Generell Topologi

Richard Williamson

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20 Thursday 21st March

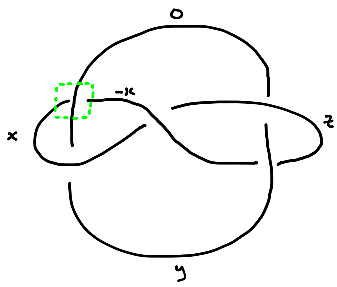
20.1 Link colourability, continued

Examples 20.1.

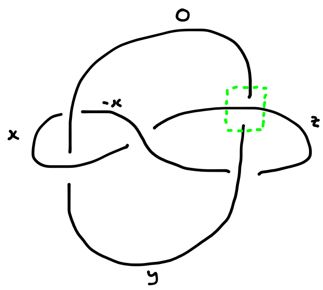
(4) Let us prove that the Whitehead link is not p-colourable for any odd prime p.

Suppose that we have a p-colouring of the Whitehead link. By Lemma 19.8 we can fix the integer assigned to one of the arcs to be 0. Let us denote the integers assigned to three of the other arcs by x, y, and z.

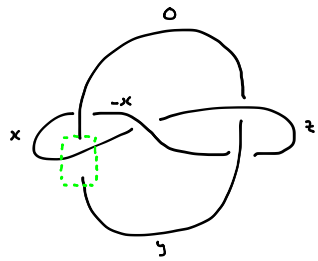
By condition (1) for an m-colouring applied to the crossing indicated below, the integer assigned to the remaining arc must be equal to $-x$ mod p.



By condition (1) for an m-colouring applied to the crossing indicated below, we must have that $y \equiv 2z \pmod{p}$.

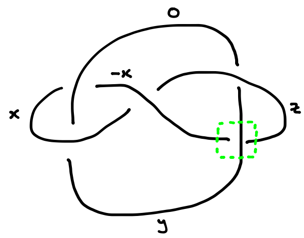


By condition (1) for an m-colouring applied to the crossing indicated below, we must have that $y \equiv 2x \pmod{p}$.



Thus we have that $2x \equiv 2z \pmod{p}$. We deduce since p is a prime that $p \mid 2(x-z)$. Since p is odd we conclude that $p \mid (x-z)$, or in other words that $x \equiv z \pmod{p}$.

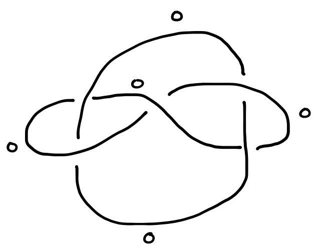
By condition (1) for an m-colouring applied to the crossing indicated below, we must have that $z - x \equiv 2y \pmod{p}$.



Since $x \equiv z \pmod{p}$ we deduce that $2y \equiv 0 \pmod{p}$. Since p is an odd prime, we deduce that $y \equiv 0 \pmod{p}$.

Then since $y \equiv 2x \pmod{p}$ we have that $2x \equiv 0 \pmod{p}$. Since p is an odd prime, we deduce that $x \equiv 0 \pmod{p}$. Since $x \equiv z \pmod{p}$ we then have that $z \equiv 0 \pmod{p}$.

Thus our colouring would be



which would contradict condition (2) for a p-colouring.

(2) In particular, the Whitehead link is not 3-colourable. By Examples 19.7 (4) the unlink with two components is 3-colourable. We conclude by Proposition 19.5 that

the Whitehead link is not isotopic to the unlink with two components, or in other words that it is genuinely linked.

Question 20.2. Is every knot m-colourable for some m?

Answer 20.3. No, but the simplest example — denoted 10_{124} — has 10 crossings!

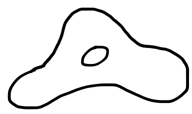
Question 20.4. Can we find an even better link invariant?

Answer 20.5. Yes! It will take us a little while to construct it.

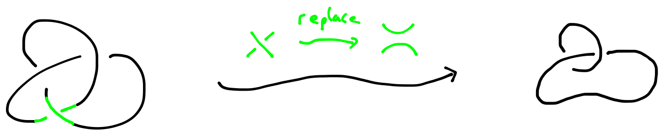
20.2 Bracket polynomial

Definition 20.6. Let $(L, \mathcal{O}_L)$ be a link, and let $\mathcal{D}$ be its corresponding link diagram. A *state* of L is a link diagram obtained by replacing every crossing of $\mathcal{D}$ with one of the following two possibilities.

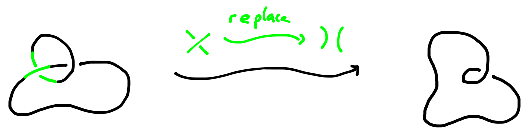
Example 20.7. The link diagram



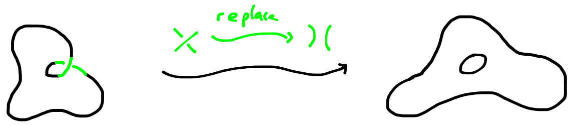
is a state of the trefoil knot. It can be obtained by first replacing a crossing as follows,



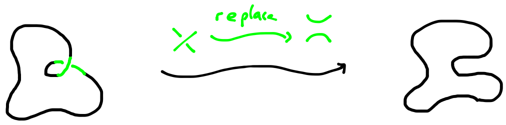
then replacing a crossing as follows,



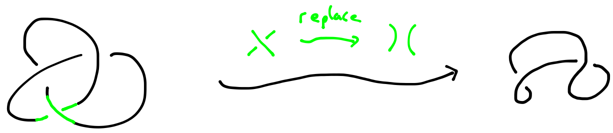
and finally replacing a crossing as follows.



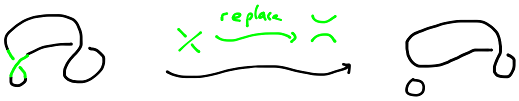
There are several other states. We shall see them all shortly. For example, at the third step we could instead proceed as follows, obtaining a different state.



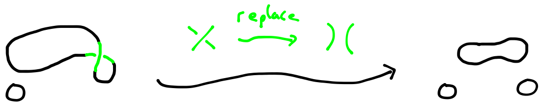
At the first step we could instead proceed as follows.



Next we could proceed as follows.



Finally we could proceed as follows, obtaining a different state to the two we have already encountered.



Definition 20.8. Let $(L, \mathcal{O}_L)$ be a link, and let $\mathcal{D}$ be its corresponding link diagram. We inductively associate to a state S of L an expression $\ll S \gg = A^i B^j$ in two variables A and B as follows.

(1) Begin by defining $\ll S \gg = 1$.

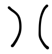
(2) If we replaced a crossing

by

when obtaining S from $\mathcal{D}$, we multiply $\ll S \gg$ by A.

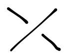
(3) If we replaced a crossing

by

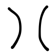


when obtaining S from $\mathcal{D}$, we multiply $\ll S \gg$ by B.

(4) If we replaced a crossing

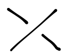


by



when obtaining S from $\mathcal{D}$, we multiply $\ll S \gg$ by A.

(5) If we replaced a crossing

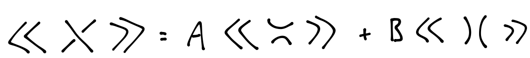


by

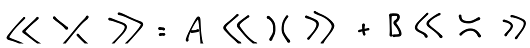
when obtaining S from $\mathcal{D}$, we multiply $\ll S \gg$ by B.

Remark 20.9. Since there are no orientations involved in Definition 20.8, rules (4) and (5) can be viewed as obtained from rules (2) and (3) by rotating, and vice versa.

Remark 20.10. To express rules (2) and (3) concisely we write the following.



To express rules (2) and (3) concisely we write the following.



Notation 20.11. Let $(L, \mathcal{O}_L)$ be a link, and let S be a state of L. We denote by $|S|$ the number of components of S.

Definition 20.12. Let $(L, \mathcal{O}_L)$ be a link. The *bracket polynomial* of L is a polynomial in three variables A, B, d given by

$$\sum_{S \text{ a state of } L} \ll S \gg d^{|S|-1}.$$

Notation 20.13. Let $(L, \mathcal{O}_L)$ be a link. We denote the bracket polynomial of L by $\ll L \gg$.

Examples 20.14.

(1) The unknot has only one state S, namely itself. Since we did not replace any crossings to obtain it, we have that $\ll S \gg = 1$. Thus the bracket polynomial of the unknot is as follows.

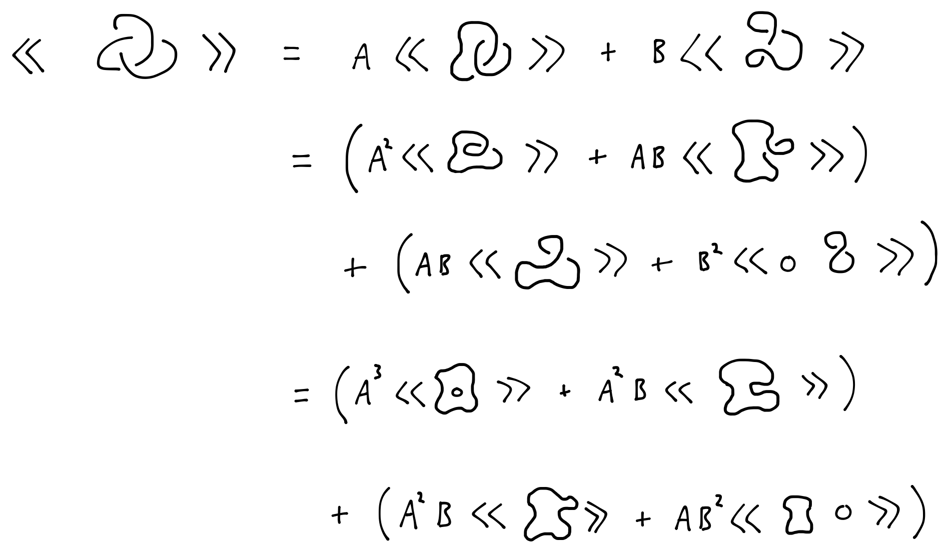
$$\ll \bigcirc \gg = 1 \cdot d^{1-1} = 1 \cdot d^0 = 1.$$

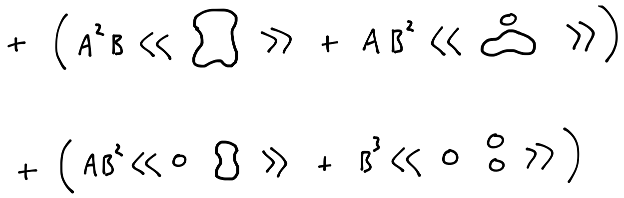
(2) The unlink with two components also has only one state S, namely itself. Again, since we did not replace any crossings to obtain it, we have that $\ll S \gg = 1$. Thus the bracket polynomial of the unlink with two components is as follows.

$$\ll \bigcirc \; \bigcirc \gg = 1 \cdot d^{2-1} = 1 \cdot d^1 = d$$

Similarly the bracket polynomial of the unlink with n components will be d^{n-1}.

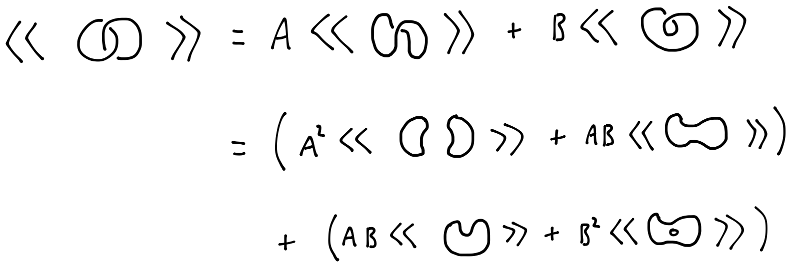
(3) Let us now calculate the bracket polynomial of the trefoil.





$$= A^3 d + A^2 B$$
$$+ A^2 B + AB^2 d$$
$$+ A^2 B + AB^2 d$$
$$+ AB^2 d + B^3 d^2$$
$$= A^3 d + 3A^2 B + 3AB^2 d + B^3 d^2$$

(4) Let us also calculate the bracket polynomial of the Hopf link.



$$= A^2 d + AB$$
$$+ AB + B^2 d$$
$$= A^2 d + 2AB + B^2 d$$

(5) Let us also calculate the bracket polynomial of the following knot.

$$= Ad + B$$

Definition 20.15. Let $(L, \mathcal{O}_L)$ be a link. We denote by $\langle L \rangle$ the polynomial obtained from $\ll L \gg$ by replacing d by $-A^2 - A^{-2}$ and replacing B by A^{-1}.

Remark 20.16. The reason for making this definition will become clear during the proof of Proposition 20.20.

Terminology 20.17. A polynomial such as $\langle L \rangle$ in which we have positive and negative powers of a single variable — in this case the variable A — is sometimes known as a *Laurent polynomial*.

Examples 20.18.

(1) Let O denote the unknot. By Examples 20.14 (1) we have that $\langle O \rangle = 1$.

(2) Let O_n denote the unlink with n components. By Examples 20.14 (2) we have that $\langle O_n \rangle = (-A^2 - A^{-2})^{n-1}$.

(3) Let 3_1 denote the trefoil. By Examples 20.14 (3) we have that

$$\begin{aligned}\langle 3_1 \rangle &= A^3(-A^2 - A^{-2}) + 3A^2A^{-1} + 3AA^{-2}(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})^2 \\ &= -A^5 - A + 3A + 3A^{-1}(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})(-A^2 - A^{-2}) \\ &= -A^5 + 2A - 3A - 3A^{-3} + A^{-3}(A^4 + 2 + A^{-4}) \\ &= -A^5 - A - 3A^{-3} + A + 2A^{-3} + A^{-7} \\ &= -A^5 - A^{-3} + A^{-7}.\end{aligned}$$

(4) Let 2_1^2 denote the Hopf link. By Examples 20.14 (4) we have that

$$\begin{aligned}\langle 2_1^2 \rangle &= A^2(-A^2 - A^{-2}) + 2AA^{-1} + A^{-2}(-A^2 - A^{-2}) \\ &= -A^4 - 1 + 2 - 1 - A^{-4} \\ &= -A^4 - A^{-4}.\end{aligned}$$

(5) Let K denote the knot of Examples 20.14 (5). We have that

$$\begin{aligned}\langle K \rangle &= A(-A^2 - A^{-2}) + A^{-1} \\ &= -A^3 - A^{-1} + A^{-1} \\ &= -A^3.\end{aligned}$$

Notation 20.19. Let $(L, \mathcal{O}_L)$ be a link. When working with $\ll L \gg$ or $\langle L \rangle$ we frequently depict only part of L inside the brackets. For example, in the proof of Proposition 20.20 we write the following.

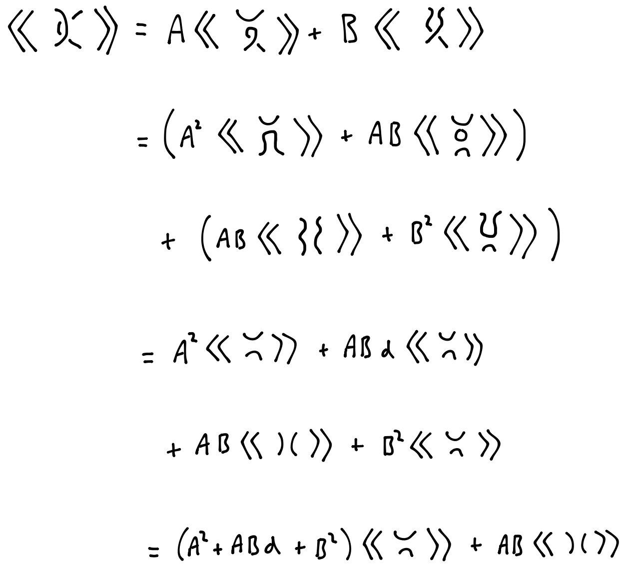
We do not typically mean to refer to the link

itself, but rather to a larger link, one part of which looks like this.

Proposition 20.20. Let $(L, \mathcal{O}_L)$ be a link. Then $\langle L \rangle$ is unchanged under the Reidemeister moves R2 and R3.

Proof. The proof for R3 relies upon the proof for R2.

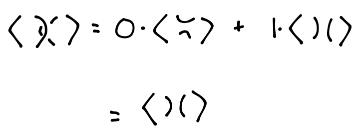
R2 We make the following calculation.



Letting $d = -A^2 - A^{-2}$ and $B = A^{-1}$ we have that

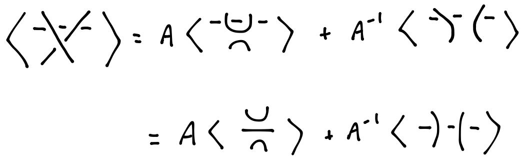
$$\begin{aligned} A^2 + ABd + B^2 &= A^2 + AA^{-1}(-A^2 - A^{-2}) + A^{-2} \\ &= A^2 - A^2 - A^{-2} + A^{-2} \\ &= 0. \end{aligned}$$

Moreover we have that $AB = AA^{-1} = 1$. We conclude that



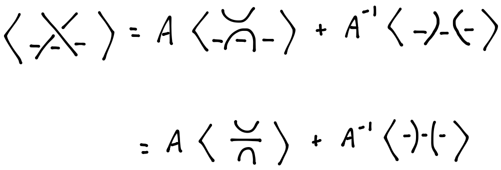
as required.

R3 We make the following calculation.



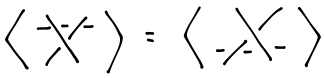
For the second equality we appeal to the fact that $\langle L \rangle$ is unchanged by an R2 move.

We also make the following calculation.



Again for the second equality we appeal to the fact that $\langle L \rangle$ is unchanged by an R2 move.

We conclude that



as required.

$\square$

Remark 20.21. This proof is not quite complete. There is another R2 and another R3 move which must be considered. However, just as in the proof of Proposition 18.14 and the proof of Proposition 19.5 it is the idea that is important. It adapts in a straightforward way to a proof for the other cases.

Remark 20.22. Let $(L, \mathcal{O}_L)$ be a link. By Examples 20.18 (1) and (5) we see that $\langle L \rangle$ is not necessarily unchanged under an R1 move, since the knot in Examples 20.18 (5) is isotopic to the unknot.

In the next lecture we will see that we can repair this. We shall introduce a tool — the writhe of L — which roughly speaking counts twists

in L, and modify the definition of $\langle L \rangle$ to take it into account.