Generell Topologi

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21 Thursday 4th April

21.1 Writhe

Definition 21.1. Let $(L, \mathcal{O}_L)$ be an oriented link. The *writhe* of L is

$$\sum_{\text{crossings } C \text{ of } L} \operatorname{sign}(C).$$

We denote it by $w(L)$.

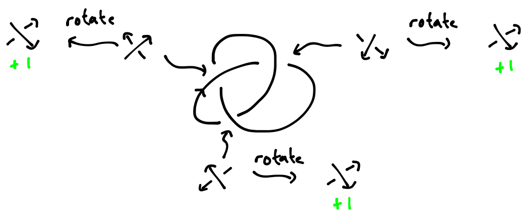
If there are no crossings, we adopt the convention that $w(L) = 0$.

Remark 21.2. Let $(L, \mathcal{O}_L)$ be an oriented link. The most important difference between the writhe of L and the linking number of L, which was introduced in Definition 18.10, is that in the definition of a linking number we consider only crossings between distinct components. Here we consider all crossings.

Examples 21.3.

(1) The writhe of the unknot is 0, since the unknot has no crossings. Similarly the writhe of the unlink with n components is 0.

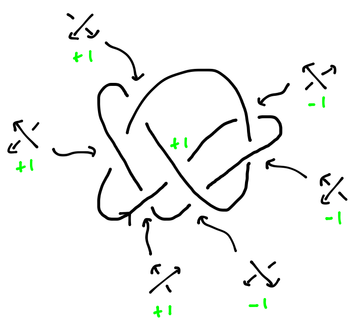
(2) The signs of the crossings of the oriented trefoil below are as shown.



Thus its writhe is $1 + 1 + 1 = 3$.

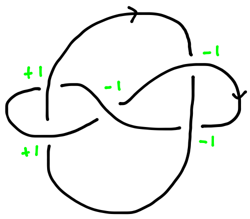
Reversing the orientation changes the sign of all three crossings, giving writhe of -3.

(3) The signs of the crossings of the oriented knot below are as shown.



Thus its writhe is $4 - 3 = 1$.

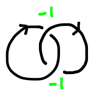
(4) The signs of the crossings of the oriented Whitehead link below are shown below.



Thus its writhe is $2 - 3 = -1$.

Compare this with Examples 18.12 (4). This time we do count the sign of the middle crossing!

(5) The signs of the crossings of the oriented Hopf link below are as shown.



Thus its writhe is -2.

21.2 Jones polynomial

Remark 21.4. The Jones polynomial was discovered in the 1980s — it is probably the most recent mathematics that you will come across in your undergraduate studies! It is of deep significance.

Definition 21.5. Let $(L, \mathcal{O}_L)$ be an oriented link. The *Jones polynomial* of L is

$$(-A)^{-3w(L)}\langle L\rangle.$$

We denote it by $V_L(A)$.

Examples 21.6.

(1) Let O be the unknot. By Examples 20.18 (1) we have that $\langle O\rangle = 1$. By Examples 21.3 (1) we have that $w(O) = 0$. Thus we have that $V_O(A) = (-A)^0 = 1$.

(2) Let O_n be the unlink with n components.. By Examples 20.18 (2) we have that $\langle O_n\rangle = (-A^{-2} - A^2)^{n-1}$. By Examples 21.3 (1) we have that $w(O_n) = 0$.

Thus we have that

$$\begin{aligned} V_{O_n}(A) &= (-A)^0(-A^{-2} - A^2)^{n-1} \\ &= (-A^{-2} - A^2)^{n-1}. \end{aligned}$$

(3) Let 3_1 be a trefoil with the orientation indicated below.

By Examples 20.18 (3) we have that $\langle 3_1 \rangle = A^{-7} - A^{-3} - A^{5}$. By Examples 21.3 (2) we have that the writhe of 3_1 with this orientation is 3.

Thus we have that

$$\begin{aligned} V_{3_1}(A) &= (-A)^{-9}(A^{-7} - A^{-3} - A^{5}) \\ &= -A^{-16} + A^{-12} + A^{-4}. \end{aligned}$$

We will later prove that the Jones polynomial of a knot does not depend on the choice of orientation.

(4) Let 2^2_1 be a Hopf link with the orientation indicated below.

By Examples 20.18 (4) we have that $\langle 2^2_1 \rangle = -A^{-4} - A^{4}$. By Examples 21.3 we have that the writhe of 2^2_1 with this orientation is -2.

Thus we have that

$$V_{2^2_1}(A) = (-A)^{6}(-A^{-4} - A^{4}) = -A^{2} - A^{10}.$$

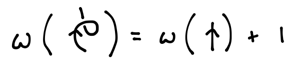
Proposition 21.7. Let $(L, \mathcal{O}_L)$ and $(L', \mathcal{O}_{L'})$ be oriented links. If L is isotopic to L' then $V_L(A) = V_{L'}(A)$.

Proof. We know by Theorem 17.16 that two links are isotopic if and only if one can be obtained from the other by a finite sequence of Reidemeister moves. Thus it suffices to prove that the Jones polynomial of an oriented link $(L, \mathcal{O}_L)$ is not changed by applying the Reidemeister moves.

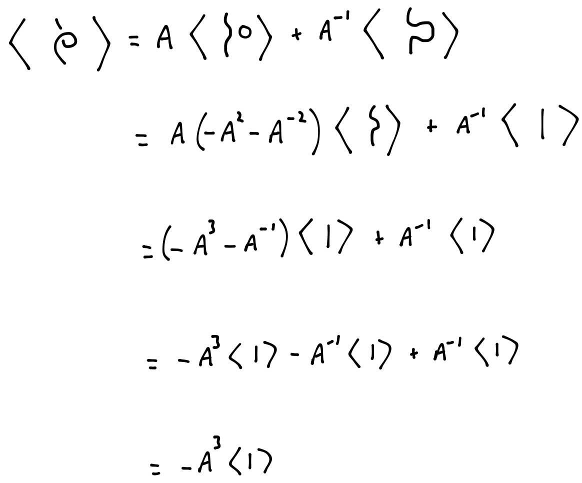
By Proposition 20.20 we have that $\langle L \rangle$ is not changed by an R2 move or an R3 move. Moreover the same argument as was given in the proof of Proposition 18.14 demonstrates that $w(L)$ is not changed by an R2 move or an R3 move. We deduce that $V_L(A)$ is not changed by an R2 move or an R3 move.

It remains to prove that $V_L(A)$ is not changed by an R1 move. We will adopt Notation 20.19 when working with the writhe of L in the following proof as well as when working with $\langle L \rangle$.

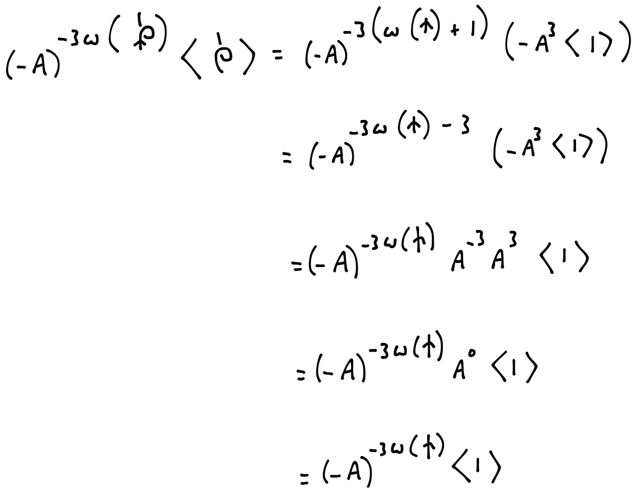
We begin by making the following observation.



Next we make the following calculation.



We deduce that the following holds, as required.



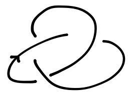
$\square$

Examples 21.8.

(1) By Examples 21.6 (2) and (4) we have that the Jones polynomial of the oriented Hopf link

is different to the Jones polynomial of the unlink with two components. By Proposition 21.7 we conclude that the Hopf link is not isotopic to the unlink with two components, which we observed via linking numbers in Examples 18.16.

(2) By Examples 21.6 (1) and (3) we have that the Jones polynomial of the trefoil



is different to the Jones polynomial of the unknot. By Proposition 21.7 we conclude that the trefoil is not isotopic to the unknot, which we observed via 3-colourability in Examples 19.9.

Definition 21.9. Let $(L, \mathcal{O}_L)$ be an oriented link. We denote by $V_L(t)$ the polynomial obtained from $V_L(A)$ by replacing A by $t^{-\frac{1}{4}}$. We refer to $V_L(t)$ also as the Jones polynomial of L.

Remark 21.10. Passing from $V_L(A)$ to $V_L(t)$ has no deep meaning! The Jones polynomial was originally constructed in the form $V_L(t)$ by different methods.

Examples 21.11.

(1) Let O be the unknot. By Examples 21.6 (1) we have that $V_O(A) = 1$. Thus we have that $V_O(t) = 1$.

(2) Let O_n be the unlink with n components. By Examples 21.6 (2) we have that $V_{O_n}(A) = (-A^{-2} - A^2)^{n-1}$. Thus we have that $V_{O_n}(t) = (-A^{-\frac{1}{2}} - A^{\frac{1}{2}})^{n-1}$.

(3) Let 3_1 be a trefoil with the orientation indicated below.

By Examples 21.6 (1) we have that $V_{3_1}(A) = -A^{-16} + A^{-12} + A^{-4}$. Thus we have that $V_{3_1}(t) = t + t^3 - t^4$.

(4) Let 2^2_1 be a Hopf link with the orientation indicated below.

By Examples 21.6 (2) we have that $V_{2^2_1}(A) = -A^2 - A^{10}$. Thus we have that $V_{2^2_1}(t) = -t^{-\frac{5}{2}} - t^{-\frac{1}{2}}$.

21.3 Skein relations

Definition 21.12. The *skein relations* for the Jones polynomial are as follows.

1) $V_{O}(t) = 1$

unknot

2) $t^{-1} V(t) - t V(t) = (t^{1/2} - t^{-1/2}) V(t)$

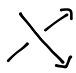
Remark 21.13. By Examples 21.11 we have that the Jones polynomial satisfies 1). We will prove later that it also satisfies 2). Before we do so we shall explore the meaning of 2) by using the skein relations to calculate inductively the Jones polynomials of a few links.

Examples 21.14.

(1) Let us calculate the Jones polynomial of the unlink with two components via the skein relations.

We begin with the following oriented knot.

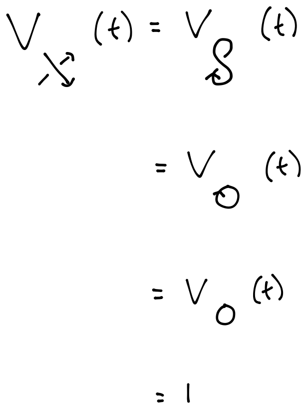
We choose a crossing. Here there is only one possibility! The way in which the skein relations work is that since the crossing is of the form



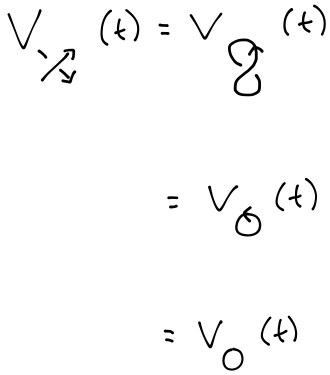
we take

$$V(t)$$

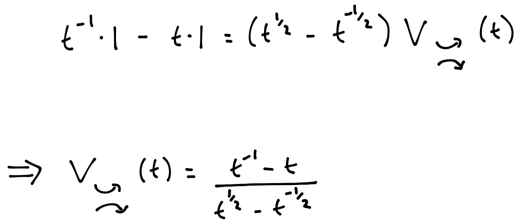
to be the Jones polynomial of the knot with which we began.



We also make the following calculation.



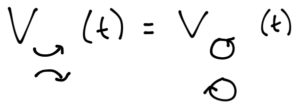
Thus by the second skein relation we have the following.

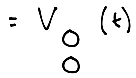


$$= \frac{(t^{1/2} - t^{-1/2})(-t^{1/2} - t^{-1/2})}{t^{1/2} - t^{-1/2}}$$

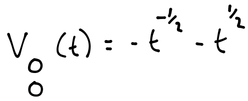
$$= -t^{1/2} - t^{-1/2}$$

In addition we have the following.



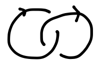


Putting everything together we have the following.

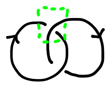


This agrees with our calculation in Examples 21.11 (2).

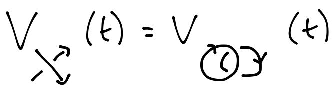
(2) Let us now use the skein relations to calculate the Jones polynomial of an oriented Hopf link.



Let us work at the indicated crossing.



We have the following, appealing to our calculation in (1).



$$= -t^{-1/2} - t^{1/2}$$

We also have the following.

V  $(t) = V$  (t)

$= V_{\bigcirc}(t)$

$= 1$

Thus by the second skein relation we have the following.

$t^{-1}\left(-t^{-1/2} - t^{1/2}\right) - t\,V$  $(t) = \left(t^{1/2} - t^{-1/2}\right)\cdot 1$

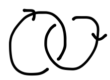
$\Rightarrow -t^{-3/2} - t^{-1/2} - t\,V$  $(t) = t^{1/2} - t^{-1/2}$

$\Rightarrow -t\,V$  $(t) = t^{-3/2} + t^{1/2}$

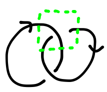
$\Rightarrow V$  $(t) = -t^{-5/2} - t^{-1/2}$

This agrees with our calculation in Examples 21.11 (4).

(3) Let us also use the skein relations to calculate the Jones polynomial of the mirror image of the oriented Hopf link of (2).



Let us work at the indicated crossing.



We have the following, appealing to our calculation in (1).

$$V(t) = V(t)$$

$$= V_{\bigcirc\bigcirc}(t)$$

$$= -t^{-1/2} - t^{1/2}$$

We also have the following.

$$V(t) = V(t)$$

$$= V_{\bigcirc}(t)$$

$$= 1$$

Thus by the second skein relation we have the following.

$$t^{-1}V(t) - t\left(-t^{-1/2} - t^{1/2}\right) = \left(t^{1/2} - t^{-1/2}\right)\cdot 1$$

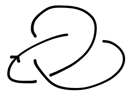
$$\Rightarrow \quad t^{-1}V(t) + t^{1/2} + t^{3/2} = t^{1/2} - t^{-1/2}$$

$$\Rightarrow \quad t^{-1}V(t) = -t^{3/2} - t^{-1/2}$$

$$\Rightarrow \quad V(t) = -t^{5/2} - t^{1/2}$$

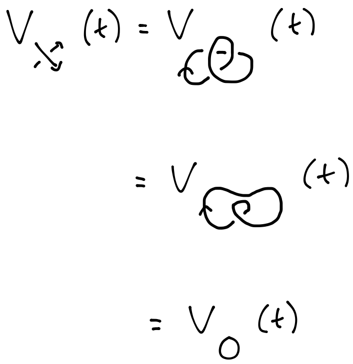
In particular this polynomial is not equal to the Jones polynomial we calculated in (2). We conclude by Proposition 21.7 that the oriented Hopf link of (2) is not isotopic to its mirror image.

(4) Let us now use the skein relations to calculate the Jones polynomial of an oriented trefoil.



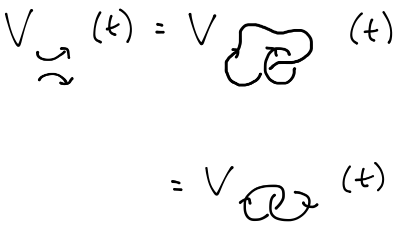
Let us work at the indicated crossing.

We have the following.



$$= 1$$

We also have the following, appealing to our calculation in (3).



$$= -t^{1/2} - t^{5/2}$$

Thus by the second skein relation we have the following.

$t^{-1} V$  $(t) - t \cdot 1 = (t^{1/2} - t^{-1/2})(-t^{1/2} - t^{5/2})$

$\Rightarrow t^{-1} V$  $(t) - t = -t + 1 - t^3 + t^2$

$\Rightarrow t^{-1} V$  $(t) = 1 + t^2 - t^3$

$\Rightarrow V$  $(t) = t + t^3 - t^4$

This agrees with our calculation in Examples 21.11 (3).