Generell Topologi — Solutions to Exercise Sheet 3

Richard Williamson

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Question.

(a) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(A)$ is closed in X for every closed subset A of Y.

(b) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if for every $x \in X$ and every neighbourhood U of $f(x)$ in $(Y, \mathcal{O}_Y)$, there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that $f(U') \subset U$.

(c) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let $\mathcal{O}'_Y$ be a basis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(U) \in \mathcal{O}_X$ for every $U \in \mathcal{O}'_Y$.

(d) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let $\mathcal{O}'_Y$ be a sub-basis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(U) \in \mathcal{O}_X$ for every $U \in \mathcal{O}'_Y$.

(e) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\{U_j\}_{j \in J}$ be a basis for $(X, \mathcal{O}_X)$, and let $\{U'_{j'}\}_{j' \in J'}$ be a basis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if for each $x \in X$ and each $j' \in J'$ such that $f(x) \in U'_{j'}$ there is a $j \in J$ such that $x \in U_j$ and $f(U_j) \subset U'_{j'}$.

Solution.

(a) Let A be a closed subset of Y with respect to $\mathcal{O}_Y$. By definition, we have that $Y \setminus A$ is open in $(Y, \mathcal{O}_Y)$. If f is continuous, we deduce that

$$X \setminus \big(f^{-1}(A)\big) = f^{-1}(Y \setminus A)$$

is open in $(X, \mathcal{O}_X)$. Thus $f^{-1}(A)$ is closed in $(X, \mathcal{O}_X)$.

Suppose that $f^{-1}(A)$ is closed in $(X, \mathcal{O}_X)$ for every closed subset A of Y with respect to $\mathcal{O}_Y$. Let U be an open subset of Y with respect to $\mathcal{O}_Y$. Then $Y \setminus U$ is closed in $(Y, \mathcal{O}_Y)$, and we deduce that

$$X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$$

is closed in $(X, \mathcal{O}_X)$. Thus $U = X \setminus (X \setminus U)$ is open in $(X, \mathcal{O}_X)$.

(b) Suppose that f is continuous. Let $x \in X$, and let U be a neighbourhood of $f(x)$ in $(Y, \mathcal{O}_Y)$. Let $U' = f^{-1}(U)$. Since f is continuous, $U' \in \mathcal{O}_X$. Moreover, we have that $f(U') \subset U$.

Conversely, suppose that for every $x \in X$ and every neighbourhood U of $f(x)$ in $(Y, \mathcal{O}_Y)$, there is a neighbourhood U' of x in $(X, \mathcal{O}_X)$ such that $f(U') \subset U$. Let $U'' \in \mathcal{O}_Y$, and let $x \in f^{-1}(U'')$. Then U'' is a neighbourhood of $f(x)$ in $(Y, \mathcal{O}_Y)$. By assumption, we deduce that there is a neighbourhood U_x of x in $(X, \mathcal{O}_X)$ such that $f(U_x) \subset U''$.

Then $U_x \subset f^{-1}(U'')$, and thus $\bigcup_{x \in f^{-1}(U'')} U_x \subset f^{-1}(U'')$. Since $x \in U_x$, we also have that $f^{-1}(U'') \subset \bigcup_{x \in f^{-1}(U'')} U_x$,. We deduce that $\bigcup_{x \in f^{-1}(U'')} U_x = f^{-1}(U'')$. Since $U_x \in \mathcal{O}_X$ for all $x \in f^{-1}(U'')$, we have that $\bigcup_{x \in f^{-1}(U'')} U_x$ is open in $(X, \mathcal{O}_X)$. We conclude that $f^{-1}(U'') \in \mathcal{O}_X$, as required.

(c) Since $\mathcal{O}'_Y$ is a basis for $(Y, \mathcal{O}_Y)$, we have that $U \in \mathcal{O}_Y$ for every $U \in \mathcal{O}'_Y$. If f is continuous, we thus have that $f^{-1}(U) \in \mathcal{O}_X$ for all $U \in \mathcal{O}'_Y$.

Conversely, suppose that $f^{-1}(U') \in \mathcal{O}_X$ for every $U' \in \mathcal{O}'_Y$. Let $U \in \mathcal{O}_Y$. Since $\mathcal{O}'_Y$ is a basis for $(Y, \mathcal{O}_Y)$, we have that $U = \bigcup_{j \in J} U'_j$ for a set $\{U'_j\}_{j \in J}$ of subsets of Y which belong to $\mathcal{O}'_Y$. Then

$$f^{-1}(U) = f^{-1}\big(\bigcup_{j \in J} U'_j\big) = \bigcup_{j \in J} f^{-1}(U'_j).$$

By assumption, we have that $f^{-1}(U_j) \in \mathcal{O}_X$ for all $j \in J$. Since $\mathcal{O}_X$ defines a topology on X, we deduce that $\bigcup_{j \in J} f^{-1}(U_j)$ belongs to $\mathcal{O}_X$¡ and thus that $f^{-1}(U) \in \mathcal{O}_X$.

(d) Since $\mathcal{O}'_Y$ is a sub-basis for $(Y, \mathcal{O}_Y)$, we have that $U \in \mathcal{O}_Y$ for every $U \in \mathcal{O}'_Y$. If f is continuous, we thus have that $f^{-1}(U) \in \mathcal{O}_X$ for all $U \in \mathcal{O}'_Y$.

Conversely, let $\mathcal{O}''_Y$ denote the set of subsets of Y obtained by taking finite intersections of subsets of Y which belong to $\mathcal{O}'_Y$. Then $\mathcal{O}''_Y$ defines a basis for $(Y, \mathcal{O}_Y)$. We deduce by (1) that it suffices to show that $f^{-1}(\bigcap_{j \in J} U_j) \in \mathcal{O}_X$ for any finite set $\{U_j\}_{j \in J}$ of subsets of Y which belong to $\mathcal{O}'_Y$.

We have that $f^{-1}\big(\bigcap_{j \in J} U_j\big) = \bigcap_{j \in J} f^{-1}(U_j)$. By assumption, we have that $f^{-1}(U_j) \in \mathcal{O}_X$ for all $j \in J$. Since J is finite and since $\mathcal{O}_X$ defines a topology on X, we deduce that $\bigcap_{j \in J} f^{-1}(U_j) \in \mathcal{O}_X$, and thus that $f^{-1}(\bigcap_{j \in J} U_j) \in \mathcal{O}_X$.

(e) Suppose that f is continuous. Let $x \in X$ and $j' \in J'$ be such that $f(x) \in U'_{j'}$. By part (b), we have that there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ such that $f(U) \subset U'_{j'}$. By Question 3 (b) on Exercise Sheet 2, there is a $j \in J$ such that $x \in U_j$ and $U_j \subset U$. We have that $f(U_j) \subset f(U) \subset U'_{j'}$, and thus that $f(U_j) \subset U'_{j'}$.

Conversely, suppose that for each $x \in X$ and each $j' \in J'$ such that $f(x) \in U'_{j'}$ there is a $j \in J$ such that $x \in U_j$ and $f(U_j) \subset U'_{j'}$. Let $x \in X$, and let U' be a neighbourhood of $f(x)$ in $(Y, \mathcal{O}_Y)$. By Question 3 (b) of Exercise Sheet 2, there is a $j' \in J'$ such that $f(x) \in U'_{j'}$ and $U'_{j'} \subset U'$. By assumption, there is a $j \in J$ such that $x \in U_j$ and $f(U_j) \subset U'_{j'}$. Thus $f(U_j) \subset U'$. We deduce by part (b) that f is continuous.

2

Question.

Let $X = \{a, b, c\}$ be equipped with the topology

$$\mathcal{O}_X = \big\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\big\},$$

and let $Y = \{a', b', c', d', e'\}$ be equipped with the topology

$$\mathcal{O}_Y = \big\{\emptyset, \{a'\}, \{e\}, \{a', e'\}, \{b', c'\}, \{a', b', c'\}, \{b', c', e'\}, \{a', b', c', e'\}, \{b', c', d', e'\}, Y\big\}.$$

Which of the following maps

$$X \xrightarrow{f} Y$$

are continuous?

(1) $a \mapsto d'$, $b \mapsto e'$, $c \mapsto d'$.

(2) $a \mapsto e'$, $b \mapsto e'$, $c \mapsto c'$.

(3) $a \mapsto c'$, $b \mapsto a'$, $c \mapsto d'$.

(4) $a \mapsto b'$, $b \mapsto c'$, $c \mapsto d'$.

Solution.

Note that $\mathcal{O}'_Y = \big\{\{a', e'\}, \{a', b', c'\}, \{b', c', d', e'\}\big\}$ defines a sub-basis for $\mathcal{O}_Y$. By Question 1 (d), it suffices to check that $f^{-1}(U) \in \mathcal{O}_X$ for each of the three sets $U \in \mathcal{O}'_Y$.

(1) We have that $f^{-1}\big(\{a', e'\}\big) = \{b\}$, $f^{-1}\big(\{a', b', c'\}\big) = \emptyset$, and $f^{-1}\big(\{b', c', d', e'\}\big) = X$, all of which belong to $\mathcal{O}_X$. Thus f is continuous.

(2) We have that $f^{-1}\big(\{a', b', c'\}\big) = \{c\}$, which does not belong to $\mathcal{O}_X$. Thus f is not continuous.

(3) We have that $f^{-1}\big(\{b', c', d', e'\}\big) = \{a, c\}$, which does not belong to $\mathcal{O}_X$. Thus f is not continuous.

(4) We have that $f^{-1}\big(\{a', e'\}\big) = \emptyset$, $f^{-1}\big(\{a', b', c'\}\big) = \{a, b\}$, and $f^{-1}\big(\{b', c', d', e'\}\big) = X$, all of which belong to $\mathcal{O}_X$. Thus f is continuous.

3

Question.

Let $(X, \mathcal{O}_X)$ be a topological space, and let $\mathbb{R}$ be equipped with its standard topology $\mathcal{O}_{\mathbb{R}}$.

(a) Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Prove that the map

$$X \xrightarrow{|f|} \mathbb{R}$$

given by $x \mapsto |f(x)|$ is continuous.

(b) Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Prove that for any $k \in \mathbb{R}$, the map

$$X \xrightarrow{kf} \mathbb{R}$$

given by $x \mapsto k \cdot f(x)$ is continuous.

(c) Let

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R} \end{tikzcd}$$

be continuous maps. Prove that the map

$$X \xrightarrow{f+g} \mathbb{R}$$

given by $x \mapsto f(x) + g(x)$ is continuous.

Hint:

(i) Appeal to Question 7 (a) of Exercise Sheet 2.

(ii) For any $b \in \mathbb{R}$, appeal to the fact that

$$\{x \in X \mid f(x)+g(x) < b\} = \bigcup_{y \in \mathbb{R}} \Big(\{x \in X \mid f(x) < b-y\} \cap \{x \in X \mid g(x) < y\} \Big).$$

(iii) For any $a \in \mathbb{R}$, appeal to an analogous expression of

$$\{x \in X \mid f(x) + g(x) > a\}$$

as a union of intersections.

(d) Let

$$X \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{R}$$

be continuous maps. Prove that the map

$$X \xrightarrow{fg} \mathbb{R}$$

given by $x \mapsto f(x) \cdot g(x)$ is continuous.

Hint:

(i) Prove that if $f(x) \geq 0$ for all $x \in X$, then the map

$$X \xrightarrow{f^2} \mathbb{R}$$

given by $x \mapsto f(x) \cdot f(x)$ is continuous.

(ii) Find an expression for fg which allows you to deduce the continuity of fg from (i) and parts (a)–(c).

(e) Let

$$X \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{R}$$

be continuous maps, and suppose that $g(x) \neq 0$ for all $x \in X$. Prove that

$$X \xrightarrow{\frac{f}{g}} \mathbb{R}$$

given by $x \mapsto \frac{f(x)}{g(x)}$ is continuous.

Hint:

For any $a, b \in \mathbb{R}$, find an expression for $\left\{x \in X \mid \frac{1}{f(x)} < b\right\}$ and an expression for $\left\{x \in X \mid \frac{1}{f(x)} > a\right\}$ which allows you to deduce that $f^{-1}\big((-\infty, b)\big)$ and $f^{-1}\big((a, \infty)\big)$ are open in $(X, \mathcal{O}_X)$ from the continuity of f and the continuity of bf and af.

(f) Deduce that a map

$$\mathbb{R} \longrightarrow \mathbb{R}$$

which is a quotient of polynomials, namely a map of the form

$$x \mapsto \frac{k_0 + k_1 x + k_2 x^2 + \ldots + k_m x^m}{l_0 + l_1 x + l_2 x^2 + \ldots + l_n x^n}$$

where $m, n \in \mathbb{N}$, $k_i \in \mathbb{R}$ for all $0 \leq i \leq m$, and $l_j \in \mathbb{R}$ for all $0 \leq j \leq n$, is continuous.

Here we assume that

$$0 \neq l_0 + l_1 x + l_2 x^2 + \ldots + l_n x^n$$

for all $x \in \mathbb{R}$.

(g) Prove that the map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{\times} \mathbb{R}$$

given by $(x, y) \mapsto xy$ is continuous.

(h) Prove that the map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$$

given by $(x, y) \mapsto x + y$ is continuous.

Solution.

(a) By definition of $\mathcal{O}_{\mathbb{R}}$, the set of open intervals (a, b) in $\mathbb{R}$ defines a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Question 1 (c) it is sufficient to check that $|f|^{-1}\big((a, b)\big)$ is open in $(X, \mathcal{O}_X)$ for any $a, b \in \mathbb{R}$ with $a < b$.

Indeed, we have that

$$\begin{aligned}
|f^{-1}|\big((a,b)\big) &= \{x \in X \mid a < |f(x)| < b\} \\
&= \{x \in X \mid -b < f(x) < -a\} \cup \{x \in X \mid a < f(x) < b\} \\
&= f^{-1}\big((-b,-a)\big) \cup f^{-1}\big((a,b)\big) \\
&= f^{-1}\big((-b,-a) \cup (a,b)\big).
\end{aligned}$$

Since f is continuous, $f^{-1}\big((-b,-a)\cup(a,b)\big)$ is open in $(X, \mathcal{O}_X)$. Thus $|f^{-1}|\big((a,b)\big)$ is open in $(X, \mathcal{O}_X)$.

(b) Again, it is sufficient to check that $f^{-1}\big((a,b)\big)$ is open in $(X, \mathcal{O}_X)$ for any $a, b \in \mathbb{R}$ with $a < b$.

If $k = 0$, then kf is the constant map $x \mapsto 0$ for all $x \in X$. By Proposition 2.18 in the Lecture Notes, we deduce that kf is continuous.

If $k > 0$, we have that

$$\begin{aligned}
(kf)^{-1}\big((a,b)\big) &= \{x \in X \mid a < kf(x) < b\} \\
&= \{x \in X \mid \frac{a}{k} < f(x) < \frac{b}{k}\} \\
&= f^{-1}\big((ka, kb)\big).
\end{aligned}$$

Since f is continuous, $f^{-1}\big((ka,kb)\big)$ is open in $(X, \mathcal{O}_X)$. Thus $(kf)^{-1}\big((a,b)\big)$ is open in $(X, \mathcal{O}_X)$.

If $k < 0$, we have that

$$\begin{aligned}
(kf)^{-1}\big((a,b)\big) &= \{x \in X \mid a < kf(x) < b\} \\
&= \{x \in X \mid \frac{b}{k} < f(x) < \frac{a}{k}\} \\
&= f^{-1}\big((kb, ka)\big).
\end{aligned}$$

Since f is continuous, $f^{-1}\big((kb,ka)\big)$ is open in $(X, \mathcal{O}_X)$. Thus $(kf)^{-1}\big((b,a)\big)$ is open in $(X, \mathcal{O}_X)$.

(c) By Question 7 (a) of Exercise Sheet 2,

$$\{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$$

defines a sub-basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Question 1 (d) it is therefore sufficient to check that $(f+g)^{-1}\big((-\infty, b)\big) \in \mathcal{O}_X$ for all $b \in \mathbb{R}$ and that $(f+g)^{-1}\big((a,\infty)\big) \in \mathcal{O}_X$ for all $a \in \mathbb{R}$.

Let us prove that

$$\{x \in X \mid f(x) + g(x) < b\} = \bigcup_{y \in \mathbb{R}} \Big(\big\{x \in X \mid f(x) < b - y\big\} \cap \big\{x \in X \mid g(x) < y\big\} \Big).$$

Suppose that $x \in X$ has the property that $f(x) < b - y$ and $g(x) < y$ for some $y \in \mathbb{R}$. Then $f(x) + g(x) < (b - y) + y = b$. Thus we have that

$$\bigcup_{y \in \mathbb{R}} \Big(\big\{x \in X \mid f(x) < b - y\big\} \cap \big\{x \in X \mid g(x) < y\big\}\Big) \subset \{x \in X \mid f(x) + g(x) < b\}.$$

Conversely, suppose that $x' \in X$ has the property that $f(x') + g(x') < b$. Take any $y \in \mathbb{R}$ be such that $g(x') < y < b - f(x')$. Then

$$x' \in \big\{x \in X \mid f(x) < b - y\big\} \cap \big\{x \in X \mid g(x) < y\big\}.$$

We deduce that

$$\{x \in X \mid f(x) + g(x) < b\} \subset \bigcup_{y \in \mathbb{R}} \Big(\big\{x \in X \mid f(x) < b - y\big\} \cap \big\{x \in X \mid g(x) < y\big\}\Big).$$

This completes the proof that

$$\{x \in X \mid f(x) + g(x) < b\} = \bigcup_{y \in \mathbb{R}} \Big(\big\{x \in X \mid f(x) < b - y\big\} \cap \big\{x \in X \mid g(x) < y\big\}\Big).$$

We have that

$$\begin{aligned} (f+g)^{-1}\big((-\infty, b)\big) &= \{x \in X \mid f(x) + g(x) < b\} \\ &= \bigcup_{y \in \mathbb{R}} \Big(\big\{x \in X \mid f(x) < b - y\big\} \cap \big\{x \in X \mid g(x) < y\big\}\Big) \\ &= \bigcup_{y \in \mathbb{R}} \Big(f^{-1}\big((-\infty, b - y)\big) \cap g^{-1}\big((-\infty, y)\big)\Big). \end{aligned}$$

Since f is continuous, $f^{-1}\big((-\infty, b-y)\big)$ is open in $(X, \mathcal{O}_X)$. Since g is continuous, $g^{-1}\big((-\infty, y)\big)$ is open in $(X, \mathcal{O}_X)$. Thus $f^{-1}\big((-\infty, b - y)\big) \cap g^{-1}\big((-\infty, y)\big)$ is open in $(X, \mathcal{O}_X)$ for all $y \in \mathbb{R}$, and hence

$$\bigcup_{y \in \mathbb{R}} \Big(f^{-1}\big((-\infty, b - y)\big) \cap g^{-1}\big((-\infty, y)\big)\Big)$$

is open in $(X, \mathcal{O}_X)$. Thus $(f+g)^{-1}\big((-\infty, b)\big)$ is open in $(X, \mathcal{O}_X)$.

Similarly we have that

$$\begin{aligned} (f+g)^{-1}\big((a, \infty)\big) &= \{x \in X \mid f(x) + g(x) > a\} \\ &= \bigcup_{y \in \mathbb{R}} \Big(\big\{x \in X \mid f(x) > a - y\big\} \cap \big\{x \in X \mid g(x) > y\big\}\Big) \\ &= \bigcup_{y \in \mathbb{R}} \Big(f^{-1}\big((a - y, \infty)\big) \cap g^{-1}\big((a, \infty)\big)\Big). \end{aligned}$$

Since f is continuous, $f^{-1}\big((a-y,\infty)\big)$ is open in $(X,\mathcal{O}_X)$. Since g is continuous, $g^{-1}\big((y,\infty)\big)$ is open in $(X,\mathcal{O}_X)$. Thus $f^{-1}\big((a-y,\infty)\big) \cap g^{-1}\big((y,\infty)\big)$ is open in $(X,\mathcal{O}_X)$ for all $y \in \mathbb{R}$, and hence

$$\bigcup_{y\in\mathbb{R}} \Big(f^{-1}\big((a-y,\infty)\big) \cap g^{-1}\big((y,\infty)\big) \Big)$$

is open in $(X,\mathcal{O}_X)$. Thus $(f+g)^{-1}\big((a,\infty)\big)$ is open in $(X,\mathcal{O}_X)$.

(d) Let us first prove that if $f(x) \geq 0$ for all $x \in X$, then f^2 is continuous. As in parts (a) and (b), it is sufficient to check that $(f^2)^{-1}\big((a,b)\big)$ is open in $(X,\mathcal{O}_X)$ for any $a,b \in \mathbb{R}$ with $a<b$. Indeed we have that

$$\begin{aligned}(f^2)^{-1}\big((a,b)\big) &= \{x \in X \mid \{x \in X \mid a < f(x)\cdot f(x) < b\} \\ &= \{x \in X \mid \sqrt{a} < f(x) < \sqrt{b}\} \\ &= f^{-1}\big((\sqrt{a},\sqrt{b})\big).\end{aligned}$$

Since f is continuous, $f^{-1}\big((\sqrt{a},\sqrt{b})\big)$ is open in $(X,\mathcal{O}_X)$, and thus $(f^2)^{-1}\big((a,b)\big)$ is open in $(X,\mathcal{O}_X)$.

Next, note that $f(x)\cdot g(x) = \frac{1}{4}\Big(\big|f(x)+g(x)\big|^2 - \big|f(x)-g(x)\big|^2\Big)$. By part (c) we have that $f+g$ is continuous. By part (a) we deduce that $|f+g|$ is continuous. By part (b) we have that $-g$ is continuous. Thus by part (c) we have that $f-g$ is continuous. By part (a) we deduce that $|f-g|$ is continuous.

Hence $|f+g|^2$ and $|f-g|^2$ are continuous. By part (c) we deduce that $|f+g|^2+|f-g|^2$ is continuous. By part (b) we conclude that $\frac{1}{4}\Big(\big|f(x)+g(x)\big|^2 - \big|f(x)-g(x)\big|^2\Big)$ is continuous, and thus that fg is continuous.

(e) Let us first prove that the map

$$X \xrightarrow{\frac{1}{g}} \mathbb{R}$$

given by $x \mapsto \frac{1}{g(x)}$ is continuous. We proceed as in (c). By Question 7 (a) of Exercise Sheet 2,

$$\{(-\infty,b) \mid b \in \mathbb{R}\} \cup \{(a,\infty) \mid a \in \mathbb{R}\}$$

defines a sub-basis for $(\mathbb{R},\mathcal{O}_{\mathbb{R}})$. By Question 1 (d) it is therefore sufficient to check that $(\frac{1}{g})^{-1}\big((-\infty,b)\big) \in \mathcal{O}_X$ for all $b \in \mathbb{R}$ and that $(\frac{1}{g})^{-1}\big((a,\infty)\big) \in \mathcal{O}_X$ for all $a \in \mathbb{R}$.

Note that

$$\{x \in X \mid \frac{1}{g(x)} > a\}$$

is the union of

$$\{x \in X \mid g(x) > 0\} \cap \{x \in X \mid ag(x) < 1\}$$

and

$$\{x \in X \mid g(x) < 0\} \cap \{x \in X \mid ag(x) > 1\}.$$

Since g is continuous, $g^{-1}\big((0,\infty)\big) = \{x \in X \mid g(x) > 0\}$ is open in X, and $g^{-1}\big((-\infty,0)\big) = \{x \in X \mid g(x) < 0\}$ is open in X.

Moreover, by (b), the map

$$X \xrightarrow{ag} \mathbb{R}$$

is continuous, since g is continuous. Hence

$$(ag)^{-1}\big((-\infty,1)\big) = \{x \in X \mid ag(x) < 1\}$$

is open in X, and

$$(ag)^{-1}\big((1,\infty)\big) = \{x \in X \mid ag(x) > 1\}$$

is open in X. We conclude that

$$\Big(\frac{1}{g}\Big)^{-1}\big((a,\infty)\big) = \{x \in X \mid \frac{1}{g(x)} > a\}$$

is open in X.

Similarly, note that

$$\{x \in X \mid \frac{1}{g(x)} < b\}$$

is the union of

$$\{x \in X \mid g(x) > 0\} \cap \{x \in X \mid ag(x) > 1\}$$

and

$$\{x \in X \mid g(x) < 0\} \cap \{x \in X \mid ag(x) < 1\}.$$

We deduce in the same way as above that

$$\Big(\frac{1}{g(x)}\Big)^{-1}\big((-\infty,b)\big) = \{x \in X \mid \frac{1}{g(x)} < b\}$$

is open in X.

This completes the proof that $\frac{1}{g}$ is continuous. Since $\frac{f}{g} = f \cdot \big(\frac{1}{g}\big)$, we deduce from (d) that $\frac{f}{g}$ is continuous.

(f) Note first that the identity map

$$\mathbb{R} \xrightarrow{id} \mathbb{R},$$

namely the map given by $x \mapsto x$, is continuous. Indeed, if U is open in $\mathbb{R}$, then $id^{-1}(U) = U$ is open in $\mathbb{R}$. By (d) and induction, we deduce that for any $n \geq 1$ the map

$$\mathbb{R} \longrightarrow \mathbb{R}$$

given by $x \mapsto x^n$ is continuous. By (b), we deduce that for any $n \geq 0$ and any $k_n \in \mathbb{R}$ the map

$$X \xrightarrow{\mathbb{R}} X$$

given by $x \mapsto k_n x^n$ is continuous.

By Proposition 2.18 in the Lecture Notes, we also have that the constant map

$$\mathbb{R} \longrightarrow \mathbb{R}$$

given by $x \mapsto k_0$ for all $x \in X$ is continuous, for any $k_0 \in \mathbb{R}$. By (c), we deduce that a polynomial map

$$\mathbb{R} \longrightarrow \mathbb{R},$$

namely a map given by

$$x \mapsto k_0 + k_1 x + k_2 x^2 + \ldots + k_n x^n$$

for some $n \geq 0$ and $k_n \in \mathbb{R}$ is continuous. By (e), we conclude that a quotient of polynomials as in the question is continuous.

(g) The map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{\times} \mathbb{R}$$

is the product $p_1 \cdot p_2$ of the maps

$$\mathbb{R} \times \mathbb{R} \xrightarrow{p_1} \mathbb{R}$$

and

$$\mathbb{R} \times \mathbb{R} \xrightarrow{p_2} \mathbb{R}.$$

By Proposition 3.2 in the Lecture Notes, we have that p_1 and p_2 are continuous. By (d), we deduce that $\times$ is continuous.

(h) The map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{\times} \mathbb{R}$$

is the sum $p_1 + p_2$ of the maps

$$\mathbb{R} \times \mathbb{R} \xrightarrow{p_1} \mathbb{R}$$

and

$$\mathbb{R} \times \mathbb{R} \xrightarrow{p_2} \mathbb{R}.$$

Again, by Proposition 3.2 in the Lecture Notes, we have that p_1 and p_2 are continuous. By (c), we deduce that $\times$ is continuous.

4

Question.

(a) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces, and let

$$Z \xrightarrow{f} X$$

and

$$Z \xrightarrow{g} Y$$

be continuous maps. Prove that the map

$$Z \xrightarrow{f \times g} X \times Y$$

given by $z \mapsto \big(f(z), g(z)\big)$ is continuous.

(b) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Prove that a map

$$Z \xrightarrow{f} X \times Y$$

is continuous if and only if the maps

$$Z \xrightarrow{p_1 \circ f} X$$

and

$$Z \xrightarrow{p_2 \circ f} Y$$

are continuous.

(c) Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $(X', \mathcal{O}_{X'})$, and $(Y', \mathcal{O}_{Y'})$ be topological spaces, and let

$$X \xrightarrow{f} X'$$

and

$$Y \xrightarrow{g} Y'$$

be continuous maps. Prove that the map

$$X \times Y \xrightarrow{f \times g} X' \times Y'$$

given by $(x, y) \mapsto \big(f(x), g(y)\big)$ is continuous.

(d) Let $(X, \mathcal{O}_X)$ be a topological space. Prove that the map

$$X \xrightarrow{\Delta} X \times X$$

given by $x \mapsto (x, x)$ is continuous.

(e) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that the map

$$X \times Y \xrightarrow{\tau} Y \times X$$

given by $(x, y) \mapsto (y, x)$ is continuous.

Solution.

(a) By Question 4 (a) of Exercise Sheet 2, we have that $\{U \times U' \mid U \in \mathcal{O}_X,\ U' \in \mathcal{O}_Y\}$ defines a basis for $\mathcal{O}_{X \times Y}$. By Question 1 (c), it therefore suffices to prove that $(f \times g)^{-1}(U \times U') \in \mathcal{O}_Z$ for any $U \in \mathcal{O}_X$ and $U' \in \mathcal{O}'_Y$.

Indeed, we have that $(f \times g)^{-1}(U \times U') = f^{-1}(U) \cap g^{-1}(U')$. Since f is continuous, $f^{-1}(U) \in \mathcal{O}_Z$. Since g is continuous, $g^{-1}(U') \in \mathcal{O}_Z$. Hence $f^{-1}(U) \cap g^{-1}(U') \in \mathcal{O}_Z$.

(b) By Proposition 3.2 of the Lecture Notes, we have that p_1 and p_2 are continuous. Hence, by Proposition 2.16 of the Lecture Notes, if f is continuous then $p_1 \circ f$ and $p_2 \circ f$ are continuous.

Conversely, suppose that $p_1 \circ f$ and $p_2 \circ f$ are continuous. We have that $f = (p_1 \circ f) \times (p_2 \times f)$. We deduce from (a) that f is continuous.

(c) We have that $f \times g = \big(p'_1 \circ (f \times g)\big) \times \big(p'_2 \circ (f \times g)\big)$, where

$$X' \times Y' \xrightarrow{p'_1} X'$$

and

$$X' \times Y' \xrightarrow{p'_2} Y'$$

are the projection maps. By (b), we deduce that $f \times g$ is continuous.

(d) We have that $\Delta = id \times id$. Since id is continuous, Δ is continuous by (a).

(e) We have that $\tau = p_2 \times p_1$, where

$$X \times Y \xrightarrow{p_1} X$$

and

$$X \times Y \xrightarrow{p_2} Y$$

are the projection maps. Since p_1 and p_2 are continuous by Proposition 3.2 in the Lecture Notes, we deduce that τ is continuous by (a).

5

Question.

Let $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$ be topological spaces. Let

$$X \times Y \xrightarrow{p_1} X$$

and

$$X \times Y \xrightarrow{p_2} Y$$

denote the projection maps.

Let A be a closed subset of $(X, \mathcal{O}_X)$, and let A' be a closed subset of $(X', \mathcal{O}_{X'})$. By Proposition 3.2 in the Lecture Notes we have that p_1 and p_2 are continuous. Use this to prove that $A \times A'$ is a closed subset of $(X \times X', \mathcal{O}_{X \times X'})$.

Solution.

Since p_1 is continuous and A is closed in $(X, \mathcal{O}_X)$ we have by Question 1 (a) that $p_1^{-1}(A)$ is closed in $(X \times Y, \mathcal{O}_{X \times Y})$. In addition we have that $A \times X' = p_1^{-1}(A)$. Thus $A \times X'$ is closed in $(X \times Y, \mathcal{O}_{X \times Y})$.

Since p_2 is continuous and A' is closed in $(X', \mathcal{O}_{X'})$ we have by Question 1 (a) that $p_2^{-1}(A')$ is closed in $X \times Y$. In addition we have that $X \times A' = p_2^{-1}(A')$. Thus $X \times A'$ is closed in $(X \times Y, \mathcal{O}_{X \times Y})$.

We have that $A \times A' = (A \times X') \cap (X \times A')$. Since both $A \times X'$ and $X \times A'$ are closed in $(X \times Y, \mathcal{O}_{X \times Y})$ we deduce that $A \times A'$ is closed in $(X \times Y, \mathcal{O}_{X \times Y})$.

6

Question.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let A be a subset of X equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$.

(a) Let

$$X \xrightarrow{f} Y$$

be a continuous map. Prove that the restriction of f to A defines a continuous map

$$A \longrightarrow Y.$$

(b) Let

$$A \xhookrightarrow{i} X$$

denote the inclusion map. Prove that a map

$$Y \xrightarrow{f} A$$

is continuous if and only if the map

$$Y \xrightarrow{i \circ f} X$$

is continuous.

(c) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let A be a subset of X equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Give an example to show that a continuous map

$$A \xrightarrow{f} Y$$

need not extend to a continuous map

$$X \longrightarrow Y.$$

In other words, find topological spaces $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(A, \mathcal{O}_A)$ and a continuous map

$$A \xrightarrow{f} Y$$

which cannot be the restriction of any continuous map

$$X \longrightarrow Y.$$

Solution.

(a) Let f' denote the restriction of f to A. Let $U \in \mathcal{O}_Y$. Then $(f')^{-1}(U) = A \cap f^{-1}(U)$. Since f is continuous, $f^{-1}(U) \in \mathcal{O}_X$. Hence, by definition of $\mathcal{O}_A$, we have that $A \cap f^{-1}(U)$ is open in $(A, \mathcal{O}_A)$. Thus $(f')^{-1}(U)$ is open in $(A, \mathcal{O}_A)$.

(b) By Proposition 2.15 in the Lecture Notes, i is continuous. Thus if f is continuous, then $i \circ f$ is continuous by Proposition 2.16 in the Lecture Notes.

Conversely, suppose that $i \circ f$ is continuous. Let $U \in \mathcal{O}_A$. Then $U = A \cap U'$ for some $U' \in \mathcal{O}_X$. We have that

$$\begin{aligned}(i \circ f)^{-1}(U') &= f^{-1}\big(i^{-1}(U')\big) \\ &= f^{-1}(A \cap U') \\ &= f^{-1}(U).\end{aligned}$$

If $i \circ f$ is continuous, then $(i \circ f)^{-1}(U')$ is open in Y, and hence $f^{-1}(U)$ is open in Y.

(c) We can for instance take both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ to be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, let $A = (-\infty, 0) \sqcup (0, \infty)$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, and define

$$A \xrightarrow{f} \mathbb{R}$$

to be the map given by $x \mapsto 0$ if $x \in (-\infty, 0)$ and $x \mapsto 1$ if $x \in (0, \infty)$.

Then f is continuous. After we have explored 'coproduct topologies' we will be able to see this immediately, but let us here verify it by hand. Let $U \in \mathcal{O}_{\mathbb{R}}$. If $0 \in U$ and $1 \notin U$, then $f^{-1}\big(U\big) = (-\infty, 0) \in \mathcal{O}_A$. If $1 \in U$ and $0 \notin U$, then $f^{-1}\big(U\big) = (0, \infty) \in \mathcal{O}_A$. If $0 \in U$ and $1 \in U$, then $f^{-1}\big(U\big) = (-\infty, 0) \cup (0, \infty)$, which is open in $\mathcal{O}_A$. Finally, if $0 \notin U$ and $1 \notin U$, then $f^{-1}\big(U\big) = \emptyset \in \mathcal{O}_A$.

Let

$$\mathbb{R} \xrightarrow{f'} \mathbb{R}$$

be a map whose restriction to A is f. If $f'(0) \neq 0$ and $f' \neq 1$, there is an open interval (a, b) such that $f'(0) \in (a, b)$ and $0 \notin (a, b)$ and $1 \notin (a, b)$. Then $f'\big((a, b)\big) = \{0\}$, which is not open in $\mathbb{R}$.

If $f'(0) = 0$, then for any $U \in \mathcal{O}_{\mathbb{R}}$ such that $0 \in U$ we have that $(f')^{-1}(U) = (-\infty, 0]$, which is not open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ since it cannot be obtained as a union of open intervals (check that you can rigorously prove this — it is not difficult!).

If $f'(0) = 1$, then for any $U \in \mathcal{O}_{\mathbb{R}}$ such that $0 \in U$ we have that $(f')^{-1}(U) = [0, \infty)$, which similarly is not open in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

This proves that f' cannot be continuous.

7

Question.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, and let A be a subset of Y. Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(Y, \mathcal{O}_Y)$.

(a) Prove that if

$$X \xrightarrow{f} Y$$

is a continuous map such that $f(X) \subset A$, then the map

$$X \longrightarrow A$$

given by $x \mapsto f(x)$ is continuous.

(b) Prove that if

$$X \xrightarrow{f} A$$

is a continuous map, then the map

$$X \longrightarrow Y$$

given by $x \mapsto f(x)$ is continuous.

Solution.

(a) Let

$$X \xrightarrow{f'} A$$

denote the map given by $x \mapsto f(x)$. Let $U \in \mathcal{O}_A$. By definition of $\mathcal{O}_A$, we have that $U = A \cap U'$ for some $U' \in \mathcal{O}_Y$. Since $f(X) \subset A$, we have that

$$\begin{aligned} f^{-1}(U) &= f^{-1}(A \cap U') \\ &= f^{-1}(A) \cap f^{-1}(U') \\ &= X \cap f^{-1}(U') \\ &= f^{-1}(U'). \end{aligned}$$

Since $(f')^{-1}(U) = f^{-1}(U)$, we deduce that $(f')^{-1}(U) = f^{-1}(U')$. Since f is continuous, $f^{-1}(U') \in \mathcal{O}_X$. Thus $(f')^{-1}(U) \in \mathcal{O}_X$.

(b) Let

$$X \xrightarrow{f'} Y$$

denote the map given by $x \mapsto f(x)$. Let $U \in \mathcal{O}_Y$. By definition of $\mathcal{O}_A$, we have that $A \cap U$ is open in $(A, \mathcal{O}_A)$. Since f is continuous, we deduce that $f^{-1}(A \cap U)$ is open in $(X, \mathcal{O}_X)$. We have that $f^{-1}(A \cap U) = f^{-1}(A) \cap f^{-1}(U) = X \cap f^{-1}(U) = f^{-1}(U) = (f')^{-1}(U)$. Thus $(f')^{-1}(U)$ is open in $(X, \mathcal{O}_X)$.

8

Let X and Y be sets, and let $\{A_j\}_{j \in J}$ be a set of subsets of $(X, \mathcal{O}_X)$ such that $X = \bigcup_{j \in J} A_j$. Let $A = \bigcap_{j \in J} A_j$.

Suppose that for every $j \in J$ we have a map

$$A_j \xrightarrow{f_j} Y$$

such that the restriction of f_j to A' is equal to the restriction of $f_{j'}$ to A for all $(j, j') \in J \times J$. Then the map

$$X \xrightarrow{g} Y$$

given by $x \mapsto f_j(x)$ if $x \in A_j$ is well-defined.

Now let $\mathcal{O}_X$ be a topology upon X, and let $\mathcal{O}_Y$ be a topology upon Y. Equip every A_j for $j \in J$ with the subspace topology with respect $(X, \mathcal{O}_X)$. Suppose that f_j is continuous for every $j \in J$.

Question.

(a) Prove that if A_j is open in $(X, \mathcal{O}_X)$ for every $j \in J$, then g is continuous.

(b) Prove that if J is finite and A_j is closed in $(X, \mathcal{O}_X)$ for every $j \in J$, then g is continuous.

(c) Find an example to show that for an arbitrary finite set $\{A_j\}$, it need not be the case that g is continuous.

(d) Find an example to show that when J is infinite, then g need not be continuous even if A_j is closed in $(X, \mathcal{O}_X)$ for every $j \in J$.

Remark 0.1. The result of (a) and (b) is known as the *glueing lemma* or *pasting lemma.*

Solution.

(a) Let $U \in \mathcal{O}_Y$. We have that $g^{-1}(U) = \bigcup_{j \in J} f_j^{-1}(U)$. Since f_j is continuous for all $j \in J$ we have that $f_j^{-1}(U) \in \mathcal{O}_{A_j}$ for all $j \in J$. Since A_j is open in X, we deduce that $\bigcup_{j \in J} f_j^{-1}(U)$ is open in $(X, \mathcal{O}_X)$. Thus $g^{-1}(U)$ is open in X.

(b) By induction, it suffices to consider the case that $X = A_1 \cup A_2$ for subsets A_1 and A_2 of X. Let

$$A_1 \xrightarrow{f_1} Y$$

denote the restriction of f to A_1, and let

$$A_1 \xrightarrow{f_1} Y$$

denote the restriction of f to A_2.

Let V be a closed subset of Y. Since f_1 is continuous, by Question 1 (a) we have that $f_1^{-1}(V)$ is closed in A_1. Since A_1 is closed in X, we deduce that $f_1^{-1}(V)$ is closed in X.

Similarly, since f_2 is continuous, by Question 2 (a) we have that $f_2^{-1}(V)$ is closed in A_2. Since A_2 is closed in X, we deduce that $f_2^{-1}(V)$ is closed in X.

Note that

$$f^{-1}(V) = f_1^{-1}(V) \cup f_2^{-1}(V).$$

Since $f_1^{-1}(V)$ and $f_2^{-1}(V)$ are closed in X, we deduce that $f^{-1}(V)$ is closed in X.

(c) Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, and let $X = (-\infty, 0) \cup [0, \infty)$. Let

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

denote the map given by

$$x \mapsto \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

The restriction f_1 of f to $(-\infty, 0)$ is the constant map given by $x \mapsto 0$ for all $x \in (-\infty, 0)$. The restriction f_2 of f to $[0, \infty)$ is the constant map given by $x \mapsto 1$ for all $x \in [0, \infty)$. By Proposition 2.18 in the Lecture Notes, we have that both f_1 and f_2 are continuous.

But f is not continuous, since $f^{-1}(U) = [0, \infty)$ for any $U \in \mathcal{O}_{\mathbb{R}}$ such that $1 \in U$ and $0 \notin U$. As we already observed in the solution to Question 6 (c), $[0, \infty)$ is not an open subset of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

(d) Let $(X, \mathcal{O}_X) = [0, 1]$, let $A_n = [\frac{1}{n}, 1]$ for any $n \in \mathbb{N}$ with $n \geq 1$, and let $A_0 = \{0\}$. Define

$$[\tfrac{1}{n}, 1] \xrightarrow{f_n} \mathbb{R}$$

to be the constant map given by $x \mapsto 1$ for all $x \in [\frac{1}{n}, 1]$. Let

$$\{0\} \xrightarrow{f_0} \mathbb{R}$$

be the map $0 \mapsto 0$. By Proposition 2.16 of the Lecture Notes, f_n is continuous for all $n \geq 0$. The corresponding map

$$[0, 1] \xrightarrow{g} \mathbb{R}$$

is given by

$$x \mapsto \begin{cases} 1 & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

Thus g is not continuous, since for example $g^{-1}(U) = \{0\}$ for any neighbourhood U of 0 in $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ which does not contain 1, and $\{0\}$ is not open in $[0, 1]$.

9

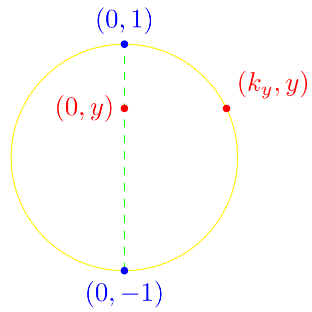
In this question, we will construct step-by-step a continuous map

$$\mathbb{R} \xrightarrow{\phi} S^1.$$

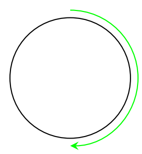
For any $y \in [-1, 1]$, there is a unique $k_y \in \mathbb{R}$ with $k_y \geq 0$ such that $\|(k_y, y)\| = 1$. We have that

$$k_y = \sqrt{1 - y^2},$$

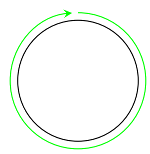
where we take the positive square root.



Given $x \in [0, \frac{1}{2}]$, let $y = 1 - 4x$, and define $\phi(x)$ to be (k_y, y). We may picture ϕ on $[0, 1]$ as follows.



Given $x \in \mathbb{R}$ such that $x \in [\frac{1}{2}, 1]$, let $y = 4x - 3$, and define $\phi(x)$ to be $(-k_y, y)$. We may picture ϕ on $[0, 1]$ as follows.



Given $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $x \in [n, n + 1]$, we define $\phi(x)$ to be $\phi(x - n)$.

Remark 0.2. The map ϕ allows us to construct paths around a circle without using trigonometric maps. Sine and cosine define continuous maps, but the proof of this is quite involved. One has two choices.

(1) Appeal to a notion of angle, which requires a rigorous definition of arc length.

(2) Appeal to analytic methods such as power series.

Both of these approaches are quite far removed from our intuitive geometric understanding of paths around a circle! Thus we will not go into this. The map ϕ is simpler, and we can construct any path around a circle that we are interested in using it!

Question.

(a) Prove that the map

$$[0, \tfrac{1}{2}] \longrightarrow \mathbb{R}$$

given by $y \mapsto k_y$ is continuous.

(b) Deduce that the maps

$$[0, \tfrac{1}{2}] \xrightarrow{\phi} S^1$$

and

$$[\tfrac{1}{2}, 1] \xrightarrow{\phi} S^1$$

are continuous.

(c) Deduce that the map

$$[0, 1] \xrightarrow{\phi} S^1$$

is continuous.

(d) Conclude that the map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is continuous.

Solution.

(a) Let us first prove that if

$$X \xrightarrow{f} \mathbb{R}$$

is a continuous map, then the map

$$X \xrightarrow{\sqrt{f}} \mathbb{R}$$

given by $x \mapsto \sqrt{f(x)}$ is continuous. Since the set of open intervals (a, b) for $a, b \in \mathbb{R}$ defines a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, by Question 1 (c) it suffices to check that $\big(\sqrt{f}\big)^{-1}(a, b)$ is open in $\mathbb{R}$ for any open interval (a, b).

Indeed,

$$\begin{aligned} \big(\sqrt{f}\big)^{-1}\big((a,b)\big) &= \{x \in X \mid a < \sqrt{f(x)} < b\} \\ &= \{x \in X \mid a^2 < f(x) < b^2\}, \end{aligned}$$

since we are taking $\sqrt{f(x)}$ to be the positive square root. Thus

$$\big(\sqrt{f}\big)^{-1}\big((a,b)\big) = f^{-1}\big((a^2, b^2)\big).$$

Since f is continuous, we have that $f^{-1}\big((a^2, b^2)\big)$ is open in X. Thus $\big(\sqrt{f}\big)^{-1}\big((a,b)\big)$ is open in X.

This completes the proof that if

$$X \xrightarrow{f} \mathbb{R}$$

is continuous, then

$$X \xrightarrow{\sqrt{f}} \mathbb{R}$$

is continuous.

By Question 3 (f) and Question 6 (a) we have that the map

$$[0, \tfrac{1}{2}] \xrightarrow{f} \mathbb{R}$$

given by $y \mapsto 1 - y^2$ is continuous. Thus the map

$$[0, \tfrac{1}{2}] \longrightarrow \mathbb{R}$$

given by $y \mapsto k_y$ is continuous, since it is exactly $\sqrt{f}$.

(b) The map

$$[0, \tfrac{1}{2}] \xrightarrow{g} \mathbb{R}^2$$

given by $x \mapsto (k_y, y)$ with $y = 1 - 4x$ is $g' \times i$, where

$$[0, \tfrac{1}{2}] \xrightarrow{g'} \mathbb{R}$$

is the map given by $x \mapsto k_y$ with $y = 1 - 4x$ and

$$[0, \tfrac{1}{2}] \xrightarrow{i} \mathbb{R}$$

is the inclusion map.

We have that $g' = f \circ f'$, where

$$[0, \tfrac{1}{2}] \xrightarrow{f'} [-1, 1]$$

is the map given by $x \mapsto 1 - 4x$, and where

$$[-1, 1] \xrightarrow{f} \mathbb{R}$$

is the map given by $y \mapsto k_y$.

By part (a), we have that f is continuous. By Question 3 (f), we have that f' is continuous. Thus, by Proposition 2.16 in the Lecture Notes, we have that g' is continuous. Moreover, by Proposition 2.15 in the Lecture Notes, we have that i is continuous. Thus by Question 4 (a) we have that $g = g' \times i$ is continuous. By Question 7 (a), we deduce that the map

$$[0, \tfrac{1}{2}] \xrightarrow{\phi} S^1$$

is continuous.

Similarly the map

$$[\tfrac{1}{2}, 1] \xrightarrow{g} \mathbb{R}^2$$

given by $x \mapsto (-k_y, y)$ with $y = 4x - 3$ is $g' \times i$, where

$$[\tfrac{1}{2}, 1] \xrightarrow{g'} \mathbb{R}$$

is the map given by $x \mapsto k_y$ with $y = 4x - 3$ and

$$[\tfrac{1}{2}, 1] \xrightarrow{i} \mathbb{R}$$

is the inclusion map.

We have that $g' = f \circ f'$, where

$$[\tfrac{1}{2}], 1 \xrightarrow{f'} [-1, 1]$$

is the map given by $x \mapsto 4x - 3$, and where

$$[-1, 1] \xrightarrow{f} \mathbb{R}$$

is the map given by $y \mapsto -k_y$.

We observed above that the map

$$[-1, 1] \longrightarrow \mathbb{R}$$

given by $y \mapsto k_y$ is continuous. By Question 3 (f) we deduce that f is continuous. By Question 3 (f), we have that f' is continuous. Thus, by Proposition 2.16 in the Lecture Notes, we have that g' is continuous. Moreover, by Proposition 2.15 in the

Lecture Notes, we have that i is continuous. Thus by Question 4 (a) we have that $g = g' \times i$ is continuous. By Question 7 (a), we deduce that the map

$$[\tfrac{1}{2}, 1] \xrightarrow{\phi} S^1$$

is continuous.

(c) It follows immediately from part (b) and Question 8 that

$$[0, 1] \xrightarrow{\phi} S^1$$

is continuous.

(d) For any $n \in \mathbb{Z}$, the map

$$[n, n+1] \xrightarrow{g} [0, 1]$$

given by $x \mapsto x - n$ is continuous by Question 3 (f). Moreover, by part (c) the map

$$[0, 1] \xrightarrow{\phi} S^1$$

is continuous. Since the map

$$[n, n+1] \longrightarrow S^1$$

given by $x \mapsto \phi(x - n)$ is $g \circ \phi$, we deduce by Proposition 2.16 in the Lecture Notes that it is continuous.

We deduce by Question 8 that

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is continuous, since we have proven that its restriction to $[n, n+1]$ is continuous for every $n \in \mathbb{Z}$.

10

Question.

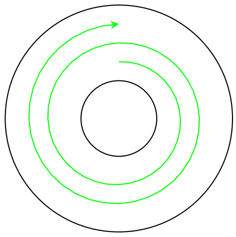
(a) Prove that the map of Example 2.13 (2) in the Lecture Notes is continuous.

(b) Prove that the map of Example 2.13 (3) in the Lecture Notes is continuous.

(c) Find a continuous map

$$I \longrightarrow A_k$$

for a fixed $0 < k < \frac{1}{2}$ which describes a 'spiral' as roughly depicted below, starting at $(0, \frac{1}{2})$, passing through $(0, \frac{5}{8})$, and ending at $(0, \frac{3}{4})$.



(d) Prove that the map

$$I^2 \longrightarrow I$$

given by $(x, y) \mapsto \min\{x, y\}$ is continuous. Also, prove that the map

$$I^2 \longrightarrow I$$

given by $(x, y) \mapsto \max\{x, y\}$ is continuous. Draw a picture of each of these maps! You may find it helpful to think of the copy of I in the target as a diagonal in I^2.

Solution.

(a) Let

$$D^2 \times I \xrightarrow{f} D^2$$

be given by $(x, y, t) \mapsto \big((1-t)x, (1-t)y\big)$. Consider the map

$$\mathbb{R}^2 \times I \xrightarrow{g \times g'} \mathbb{R}^2,$$

where

$$\mathbb{R}^2 \times I \xrightarrow{g} \mathbb{R}$$

is given by $(x, y, t) \mapsto (1-t)x$, and where

$$\mathbb{R}^2 \times I \xrightarrow{g'} \mathbb{R}$$

is given by $(x, y, t) \mapsto (1-t)y$.

By Question 3 (f) and Question 6 (a), the map

$$I \xrightarrow{u} \mathbb{R}$$

given by $t \mapsto 1-t$ is continuous. Moreover, the identity map

$$\mathbb{R} \xrightarrow{id} \mathbb{R}$$

is continuous. By Question 4 (c), we deduce that the map

$$\mathbb{R}^2 \xrightarrow{id \cdot u} \mathbb{R}$$

is continuous.

Thinking of $\mathbb{R}^2 \times I$ as $\mathbb{R} \times (\mathbb{R} \times I)$, let

$$\mathbb{R}^2 \times I \xrightarrow{p_2} \mathbb{R} \times I$$

denote the projection map. By Proposition 3.2 in the Lecture Notes, we have that p_2 is continuous. We have that $g' = (id \cdot u) \circ p_2$. By Proposition 2.16 in the Lecture Notes, we deduce that g' is continuous.

Let

$$\mathbb{R}^2 \times I \xrightarrow{q} \mathbb{R} \times I$$

denote the map given by $(x, y, t) \mapsto (x, t)$. Then $p = p_2 \circ (\tau \times id)$, where

$$\mathbb{R}^2 \xrightarrow{\tau} \mathbb{R}^2$$

is the map of Question 4 (e). By Question 4 (e), we have that τ is continuous. Since id is continuous, we deduce by Question 4 (c) that q is continuous. Observe also that $g = (id \cdot u) \circ q$. Thus, by Proposition 2.16 in the Lecture Notes, we conclude that g is continuous.

Putting everything together, by Question 4 (a) we deduce that the map

$$\mathbb{R}^2 \times I \xrightarrow{g \times g'} \mathbb{R}^2$$

is continuous. Hence, by Question 6 (a), the restriction of $g \times g'$ to $D^2 \times I$ is continuous. Since the image of this restriction is contained in (in fact equal to) D^2, we conclude by Question 7 (a) that $g \times g'$ defines a continuous map

$$D^2 \times I \longrightarrow D^2.$$

This map is exactly f.

(b) Let $k \in \mathbb{R}$, and let

$$I \xrightarrow{f} S^1$$

be given by $t \mapsto \phi(kt)$. Then $f = \phi \circ g$, where

$$I \xrightarrow{g} \mathbb{R}$$

is the map given by $t \mapsto kt$. By Question 9 (d), we have that ϕ is continuous. By Question 3 (f), we have that g is continuous. Hence, by Proposition 2.16 in the Lecture Notes, f is continuous.

(c) The map

$$I \xrightarrow{f} A_k$$

given by $t \mapsto \frac{1}{2}\phi(t) + (\frac{1}{4}t, 0)$ gives rise to a spiral with the required properties. We must show that f is continuous.

In order to do so, let us first prove that if $(X, \mathcal{O}_X)$ is a topological space and

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \mathbb{R}^2 \end{tikzcd}$$

are continuous maps, then the map

$$X \xrightarrow{u+v} \mathbb{R}^2$$

given by $(x, y) \mapsto u(x, y) + v(x, y)$ is continuous. Indeed, we have that $u + v$ is

$$\Big((p_1 \circ u) + (p_1 \circ v)\Big) \times \Big((p_2 \circ u) + (p_2 \circ v)\Big).$$

Here

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathbb{R} \end{tikzcd}$$

are the projection maps.

By Proposition 3.2 in the Lecture Notes, p_1 is continuous. Thus, by Proposition 2.16 in the Lecture Notes, $p_1 \circ u$ and $p_1 \circ v$ are continuous. Hence, by Question 3 (h), we have that $(p_1 \circ u) + (p_1 \circ v)$ is continuous.

By an entirely analogous argument, $(p_2 \circ u) + (p_2 \circ v)$ is continuous. We deduce by Question 4 (a) that $u + v$ is continuous.

We now turn to proving that f is continuous. Since the map

$$I \xrightarrow{\phi} \mathbb{R}^2$$

is continuous by Question 9 (c) and Question 7 (b), we deduce by Question 3 (b) that the map

$$I \xrightarrow{\frac{1}{2}\phi} \mathbb{R}^2$$

is continuous.

The map

$$I \xrightarrow{g} \mathbb{R}^2$$

given by $t \mapsto (\frac{1}{4}t, 0)$ is $g' \times 0$, where

$$I \xrightarrow{0} \mathbb{R}$$

is the constant map $t \mapsto 0$, and

$$I \xrightarrow{g'} \mathbb{R}$$

is the map given by $t \mapsto \frac{1}{4}t$. By Proposition 2.18 in the Lecture Notes, the map

$$I \xrightarrow{0} \mathbb{R}$$

is continuous. By Question 3 (f) and Question 6 (a), the map g' is continuous. Thus, by Question 4 (a), we have that $g = g' \times 0$ is continuous.

We deduce that the map

$$I \xrightarrow{\frac{1}{2}\phi + g} \mathbb{R}^2$$

is continuous. Since the image of this map is contained in A_k, we deduce by Question 7 (a) that $\frac{1}{2}\phi + g$ defines a continuous map

$$I \longrightarrow A_k.$$

This map is exactly f.

(d) The map

$$I^2 \xrightarrow{f} I$$

given by $(x, y) \mapsto \min\{x, y\}$ can be pictured as mapping everything below the diagonal horizontally left to the diagonal, and everything above the diagonal vertically down to the diagonal.

The map

$$I^2 \longrightarrow I$$

given by $(x, y) \mapsto \max\{x, y\}$ can be pictured as mapping everything below the diagonal horizontally right to the right vertical face, and everything above the diagonal vertically up to the upper horizontal face.

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Question.

Let $\mathbb{R}$ be equipped with its standard topology $\mathcal{O}_{\mathbb{R}}$. Prove that a map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous in the topological sense if and only if it is continuous in the $\epsilon - \delta$ sense that you have met in real analysis/calculus, namely for all $x, c, \epsilon \in \mathbb{R}$ with $\epsilon > 0$ there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

Hint:

(1) Appeal to Examples 2.9 (1).

(2) Appeal to Question 1 (e).

Solution.

By Examples 2.9 (1) in the lecture notes, $\{B_\epsilon(x)\}_{x \in \mathbb{R}, \epsilon \in \mathbb{R}, \epsilon > 0}$ defines a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Question 1 (e) we deduce that f is continuous in the topological sense if and only if for all $x, y, \epsilon \in \mathbb{R}$ with $\epsilon > 0$ such that $f(x) \in B_\epsilon(y)$ there are $y', \delta \in \mathbb{R}$ with $y' > 0$ such that $x \in B_\delta(y')$ and $f\big(B_\delta(y')\big) \subset B_\epsilon(y)$.

Suppose first that f is continuous in the topological sense. Taking y to be $f(x)$, we have that there are $y', \delta \in \mathbb{R}$ with $y' > 0$ such that $x \in B_\delta(y')$ and $f\big(B_\delta(y')\big) \subset B_\epsilon\big(f(x)\big)$.

Let $\delta' = \min\{x - (y' - \delta), (y' + \delta) - x\}$. Then $B_{\delta'}(x) \subset B_{\delta}(y')$, and hence $f\big(B_{\delta'}(x)\big) \subset B_{\epsilon}\big(f(x)\big)$, as required.

Conversely, suppose that f is continuous in the ϵ-δ sense. Let $x, y, \epsilon \in \mathbb{R}$ with $\epsilon > 0$ be such that $f(x) \in B_{\epsilon}(y)$. Let $\epsilon' = \min\{f(x) - \big(y - \epsilon), (y + \epsilon) - f(x)\}$. Then $B_{\epsilon'}\big(f(x)\big) \subset B_{\epsilon}(y)$. Take c to be x. Since f is continuous in the ϵ-δ sense, there exists $\delta \in \mathbb{R}$ with $\delta > 0$ such that $f\big(B_{\delta}(x)\big) \subset B_{\epsilon'}\big(f(x)\big) \subset B_{\epsilon}(y)$.

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Let $(X, <)$ and $(Y, <)$ be pre-orderings. A *morphism* from $(X, <)$ to $(Y, <)$ is a map

$$X \xrightarrow{f} Y$$

such that if $x < x'$ then $f(x) < f(x')$.

Question.

(a) What does this requirement correspond to if we picture $(X, <)$ and $(Y, <)$ via arrows as in Question 8 of Exercise Sheet 1?

Recall that by Question 10 of Exercise Sheet 2, Alexandroff topologies on a set X correspond bijectively to pre-orderings on X, in the following way.

(i) Let $(X, \mathcal{O}_X)$ be an Alexandroff topological space. Given $x \in X$, define U_x to be the intersection of all neighbourhoods of x in $(X, \mathcal{O}_X)$. To $(X, \mathcal{O}_X)$ we associate the pre-ordering $<$ defined by $x < x'$ if $U_x \supset U_{x'}$.

(ii) Let $(X, <)$ be a pre-ordering. We define a topology $\mathcal{O}_X$ on X by stipulating that $U \subset X$ belongs to $\mathcal{O}_X$ if for any $x \in U$ and any $x' \in X$ such that $x < x'$ we have that $x' \in U$. We have that $(X, \mathcal{O}_X)$ is an Alexandroff space.

Question.

(b) Let $(X, \mathcal{O}_X)$ be an Alexandroff topological space, and let $<_X$ denote the corresponding pre-ordering of X. Let $(Y, \mathcal{O}_Y)$ be another Alexandroff topological space, and let $<_Y$ denote the corresponding pre-ordering.

Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if f defines a morphsm from $(X, <_X)$ to $(Y, <_Y)$.

Solution.

(a) The requirement that if $x < x'$ then $f(x) < f(x')$ corresponds to requiring that if there is an arrow from x to x' in X, then there is an arrow from $f(x)$ to $f(x')$ in Y.

(b) Suppose that f is continuous. Let $x, x' \in X$ be such that $x <_X x'$. By Question 10 (b) of Exercise Sheet 2, we have that $\{U_x\}_{x \in X}$ defines a basis for $(X, \mathcal{O}_X)$, and $\{U_y\}_{y \in Y}$ defines a basis for $(Y, \mathcal{O}_Y)$. By Question 1 (e), we deduce that there is an $x'' \in X$ such that $x \in U_{x''}$ and $U_{x''} \subset f^{-1}(U_{f(x)})$.

By definition of U_x, we have that $U_x \subset U_{x''}$, and hence that $U_x \subset f^{-1}(U_{f(x)})$. Moreover, by definition of $<_X$, we have that $U_x \supset U_{x'}$. Thus we have that $U_{x'} \subset f^{-1}(U_{f(x)})$. Since $x' \in U_{x'}$, we deduce that $f(x') \in U_{f(x)}$.

By definition of $U_{f(x')}$, we conclude that $U_{f(x')} \subset U_{f(x)}$. Thus by definition of $<_Y$ we have that $f(x) \leq_Y f(x')$.

Conversely, suppose that if $x, x' \in X$ have the property that $x \leq_X x'$, then $f(x) \leq_Y f(x')$. Let $x \in X$, and let U be a neighbourhood of $f(x)$ in $(Y, \mathcal{O}_Y)$. Then by definition of $U_{f(x)}$, we have that $U_{f(x)} \subset U$.

Let $x' \in U_x$. Then by definition of $U_{x'}$ we have that $U_{x'} \subset U_x$, and hence that $x \leq_X x'$. By assumption, we deduce that $f(x) \leq_Y f(x')$. By definition of $<_Y$, we then have that $U_{f(x)} \supset U_{f(x')}$. Hence $U_{f(x')} \subset U$. In particular, since $f(x') \in U_{f(x')}$ we have that $f(x') \in U$.

This proves that $f(U_x) \subset U$. We have that $x \in U_x$. By Question 1 (b), we conclude that f is continuous.