Generell Topologi

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11.1 Separation axioms, continued

Definition 11.1. Let $(X, \mathcal{O})$ be a topological space.

(0) If $(X, \mathcal{O})$ satisfies (T0) we refer to $(X, \mathcal{O})$ as a *T0* topological space.

(1) If $(X, \mathcal{O})$ satisfies (T1) we refer to $(X, \mathcal{O})$ as a *T1* topological space.

(2) If $(X, \mathcal{O})$ satisfies (T2) we refer to $(X, \mathcal{O})$ as a *Hausdorff* topological space.

(3) If $(X, \mathcal{O})$ satisfies both (T1) and (T3) we refer to $(X, \mathcal{O})$ as a *regular* topological space.

($3\frac{1}{2}$) If $(X, \mathcal{O})$ satisfies both (T1) and (T$3\frac{1}{2}$) we refer to $(X, \mathcal{O})$ as a *completely regular* topological space.

(4) If $(X, \mathcal{O})$ satisfies both (T1) and (T4) we refer to $(X, \mathcal{O})$ as a *normal* topological space.

(6) If $(X, \mathcal{O})$ satisfies both (T1) and (T6) we refer to $(X, \mathcal{O})$ as a *perfectly normal* topological space.

11.2 T0 and T1 topological spaces

Examples 11.2.

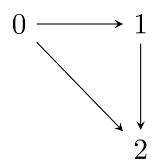
(1) Let $<$ be a pre-order on a set X. Let $\mathcal{O}$ denote the corresponding topology on X defined in Question 8 of Exercise Sheet 1.

A *partial order* is a pre-order $<$ such that if both $x < x'$ and $x' < x$ then $x = x'$. In other words there is at most one arrow between every pair of elements of X.

For instance the the pre-orders

$$0 \longrightarrow 1$$

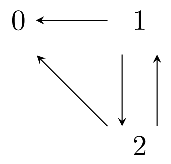
and



are partial orders. The pre-orders

$$0 \rightleftarrows 1$$

and



are not partial orders.

The topological space $(X, \mathcal{O})$ is a T0 topological space if and only if $<$ is a partial order. Let us prove this.

By definition a subset U of X belongs to $\mathcal{O}$ if and only if for all $x, x' \in X$ we have that if $x \in X$ and $x < x'$ then $x' \in X$.

Suppose that $<$ is not a partial order. Then for some $x, x' \in X$ with $x \neq x'$ we have that $x < x'$ and $x' < x$. By the previous paragraph, every neighbourhood of x in $(X, \mathcal{O})$ will contain x', and every neighbourhood of x' in $(X, \mathcal{O})$ will contain x. Hence $(X, \mathcal{O})$ is not a T0 topological space.

Suppose instead that $<$ is a partial order. Then for every $x, x' \in X$ with $x \neq x'$ we have either:

(i) $x < x'$ and $x' \nless x$

(ii) $x' < x$ and $x \nless x'$.

Without loss of generality — we may simply relabel x and x' — let us assume that (i) holds. Then the neighbourhood $U^x = \{x'' \in X \mid x < x''\}$ of x in $(X, \mathcal{O})$ contains x but not x'. We conclude that $(X, \mathcal{O})$ is T0.

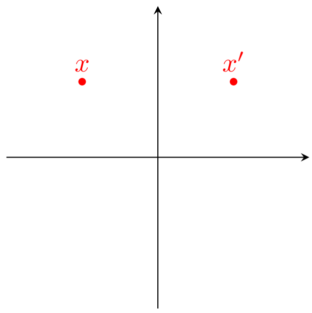
In Question 10 of Exercise Sheet 2 we saw that Alexandroff topological spaces correspond exactly to pre-orders. In this way we obtain a characterisation of T0 Alexandroff topological spaces.

(2) Let X be a set, and let $\mathcal{O}^{\text{indis}}$ denote the indiscrete topology on X. Then $(X, \mathcal{O})$ is not T0. Let us prove this.

Let $x, x' \in X$. The only neighbourhood of x in $(X, \mathcal{O}^{\text{indisc}})$ is X itself, which contains x'. Similarly the only neighbourhood of x' in $(X, \mathcal{O}^{\text{indisc}})$ is X itself, which contains x.

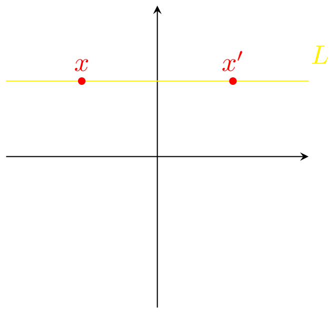
(3) Let $\mathcal{O}$ denote the topology on $\mathbb{R}^2$ generated in the sense of Question 5 of Exercise Sheet 2 by straight lines parallel to the x-axis. Then $(\mathbb{R}^2, \mathcal{O})$ is not T0.

Let us prove this. Let $(x, y) \in \mathbb{R}^2$ and $(x', y) \in \mathbb{R}^2$ be such that $x < x'$.



Every neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O})$ contains the straight line

$$L = \{(x'', y \mid x'' \in \mathbb{R}\}.$$



Moreover L is contained in every neighbourhood of (x', y) in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Thus there is no neighbourhood of (x, y) in $(\mathbb{R}^2, \mathcal{O})$ which does not contain (x', y), and no neighbourhood of (x', y) in $(\mathbb{R}^2, \mathcal{O})$ which does not contain (x, y).

Observation 11.3. Every T1 topological space is a T0 topological space.

Examples 11.4.

(1) Let X be a set, and let $<$ be a pre-order on X. Let $\mathcal{O}$ denote the corresponding topology on X defined in Question 8 of Exercise Sheet 1.

The topological space $(X, \mathcal{O})$ is T1 if and only $<$ is equality, by which we mean that $x < x'$ if and only if $x = x'$. Let us prove this.

Suppose that $<$ is not equality. Then there is a pair (x, x') of elements of X such that $x < x'$ but $x \neq x'$. By definition of $\mathcal{O}$, which we recalled in Examples 11.2 (1), every neighbourhood of x contains x'. Thus $(X, \mathcal{O})$ is not T1.

Suppose instead that $<$ is equality. Then $\mathcal{O}$ is the discrete topology on X. In particular, for any pair (x, x') of elements of X, the singleton set $\{x\}$ is a neighbourhood of x in $(X, \mathcal{O})$ which does not contain x'. Thus $(X, \mathcal{O})$ is T1.

Let us summarise.

(a) If $<$ is not a partial order, then $(X, \mathcal{O})$ is not a T0 topological space.

(b) If $<$ is a partial order which is not equality then $(X, \mathcal{O})$ is a T0 topological space but is not a T1 topological space.

(c) If $<$ is equality, then $(X, \mathcal{O})$ is a T1 topological space — and in particular a T0 topological space.

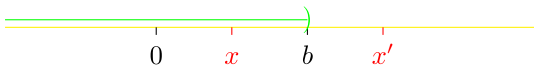
(2) Let $\mathcal{O}$ be the topology on $\mathbb{R}$ generated by

$$\mathcal{O}' = \{(a, \infty) \mid a < 0\} \cup \{(-\infty, b) \mid b > 0\}.$$

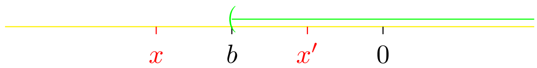
Then $(\mathbb{R}, \mathcal{O})$ is a T0 topological space. Let us prove this.

Suppose that $x, x' \in X$ and that $x \neq x'$. Without loss of generality — we may relabel x and x' if necessary — suppose that $x < x'$. We have the following two cases.

(a) If $x' > 0$ then for any $b \in \mathbb{R}$ such that $x < b$ and $0 < b < x'$ we have that $(-\infty, b)$ is a neighbourhood of x in $(\mathbb{R}, \mathcal{O})$ which does not contain x'.



(b) If $x' \leq 0$ then for any $x < a < x'$ we have that (a, ∞) is a neighbourhood of x' in $(\mathbb{R}, \mathcal{O})$ which does not contain x.



In each case we have a neighbourhood of either x or x' in $(\mathbb{R}, \mathcal{O})$ which does not contain both x and x'. Thus $(\mathbb{R}, \mathcal{O})$ is T0.

However $(\mathbb{R}, \mathcal{O})$ is not T1. For any $x \in \mathbb{R}$ we have that every neighbourhood of x in $(\mathbb{R}, \mathcal{O})$ contains 0. Thus for any $x \in \mathbb{R}$ with $x \neq 0$ we cannot find a neighbourhood of x in $(\mathbb{R}, \mathcal{O})$ which does not contain 0.

Proposition 11.5. Let $(X, \mathcal{O})$ be a topological space. The following are equivalent.

(1) $(X, \mathcal{O})$ is T1.

(2) The singleton set $\{x\}$ is closed in X for every $x \in X$.

(3) Every finite subset A of X is a closed subset of X.

(4) For every subset A of X we have that

$$A = \bigcap_{U \in \mathcal{O} \text{ and } A \subset U} U.$$

Proof. It suffices to prove the following implications.

(1) ⇒ (2) Suppose that $(X, \mathcal{O})$ is T1. Let $x' \in X$ be such that $x \neq x'$. Since $(X, \mathcal{O})$ is T1 there is a neighbourhood U of x' in $(X, \mathcal{O})$ which does not contain x. Thus x' is not a limit point of $\{x\}$ in $(X, \mathcal{O})$.

(2) ⇒ (3) Suppose that (2) holds. Let A be a finite subset of X. Let $x \in X \setminus A$. For each $a \in A$ there is by (2) a neighbourhood U_a of x in $(X, \mathcal{O})$ such that $a \notin U_x$.

Let $U = \bigcap_{a \in A} U_a$. Since A is finite we have U is open in X. Moreover we have that $x \in U$ and that $U \cap A = \emptyset$. Thus x is not a limit point of A in $(X, \mathcal{O})$.

(3) ⇒ (2) Clear.

(2) ⇒ (4) Suppose that (2) holds. Let A be a subset of X. Let

$$A' = \bigcap_{U \in \mathcal{O} \text{ and } A \subset U} U.$$

Let $x \in X \setminus A$. By (2) we have that $\{x\}$ is closed in X. Thus $X \setminus \{x\}$ is open in X. Hence

$$A' \subset \bigcap_{x \in X \setminus A} X \setminus \{x\}.$$

We have that

$$\begin{aligned} \bigcap_{x \in X \setminus A} X \setminus \{x\} &= X \setminus \bigcup_{x \in X \setminus A} \{x\} \\ &= X \setminus \big(X \setminus A\big) \\ &= A. \end{aligned}$$

Thus $A' \subset A$. It is moreover clear that $A \subset A'$. We conclude that $A = A'$.

(4) ⇒ (1) Suppose that (4) holds. Let (x, x') be an ordered pair of elements of X be such that $x \neq x'$.

By (4) we have that

$$\{x\} = \bigcap_{U \in \mathcal{O} \text{ and } x \in U} U.$$

In particular we have that

$$x' \notin \bigcap_{U \in \mathcal{O} \text{ and } x \in U} U.$$

Thus there is a neighbourhood of x in $(X, \mathcal{O})$ which does not contain x'.

□

11.3 Hausdorff topological spaces

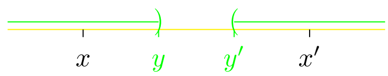
Observation 11.6. Every Hausdorff topological space is a T1 topological space. In particular every Hausdorff topological space is a T0 topological space.

Examples 11.7.

(1) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is a Hausdorff topological space. Let us prove this.

Let $x, x' \in \mathbb{R}$ be such that $x \neq x'$. Without loss of generality — we may relabel x and x' — let us assume that $x < x'$.

Let $y, y' \in \mathbb{R}$ be such that $x < y \leq y' < x'$. We then have that $x \in (-\infty, y)$, that $x' \in (y', \infty)$, and that $(-\infty, y) \cap (y', \infty) = \emptyset$.



(3) Let $\mathsf{Spec}(\mathbb{Z})$ denote the set of primes. Let $\mathcal{O}$ denote the topology on $\mathsf{Spec}(\mathbb{Z})$ defined in Question 9 of Exercise Sheet 1. Recall that

$$\mathcal{O} = \left\{ \mathsf{Spec}(\mathbb{Z}) \setminus V(n) \mid n \in \mathbb{Z} \right\}$$

where

$$V(n) = \{ p \in \mathsf{Spec}(\mathbb{Z}) \mid p \mid n \}.$$

The topological space $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ is T1. Let us prove this.

For any $n \in \mathbb{Z}$ we have that $V(n)$ is closed in $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$. In particular for any $p \in \mathsf{Spec}(\mathbb{Z})$ we have that $V(p)$ is closed in $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$. Thus since $V(p) = \{p\}$ we have that $\{p\}$ is a closed subset of $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ for every $p \in \mathsf{Spec}(\mathbb{Z})$.

By Proposition 11.5 we conclude that $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ is T1. However $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ is not Hausdorff. Let us prove this.

Let $p, p' \in \mathsf{Spec}(\mathbb{Z})$. The neighbourhoods of p in $\left(\mathsf{Spec}(\mathbb{Z}), \mathcal{O}\right)$ are the sets

$$\mathsf{Spec}(\mathbb{Z}) \setminus V(n)$$

for which $p \nmid n$. The neighbourhoods of p' in $(\mathsf{Spec}(\mathbb{Z}), \mathcal{O})$ are the sets

$$\mathsf{Spec}(\mathbb{Z}) \setminus V(n')$$

for which $p' \nmid n'$.

Let $n, n' \in \mathbb{Z}$ be such that $p \nmid n$ and $p' \nmid n'$. Let $p'' \in \mathsf{Spec}(\mathbb{Z})$ be such that $p'' \nmid n$ and $p'' \nmid n'$. For example, we may take p'' to be any prime larger than n and n'.

Then $p'' \in \mathsf{Spec}(\mathbb{Z}) \setminus V(n)$ and $p'' \in \mathsf{Spec}(\mathbb{Z}) \setminus V(n')$. Hence

$$\big(\mathsf{Spec}(\mathbb{Z}) \setminus V(n)\big) \cap \big(\mathsf{Spec}(\mathbb{Z}) \setminus V(n')\big) \neq \emptyset.$$

We have shown that for every neighbourhood U of p in $(\mathsf{Spec}(\mathbb{Z}), \mathcal{O})$ and every neighbourhood U' of p' in $(\mathsf{Spec}(\mathbb{Z}), \mathcal{O})$ we have that $U \cap U' \neq \emptyset$. Thus $(\mathsf{Spec}(\mathbb{Z}), \mathcal{O})$ is not Hausdorff.

Notation 11.8. Let X be a set. We denote by $\Delta(X)$ the subset

$$\{(x, x) \in X \times X \mid x \in X\}$$

of $X \times X$.

Proposition 11.9. Let $(X, \mathcal{O}_X)$ be a topological space. Then $(X, \mathcal{O}_X)$ is Hausdorff if and only if $\Delta(X)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$.

Proof. Suppose that $(X, \mathcal{O}_X)$ is Hausdorff. Let $(x, x') \in (X \times X) \setminus \Delta(X)$. By definition of $\Delta(X)$, we have that $x \neq x'$. Since $(X, \mathcal{O})$ is Hausdorff there is a neighbourhood U of x in $(X, \mathcal{O})$ and a neighbourhood U' of x' in $(X, \mathcal{O})$ such that $U \cap U' = \emptyset$.

We have that $\Delta(X) \cap (U \times U') = \Delta(U \cap U') = \emptyset$. Hence (x, x') is not a limit point of $\Delta(X)$. We deduce that $\Delta(X) = \overline{\Delta(X)}$. By Proposition 5.7 we conclude that $\Delta(X)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$.

Suppose instead that $\Delta(X)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$. For any $x, x' \in X$ with $x \neq x'$ we have by Proposition 5.7 that (x, x') is not a limit point of $\Delta(X)$. Hence there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ and a neighbourhood U' of x' in $(X, \mathcal{O}_X)$ such that $\Delta(X) \cap (U \times U') = \emptyset$.

Appealing again to the fact that $\Delta(X) \cap (U \times U') = \Delta(U \cap U')$ we deduce that $\Delta(U \cap U') = \emptyset$. Hence $U \cap U' = \emptyset$. We conclude that $(X, \mathcal{O})$ is Hausdorff.

□

Proposition 11.10. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology $\mathcal{O}_A$ on A with respect to $(X, \mathcal{O}_X)$. Then $(A, \mathcal{O}_A)$ is a Hausdorff topological space.

Proof. Exercise. □

Proposition 11.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then

$$(X \times Y, \mathcal{O}_{X \times Y})$$

is Hausdorff if and only if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are both Hausdorff.

Proof. Exercise. □

Examples 11.12. By Examples 11.7 (1) we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. By Proposition 11.10 and Proposition 11.11 we deduce that all the topological spaces of Examples 1.38 are Hausdorff.

Proposition 11.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a bijection which is an open map. If $(X, \mathcal{O}_X)$ is Hausdorff then $(Y, \mathcal{O}_Y)$ is Hausdorff.

Proof. Exercise. □

Corollary 11.14. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. If $(X, \mathcal{O}_X)$ is Hausdorff then $(Y, \mathcal{O}_Y)$ is Hausdorff.

Proof. Follows immediately from Proposition 11.13 since a homeomorphism is in particular a bijection which is an open map. □

Proposition 11.15. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X \sqcup Y, \mathcal{O}_{X \sqcup Y})$ is Hausdorff if and only if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are both Hausdorff.

Proof. Exercise. □