Generell Topologi

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18 Thursday 14th March

18.1 Setting the scene

Remark 18.1. Knots have been of importance throughout human history, in our everyday life and in our artistic expression.

We can think of knot theory as an art. Our mathematical understanding of knots heightens our appreciation for these mysterious gadgets which have fascinated humans from prehistoric times.

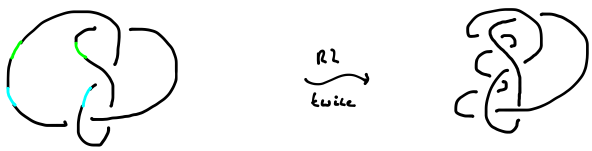
Mathematically, the theory of knots and links is of great significance in low dimensional topology, in the study of what are known as 3-manifolds and 4-manifolds. It is a bridge between representation theory, topology, and category theory which appears to be a gateway to a hitherto unexplored beautiful garden. This is of high current research interest.

18.2 Showing that two knots are isotopic through a seqeunce of Reidemeister moves

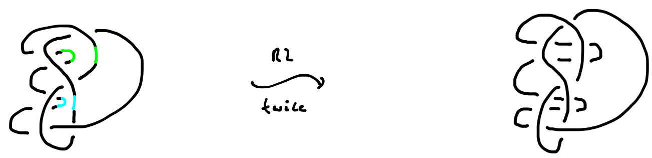
Example 18.2.

Let us explore the Reidemeister moves in practise. We shall prove that the figure of eight knot is isotopic to its mirror image.

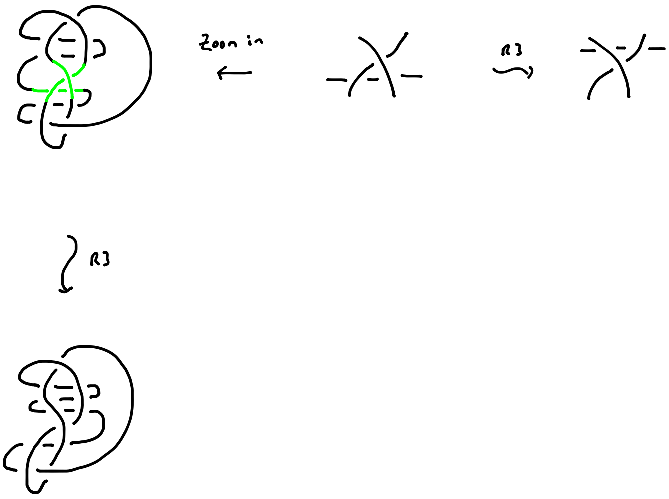
We begin with the figure of eight knot. We apply two R2 moves, one with respect to the green arcs and one with respect to the blue arcs.



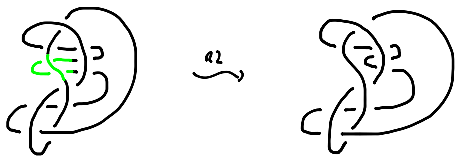
Next we once more apply two R2 moves, one with respect to the green arcs and one with respect to the blue arcs.



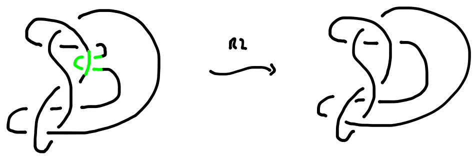
Zooming in on the green arcs below we see that we can apply an R3 move.



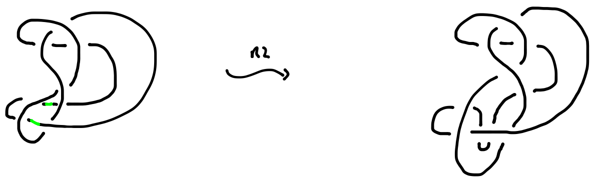
Next we apply an R2 move.



We now apply another R2 move.

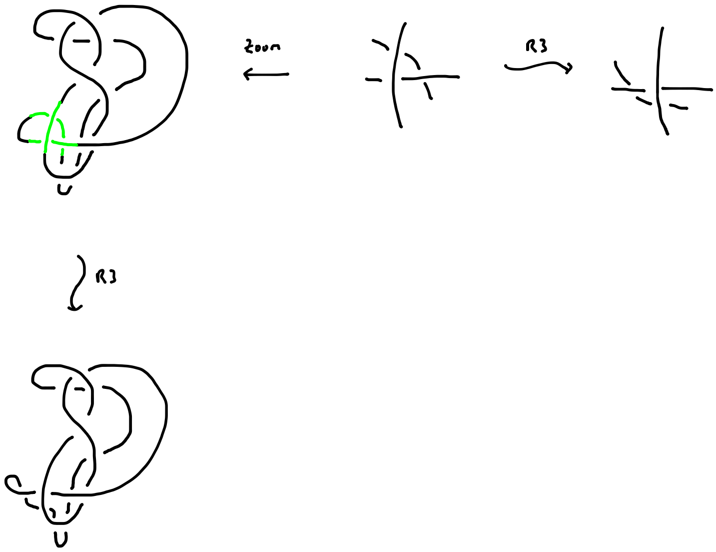


Again we apply an R2 move.

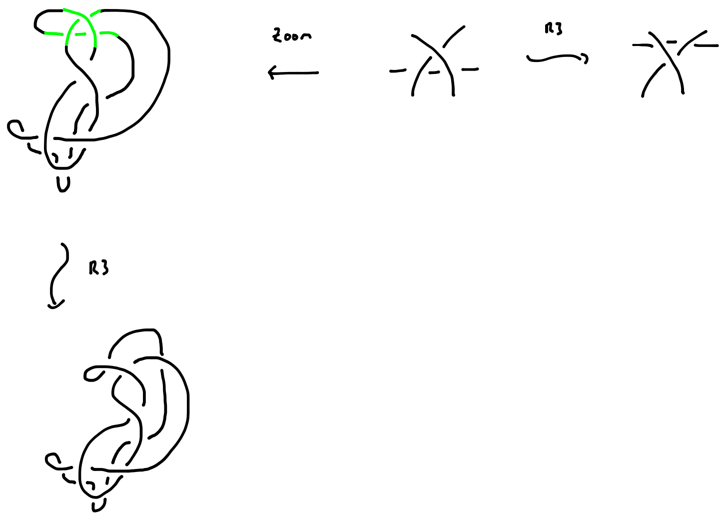


Once more we apply an R2 move.

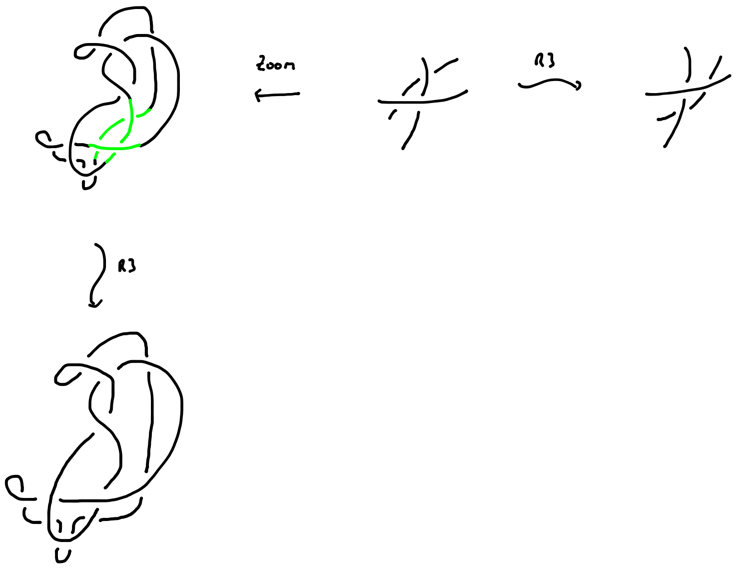
Zooming in on the green arcs below we see that we can apply an R3 move.



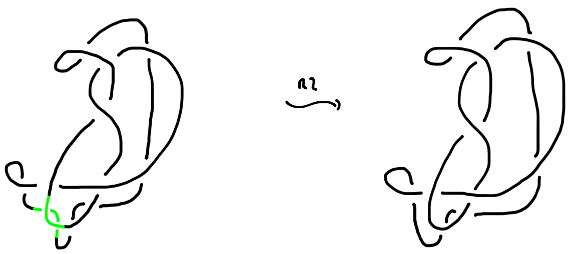
Zooming in on the green arcs below we see that we can apply another R3 move.



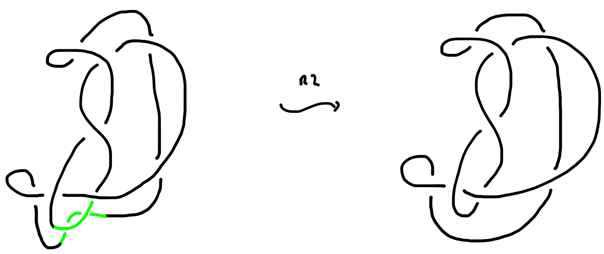
Zooming in on the green arcs below we see that we can once more apply an R3 move.



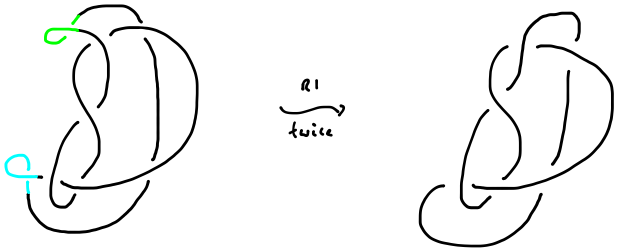
We now apply an R2 move.



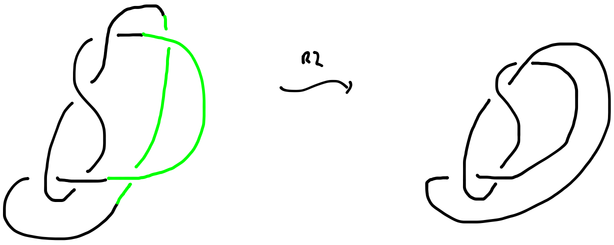
Next we apply another R2 move.



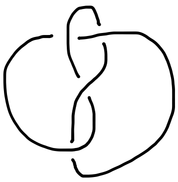
Now we apply two R1 moves, one indicated in green and one indicated in blue.



Finally we apply an R2 move.



Applying an R0 move, or in other words manipulating this knot a little, we see that we indeed have obtained the mirror image of the figure of eight knot!



Remark 18.3. This sequence of Reidemeister moves is far from the only possibility. Even for this sequence we could have carried out various moves in a different order.

Remark 18.4. A link which is isotopic to its mirror image is known as *amphichiral*. A link which is not isotopic to its mirror image is known as *chiral*.

The trefoil knot is chiral, but historically this was difficult to prove. We will later be able to prove it easily by calculating its Jones polynomial.

Remark 18.5. Note that in Example 18.2 we had to increase the number of crossings in our knot via R2 moves before we could simplify.

How do we know that we could not prove that the trefoil is isotopic to its mirror image by increasing to a very large number of crossings?

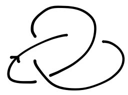
We need some tools which can tell us when one link is not isotopic to another! This is where the importance of the Reidemeister moves truly lies.

18.3 Linking number

Definition 18.6. An *oriented link* is a link $(L, \mathcal{O}_L)$ whose components have been equipped with arrows.

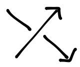
Examples 18.7.

(1) A trefoil with its two possible choices of orientation.

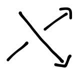


(2) A Hopf link with two choices of orientation.

Definition 18.8. Let $(L, \mathcal{O}_L)$ be an oriented link. The *sign* of a crossing



is -1, and of a crossing



is $+1$.

Remark 18.9. In Definition 18.8 the arcs of the crossings may belong to the same component of L or to distinct components of L.

Definition 18.10. Let $(L, \mathcal{O}_L)$ be an oriented link. The *linking number* of L is

$$\left| \frac{1}{2} \times \Big(\sum_{\substack{\text{crossings } C \text{ between} \\ \text{distinct components of } L}} \operatorname{sign}(C) \Big) \right|.$$

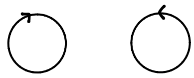
We denote it by $lk(L)$.

If there are no crossings between distinct components of L, we define $lk(L) = 0$.

Remark 18.11. It is crucial that we only allow crossings between distinct components of L in Definition 18.10. In particular, $lk(K) = 0$ for all knots $(K, \mathcal{O}_K)$.

Examples 18.12.

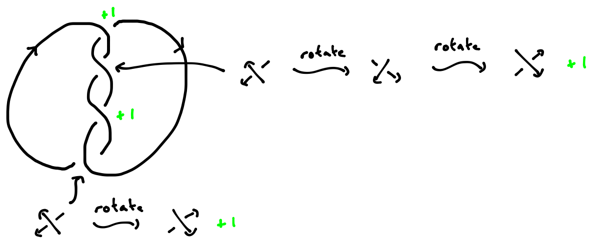
(1) The linking number of the unlink with n components is 0 for any n, since there are no crossings.



(2) The signs of the crossings for two choices of orientation on a Hopf link are as follows.

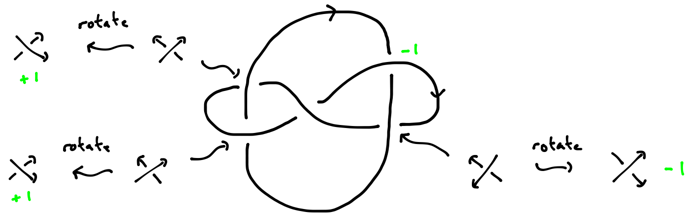
Thus the linking number of the left Hopf link is $|\frac{1}{2} \cdot (-1-1)| = 1$, and the linking number of the right Hopf link is $|\frac{1}{2} \cdot (1+1)| = 1$.

(3) Here are the signs of the crossings for another link.



Its linking number is thus $|\frac{1}{2} \cdot (1+1+1+1)| = 2$.

(4) Here are the signs of the crossings between distinct components of the Whitehead link.



Thus the linking number of the Whitehead link is $|\frac{1}{2} \cdot (1+1-1-1)| = 0$.

Let us emphasise that the sign of the middle crossing is omitted since this crossing does not involve distinct components.

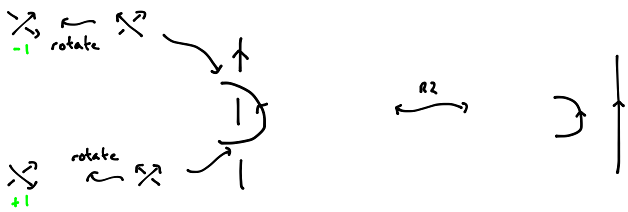
Remark 18.13. It is not a coincidence that we obtained the same linking number for the two choices of orientation on a Hopf link in Examples 18.12 (2). Though we shall not dwell on the point here, one can observe that the linking number is independent of the choice of orientation for all links.

Thus we can speak of the linking number of a link even if no orientation is specified, rather than only of the linking number of an oriented link. We just choose an orientation with which to work.

Proposition 18.14. If two links are isotopic then their linking numbers are equal.

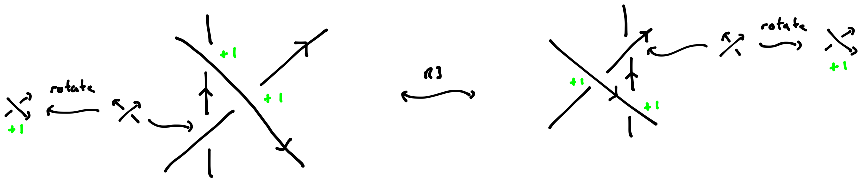
Proof. We know by Theorem 17.16 that two links are isotopic if and only if one can be obtained from the other by a finite sequence of Reidemeister moves. Thus it suffices to prove that the linking number of a link is unchanged under the Reidemeister moves.

- R1 An R1 move does not change the linking number of a link since it only involves a crossing in which both arcs belong to the same component.
- R2 The signs of the crossings in the part of a link affected by an R2 move are indicated below.



The contribution of the crossings in the left diagram to the linking number is $-1+1=0$. Thus an R2 move does not affect the linking number of a link.

- R3 The signs of the crossings in the part of a link affected by an R3 move are indicated below.



Thus the crossings in each case make the same contribution to the linking number. Hence an R3 move does not affect the linking number of a link.

□

Remark 18.15. This proof is not complete. There is another R2 and another R3 move which must be considered. Moreover the arcs could have other combinations of orientations. However, it is the idea that is important. It adapts in a straightforward way to a proof for the other cases.

Examples 18.16.

(1) We would certainly intuitively believe that the Hopf link cannot be unlinked, or in other words that it is not isotopic to the unlink with two components!

Proposition18.14 allows us to give us a rigorous proof, since by Examples 18.12 (2) the Hopf link has linking number 1, whereas the unlink with two components has linking number 0.

(2) Linking numbers do not however allow us to prove that the Whitehead link is not isotopic to the unlink with two components.

By Examples 18.12 (2) the linking number of the Whitehead link is 0, which is the same as the linking number of the unlink with two components.