Generell Topologi

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May 6, 2013

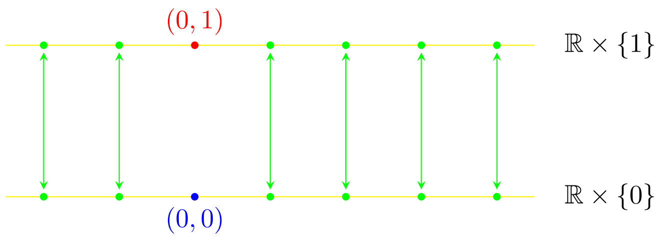
12 Thursday 21st February

12.1 Quotients of Hausdorff topological spaces

Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space, and let $\sim$ be an equivalence relation on X. Then $(X/\sim, \mathcal{O}_{X/\sim})$ is not necessarily Hausdorff.

Example 12.1. Recall from Recollection 5.17 that $\mathbb{R}\sqcup\mathbb{R}$ is the set $\big(\mathbb{R}\times\{0\}\big)\cup\big(\mathbb{R}\times\{1\}\big)$. By Examples 11.7 (1) we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. Thus by Proposition 11.15 we have that $(\mathbb{R}\sqcup\mathbb{R}, \mathcal{O}_{\mathbb{R}\sqcup\mathbb{R}})$ is Hausdorff.

Let $\sim$ be the equivalence relation on $\mathbb{R}\sqcup\mathbb{R}$ defined by $(x,0)\sim(x,1)$ for all $x\neq 0$. To put it slightly less formally, we have two copies of $\mathbb{R}$ and identify every real number except zero in the first copy of $\mathbb{R}$ with the same real number in the second copy of $\mathbb{R}$.



The topological space $\big((\mathbb{R}\sqcup\mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$ is known as the *real line with two origins*. It is not Hausdorff — indeed is not even T0. Let us prove this.

To avoid confusion let us for the remainder of this example adopt the notation $]a,b[$ for the open interval from a real number a to a real number b, rather than our usual (a,b). Let

$$\mathbb{R}\sqcup\mathbb{R} \xrightarrow{\pi} (\mathbb{R}\sqcup\mathbb{R})/\sim$$

denote the quotient map.

Let U be a neighbourhood of $\pi\big((0,0)\big)$ in $\big((\mathbb{R}\sqcup\mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$. We claim that $\pi\big((0,1)\big)\in U$. First let us make two observations.

(1) By definition of $\mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}$ we have that $\pi^{-1}(U)$ is open in $\big(\mathbb{R}\sqcup\mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$.

(2) By definition of $\mathcal{O}_{\mathbb{R}}$ we have that

$$\big\{\,]a,b[\mid a,b\in\mathbb{R}\big\}$$

is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Hence — see the Exercise Sheet — we have that

$$\big\{\,]a,b[\times\{0\} \mid a,b\in\mathbb{R}\big\}\cup\big\{\,]a,b[\times\{1\} \mid a,b\in\mathbb{R}\big\}$$

is a basis for $(\mathbb{R}\sqcup\mathbb{R}, \mathcal{O}_{\mathbb{R}\sqcup\mathbb{R}})$.

(3) We have that $(0,0) \in \pi^{-1}(U)$.

By (1) – (3) together with Question 3 (a) on Exercise Sheet 2 we deduce there are $a, b \in \mathbb{R}$ such that $0 \in \,]a,b[$ and $]a,b[\, \times \{0\} \subset \pi^{-1}(U)$.

From the latter we deduce that

$$\pi\big(\,]a,b[\, \times \{0\}\big) \subset \pi\big(\pi^{-1}(U)\big) = U.$$

Thus

$$\pi^{-1}\Big(\pi\big(\,]a,b[\, \times \{0\}\big)\Big) \subset \pi^{-1}(U).$$

Moreover

$$\pi^{-1}\Big(\pi\big(\,]a,b[\, \times \{0\}\big)\Big) = \big(\,]a,b[\, \times \{0\}\big) \sqcup \big(\,]a,b[\, \times \{1\}\big).$$

Since $0 \in \,]a,b[$ we have that $(0,1) \in \,]a,b[\, \times \{1\}$. We deduce that $(0,1) \in \pi^{-1}(U)$, and hence that $\pi\big((0,1)\big) \in U$ as claimed.

We have now proven that every neighbourhood of $\pi\big((0,0)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$ contains $\pi\big((0,1)\big)$. Thus $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$ is not T1. An entirely analogous argument demonstrates that every neighbourhood of $\pi\big((0,1)\big)$ in $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$ contains $\pi\big((0,0)\big)$. We conclude that $\big((\mathbb{R} \sqcup \mathbb{R})/\sim, \mathcal{O}_{(\mathbb{R}\sqcup\mathbb{R})/\sim}\big)$ is not T0.

Notation 12.2. Let X be a set and let $\sim$ be a relation on X. Let

$$R_\sim := \{(x, x') \in X \times X \mid x \sim x'\}.$$

Proposition 12.3. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let $\sim$ be an equivalence relation on X. If $(X/\sim, \mathcal{O}_{X/\sim})$ is a Hausdorff topological space then $R_\sim$ is a closed subset of $(X \times X, \mathcal{O}_{X \times X})$.

Proof. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map.

Let

$$X \times X \xrightarrow{\pi \times \pi} (X/\sim) \times (X/\sim)$$

be the map given by $(x, x') \mapsto \big(\pi(x), \pi(x')\big)$. By Observation 3.7 we have that π is continuous. By Question 4 (c) on Exercise Sheet 3 we deduce that $\pi \times \pi$ is continuous.

If $X/\sim$ is Hausdorff then by Proposition 11.9 we have that $\Delta(X/\sim)$ is closed in

$$(X/\sim) \times (X/\sim).$$

By Question 1 (a) on Exercise Sheet 3 we deduce that $(\pi \times \pi)^{-1}\Big(\Delta(X/\sim)\Big)$ is closed in $X \times X$. Note that $R_\sim = (\pi \times \pi)^{-1}\Big(\Delta(X/\sim)\Big)$. We conclude that $R_\sim$ is closed in $X \times X$. ☐

Remark 12.4. We will shortly introduce compact topological spaces. If $(X, \mathcal{O}_X)$ is compact we will see in a later lecture that the requirement that $R_\sim$ be closed in $(X \times X, \mathcal{O}_{X\times X})$ is not only necessary but sufficient to ensure that if $(X, \mathcal{O}_X)$ is Hausdorff then $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff.

That $R_\sim$ be closed in $(X \times X, \mathcal{O}_{X\times X})$ is not sufficient in general to ensure that $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff, as the following example demonstrates.

Example 12.5. For the purposes of this example let $\mathbb{N}$ be the set $\{1, 2, \ldots\}$, namely the set of natural numbers without 0. Let $\Sigma = \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$. Let $\mathcal{O}'$ be the set

$$\big\{(a,b) \mid a,b \in \mathbb{R}\big\} \cup \big\{(a,b) \setminus \big((a,b) \cap \Sigma\big) \mid a,b \in \mathbb{R}\big\}.$$

Then $\mathcal{O}'$ satisfies the conditions of Proposition 2.2. Let $\mathcal{O}$ denote the corresponding topology on $\mathbb{R}$ with basis $\mathcal{O}'$. Note that $\mathcal{O}_\mathbb{R} \subset \mathcal{O}$. Since $(\mathbb{R}, \mathcal{O}_\mathbb{R})$ is Hausdorff we deduce that $(\mathbb{R}, \mathcal{O})$ is Hausdorff.

Let $\sim$ be the equivalence relation on $\mathbb{R}$ generated by $1 \sim \frac{1}{n}$ for all $n \in \mathbb{N}$. Let $\mathcal{O}_\sim$ denote the quotient topology on $\mathbb{R}/\sim$ with respect to $(\mathbb{R}, \mathcal{O})$ and $\sim$. Then:

(1) $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ is not Hausdorff.

(2) $R_\sim$ is closed in $(\mathbb{R}^2, \mathcal{O}^2)$, where $\mathcal{O}^2$ denotes the product topology on $\mathbb{R}^2$ with respect to $(\mathbb{R}, \mathcal{O})$ and $(\mathbb{R}, \mathcal{O})$.

We shall first prove (1). Let

$$\mathbb{R} \xrightarrow{\ \pi\ } \mathbb{R}/\sim$$

be the quotient map. Let U be a neighbourhood of $\pi(1)$ in $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ and let U' be a neighbourhood of $\pi(0)$ in $(\mathbb{R}, \mathcal{O}_\sim)$. We claim that $U \cap U' \neq \emptyset$. Let us prove this.

Since π is continuous we have that $\pi^{-1}(U)$ is open in $(\mathbb{R}, \mathcal{O})$. Moreover we have that

$$\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} = \pi^{-1}\big(\pi(1)\big) \subset \pi^{-1}(U).$$

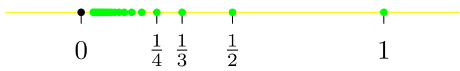
Let $n \in \mathbb{N}$. By Question 3 (a) of Exercise Sheet 2 there is a $U_n \in \mathcal{O}'$ such that $\frac{1}{n}$ in $(\mathbb{R}, \mathcal{O})$ and such that $U_n \subset \pi^{-1}(U)$. We make the following observations.

(i) Since $U_n \subset \pi^{-1}(U)$ for all $n \in \mathbb{N}$ we have that $\bigcup_{n\in\mathbb{N}} U_n \subset \pi^{-1}(U)$.

(ii) Since $\frac{1}{n}$ does not belong to $(a,b) \setminus \Sigma$ for any $a, b \in \mathbb{R}$ we have that $U_n \in \mathcal{O}_\mathbb{R}$.

By (ii) we have that $U_n = (a_n, b_n)$ for some $a_n, b_n \in \mathbb{R}$. Moreover we have that that

$$\inf\Big(\bigcup_{n\in\mathbb{N}} U_n\Big) \leq \inf\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} = 0.$$



Since π is continuous we have that $\pi^{-1}(U')$ is open in $(\mathbb{R}, \mathcal{O})$ and that $0 \in \pi^{-1}(U')$. By Question 3 (a) of Exercise Sheet 2 there is a $W \in \mathcal{O}'$ such that $0 \in W$ and such that $W \subset \pi^{-1}(U')$.

By definition of $\mathcal{O}'$ there are $a, b \in \mathbb{R}$ that $W = (a, b)$ or $W = (a, b) \setminus \Sigma$. Either way, since $0 \in W$ we must have that $a < 0$ and $b > 0$. We also have that

$$\inf\left(\pi^{-1}(U)\right) \leq \inf\Big(\bigcup_{n \in \mathbb{N}} U_n\Big) \leq 0.$$

Any $x \in \mathbb{R}$ such that $x \notin \Sigma$ and $0 < x < b$ belongs to $W \cap \pi^{-1}(U')$. Thus

$$W \cap \pi^{-1}(U') \neq \emptyset.$$

Hence $\pi^{-1}(U) \cap \pi^{-1}(U') \neq \emptyset$, Since $\pi^{-1}(U \cap U') = \pi^{-1}(U) \cap \pi^{-1}(U')$ we deduce that $\pi^{-1}(U \cap U') \neq \emptyset$. Thus $U \cap U' \neq \emptyset$ as claimed.

We have now proven that for any neighbourhood U of $\pi(0)$ in $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ and any neighbourhood U' of $\pi(1)$ in $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ we have that $U \cap U \neq \emptyset$. Thus $(\mathbb{R}/\sim, \mathcal{O}_\sim)$ is not Hausdorff.

Let us now prove (2). We claim that Σ is closed in $(\mathbb{R}, \mathcal{O})$. Let us prove this.

(i) If $x \in \mathbb{R}$ is a limit point of Σ in $(\mathbb{R}, \mathcal{O})$ then x is a limit point of Σ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The only limit point of Σ in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ which does not belong to Σ is 0.

(ii) Let $a < 0$ and $b > 0$ be real numbers. Then $(a, b) \setminus \Sigma$ is a neighbourhood of 0 in $(\mathbb{R}, \mathcal{O})$. We have that $\big((a, b) \setminus \Sigma\big) \cap \Sigma = \emptyset$. Thus 0 is not a limit point of Σ in $(\mathbb{R}, \mathcal{O})$.

Thus every limit point of Σ in $(\mathbb{R}, \mathcal{O})$ belongs to Σ. By Proposition 5.7 we deduce that Σ is closed in $(\mathbb{R}, \mathcal{O})$ as claimed. By Question 5 of Exercise Sheet 3 we thus have that $\Sigma \times \Sigma$ is closed in $(\mathbb{R}^2, \mathcal{O}^2)$. Moreover note that $R_\sim = \Sigma \times \Sigma$. We conclude that $R_\sim$ is closed in $(\mathbb{R}^2, \mathcal{O}^2)$.

12.2 Compact topological spaces

Terminology 12.6. Let $(X, \mathcal{O})$ be a topological space. An *open covering* of X is a set $\{U_j\}_{j \in J}$ of open subsets of X such that $X = \bigcup_{j \in J} U_j$.

Definition 12.7. A topological space $(X, \mathcal{O})$ is *compact* if for every open covering $\{U_j\}_{j \in J}$ of X there is a finite subset J' of J such that $X = \bigcup_{j' \in J'} U_{j'}$.

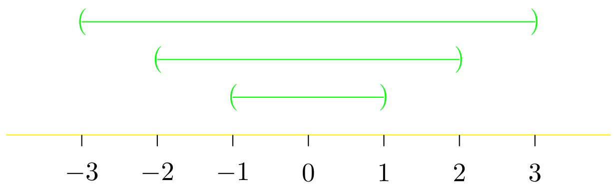
Terminology 12.8. Let $(X, \mathcal{O})$ be a topological space, and let $\{U_j\}_{j \in J}$ be an open covering of X. Suppose that there is a finite subset J' of J such that $X = \bigcup_{j' \in J'} U_{j'}$. We write that $\{U_{j'}\}_{j' \in J'}$ is a *finite subcovering* of $\{U_j\}_{j \in J}$.

Examples 12.9.

(1) Let $(X, \mathcal{O})$ be a topological space. If $\mathcal{O}$ is finite then $(X, \mathcal{O})$ is compact. For if $\mathcal{O}$ is finite then every set $\{U_j\}_{j \in J}$ of open subsets of X is finite.

In particular if X is finite then $(X, \mathcal{O})$ is compact. For if X is finite there are only finitely many subsets of X, and thus $\mathcal{O}$ is finite.

(2) $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact. The open covering $\{(-n, n)\}_{n \in \mathbb{N}}$ of $\mathbb{R}$ has no finite subcovering for instance.

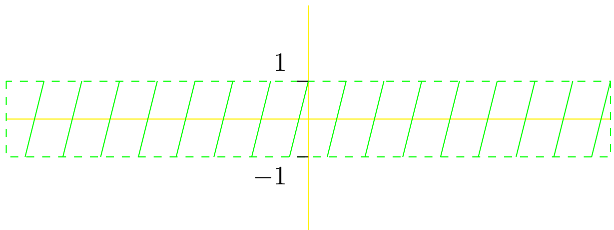


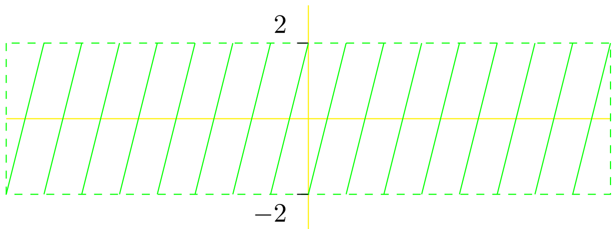
(3) $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is not compact. The open covering of $\mathbb{R}^2$ given by

$$\left\{ \mathbb{R} \times (-n, n) \right\}_{n \in \mathbb{N}}$$

has no finite subcovering for instance.

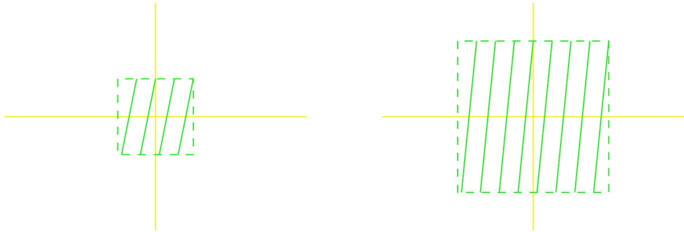
This open covering consists of horizontal strips of increasing height.





A different open covering of $\mathbb{R}^2$ which has no finite subcovering is given by

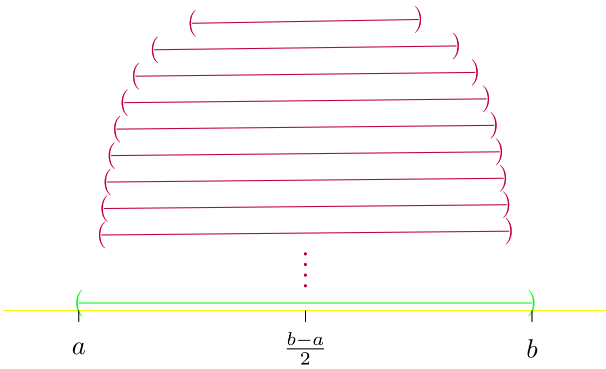
$$\left\{ (-n, n) \times (-n, n) \right\}_{n \in \mathbb{N}}.$$



(4) An open interval $\big((a,b),\mathcal{O}_{(a,b)}\big)$, where $\mathcal{O}_{(a,b)}$ denotes the subspace topology with respect to $(\mathbb{R},\mathcal{O}_{\mathbb{R}})$, is not compact for any $a,b\in\mathbb{R}$. The open covering of (a,b) given by

$$\big\{(a+\frac{1}{n},b-\frac{1}{n})\big\}_{n\in\mathbb{N}\text{ and }\frac{1}{n}<\frac{b-a}{2}}$$

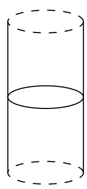
has no finite subcovering for instance.



(5) Generalising (2) let $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ be topological spaces, one of which is not compact. Then $(X\times Y,\mathcal{O}_{X\times Y})$ is not compact.

Suppose for example that $(X,\mathcal{O}_X)$ is not compact. Let $\{U_j\}_{j\in J}$ be an open covering of X which does not admit a finite subcovering. Then $\big\{U_j\times Y\big\}_{j\in J}$ is an open covering of $X\times Y$ which does not admit a finite subcovering.

For instance let $(S^1\times(0,1),\mathcal{O}_{S^1\times(0,1)})$ be a cylinder with the two circles at its ends cut out.

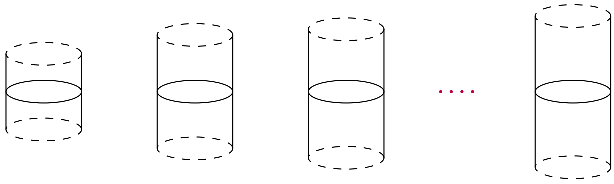


Since $(0,1)$ is not compact by (4) we have that $(S^1\times(0,1),\mathcal{O}_{S^1\times(0,1)})$ is not compact.

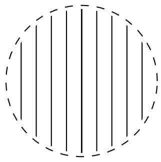
The open covering

$$\left\{ S^1 \times \left(\frac{1}{n}, 1 - \frac{1}{n}\right) \right\}_{n \in \mathbb{N} \text{ and } n \geq 2}$$

of $S^1 \times (0,1)$ is pictured below. It does not admit a finite subcovering.



(6) Let $D^2 \setminus S^1$ be the disc D^2 with its boundary circle removed.



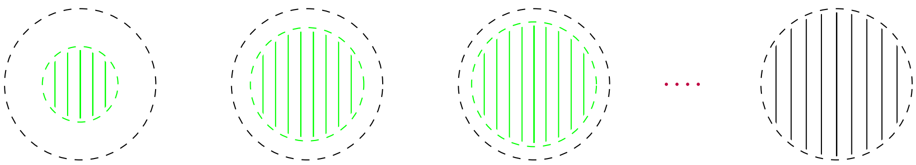
In other words $D^2 \setminus S^1$ is

$$\left\{ (x,y) \in \mathbb{R}^2 \mid \|(x,y)\| < 1 \right\}$$

equipped with the subspace topology $\mathcal{O}_{D^2 \setminus S^1}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Then $(D^2 \setminus S^1, \mathcal{O}_{D^2 \setminus S^1})$ is not compact. The open covering

$$\left\{ (x,y) \in \mathbb{R}^2 \mid \|(x,y)\| < 1 - \frac{1}{n} \right\}_{n \in \mathbb{N}}$$

of $D^2 \setminus S^1$ does not admit a finite subcovering for instance.



Proposition 12.10. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a surjective continuous map. If $(X, \mathcal{O}_X)$ is compact then $(Y, \mathcal{O}_Y)$ is compact.

Proof. Let $\{U_j\}_{j \in J}$ be an open covering of Y. Since f is continuous we have that $f^{-1}(U_j) \in \mathcal{O}_X$ for all $j \in J$. Moreover

$$\begin{aligned}\bigcup_{j \in J} f^{-1}(U_j) &= f^{-1}(\bigcup_{j \in J} U_j) \\ &= f^{-1}(Y) \\ &= X.\end{aligned}$$

Thus $\left\{f^{-1}(U_j)\right\}_{j \in J}$ is an open covering of X.

Since $(X, \mathcal{O}_X)$ is compact there is a finite subset J' of J such that $\left\{f^{-1}(U_{j'})\right\}_{j' \in J'}$ is an open covering of X. We have that

$$\begin{aligned}\bigcup_{j' \in J'} U_{j'} &= \bigcup_{j' \in J'} f\big(f^{-1}(U_{j'})\big) \\ &= f\Big(\bigcup_{j' \in J'} f^{-1}(U_{j'})\Big) \\ &= f(X) \\ &= Y.\end{aligned}$$

Thus $\{U_{j'}\}_{j' \in J'}$ is an open covering of Y. □

Corollary 12.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. If $(X, \mathcal{O}_X)$ is compact then $(Y, \mathcal{O}_Y)$ is compact.

Proof. Follows immediately from Proposition 12.10, since by Proposition 3.15 a homeomorphism is in particular surjective and continuous. □

Remark 12.12. Let the open interval (a, b) for $a, b \in \mathbb{R}$ be equipped with its subspace topology $\mathcal{O}_{(a,b)}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Examples 4.7 (6) we have that $\big((a, b), \mathcal{O}_{(a,b)}\big)$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Once we know by Examples 12.9 (2) that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact we could appeal to Corollary 12.11 to deduce that $\big((a, b), \mathcal{O}_{(a,b)}\big)$ is not compact. We observed this directly in Examples 12.9 (4).