge sum $S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1$.

(b) Draw the wedge sum $S^2 \vee S^2$. Find an equivalence relation $\sim$ on S^2 such that $(S^2, \mathcal{O}_{S^2/\sim})$ is homeomorphic to $S^2 \vee S^2$.

16

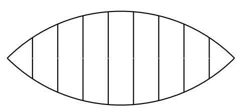
Question.
(a) Let $\sim$ denote the equivalence relation on T^2 given by $x \sim x'$ for x and x' belonging to the circle indicated below.



Draw $(T^2/\sim, \mathcal{O}_{T^2/\sim})$.

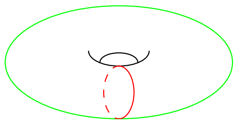
Can you find a way to rigorously define the equivalence relation $\sim$ via the definition of T^2 as a quotient of I^2?

Let G be a subset of $\mathbb{R}^2$ as pictured below. Let $\mathcal{O}_G$ denote the subspace topology on G with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



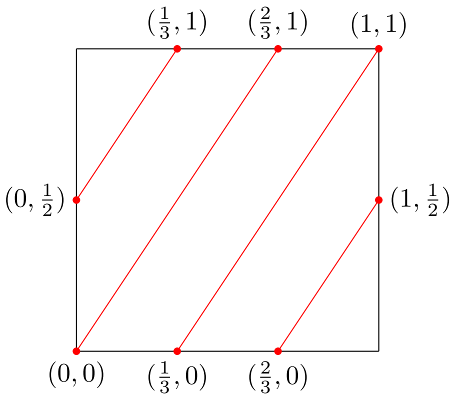
Define an equivalence relation $\approx$ on G such that $(G/\approx, \mathcal{O}_{G/\approx})$ is homeomorphic to $(T^2/\sim, \mathcal{O}_{T^2/\sim})$.

(b) Let $\sim$ denote the equivalence relation on T^2 defined by both $x \sim x'$ for x and x' belonging to the red circle indicated below and $y \sim y'$ for y and y' belonging to the green circle indicated below.



Which familiar topological space is homeomorphic to $(T^2/\sim, \mathcal{O}_{T^2/\sim})$? Can you find a way to rigorously prove it?

(c) Let K denote the subset of I^2 depicted in red below.



We can think of this as follows. Begin at $(0, 0)$, and follow a line of gradient $\frac{2}{3}$ until we hit a side of I^2. Jump over to the other side, and repeat this process. Eventually we end up at $(1, 1)$.

Prove that the image of K under the quotient map

$$I^2 \xrightarrow{\pi} T^2$$

is homeomorphic to S^1.

In fact the image of K under the quotient map π is the trefoil knot, wrapped around a torus!

Try to visualise this! If your artistic skills are good, try to draw it!

If anybody can make a nice picture of the trefoil wrapped around a torus, I would be delighted if you could scan it and send it to me — I will upload to the course webpage and to the front cover of the lecture notes!

A knot which wraps around the torus like this is known as a *torus knot*. There is a link wrapping around a torus for any rational number $\frac{p}{q}$, obtained by working with lines of gradient $\frac{p}{q}$ in place of $\frac{2}{3}$ above.

Draw the subset of I^2 which corresponds in this way to $\frac{3}{4}$. Draw the corresponding knot in the usual way — if you can visualise and/or draw it wrapped around the torus, that's even better!

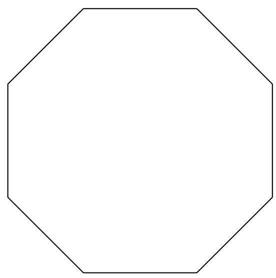
Draw the subset of I^2 which corresponds in this way to $\frac{2}{4}$. Draw the corresponding link in the usual way — again if you can visualise and/or draw it wrapped around the torus, that's even better!

Prove that if we take lines with an irrational gradient then the image under π is a dense subset of $(T^2, \mathcal{O}_{T^2})$.

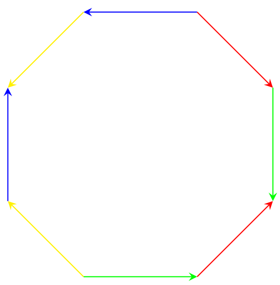
17

Question.

(a) Let X be an octagon, regarded as a subset of $\mathbb{R}^2$ as below. We equip X with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

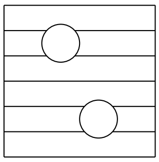


Let $\sim$ denote the equivalence relation on X which identifies the sides with the same colour.



Let $(H_2, \mathcal{O}_{H_2})$ denote the topological space obtained by drilling two disjoint holes right through $(I^3, \mathcal{O}_{I^3})$.

We can construct it by taking the product with I of a copy of I^2 with the interiors of two discs cut out as below.



Prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is homeomorphic to $(H_2, \mathcal{O}_{H_2})$.

(d) Let $(H_n, \mathcal{O}_{H_n})$ be the topological space obtained by taking a copy of I^3 and drilling n dsijoint holes right through it.

Let X denote the regular polygon X with $4n$ edges. Equip X with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Find an equivalence relation on X such that $(X/\sim, \mathcal{O}_{X/\sim})$ is homeomorphic to $(H_n, \mathcal{O}_{H_n})$.

The topological space $(H_n, \mathcal{O}_{H_n})$ is known as a *handlebody*. The reason for the name will be revealed when we discuss the classification of surfaces in the last few lectures.

18

Question. When learning topology, it is a rite of passage to consider the following: why is a coffee mug (with a handle!) homeomorphic to a doughnut?

Better not sit with a topologist in a café if you wish to avoid embarrassing glances!

What is a coffee mug without a handle homeomorphic to?

19

Question. (a) Let X be a topological space, and let A be a subset of X. Prove that there are at most 14 possible sets which can be obtained by the following procedure.

(1) Either take the complement of A in X or take the closure of A in X.

(2) Denote the resulting set by A'. Go back to Step (1), replacing A by A'.

(b) Find a subset A of $\mathbb{R}$ for which one can obtain exactly 14 sets by the procedure of (a).