MA3002 Generell Topologi — Våren 2014

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May 19, 2014

Contents

Part I.

Point-set foundations

1. Monday 6th January

1.1. Definition of a topological space

Definition 1.1.1. Let X be a set, and let $\mathcal{O}$ be a set of subsets of X. Then $(X, \mathcal{O})$ is a *topological space* if the following hold.

(1) The empty set $\emptyset$ belongs to $\mathcal{O}$.

(2) The set X belongs to $\mathcal{O}$.

(3) Let U be a union of (possibly infinitely many) subsets of X which belong to $\mathcal{O}$. Then U belongs to $\mathcal{O}$.

(4) Let U and U' be subsets of X which belong to $\mathcal{O}$. Then $U \cap U'$ belongs to $\mathcal{O}$.

Remark 1.1.2. By induction, the following holds if and only if (4) holds.

(4') Let J be a finite set, and let $\{U_j\}_{j \in J}$ be a set of subsets of X such that U_j belongs to $\mathcal{O}$ for all $j \in J$. Then $\bigcap_j U_j$ belongs to $\mathcal{O}$.

Terminology 1.1.3. Let $(X, \mathcal{O})$ be a topological space. We refer to $\mathcal{O}$ as a *topology* on X.

A set may be able to be equipped with many different topologies! See §1.4.

1.2. Open and closed subsets

Notation 1.2.1. Let X be a set. By $A \subset X$ we shall mean that A is a subset of X, allowing that A may be equal to X. In the past, you may instead have written $A \subseteq X$.

Terminology 1.2.2. Let $(X, \mathcal{O})$ be a topological space.

(1) Let U be a subset of X. Then U is *open* with respect to $\mathcal{O}$ if U belongs to $\mathcal{O}$.

(2) Let V be a subset of X. Then V is *closed* with respect to $\mathcal{O}$ if $X \setminus V$ is an open subset of X with respect to $\mathcal{O}$.

1.3. Discrete and indiscrete topologies

Example 1.3.1. We can equip any set X with the following two topologies.

(1) The *discrete topology*, consisting of all subsets of X. In other words, the power set of X.

(2) The *indiscrete topology*, given by $\{\emptyset, X\}$.

Remark 1.3.2. By (1) and (2) of Definition 1.1.1, every topology on a set X must contain both $\emptyset$ and X. Thus the indiscrete topology is the smallest topology with which X may be equipped.

1.4. Finite examples of topological spaces

Example 1.4.1. Let $X = \{a\}$ be a set with one element. Then X can be equipped with exactly one topology, given by $\{\emptyset, X\}$. In particular, the discrete topology on X is the same as the indiscrete topology on X.

Remark 1.4.2. The topological space of Example 1.4.1 is important! It is known as the *point*.

Example 1.4.3. Let $X = \{a, b\}$ be a set with two elements. We can define exactly four topologies upon X.

(1) The discrete topology, given by $\big\{\emptyset, \{a\}, \{b\}, X\big\}$.

(2) The topology given by $\big\{\emptyset, \{a\}, X\big\}$.

(3) The topology given by $\big\{\emptyset, \{b\}, X\big\}$.

(4) The indiscrete topology, given by $\big\{\emptyset, X\big\}$.

Remark 1.4.4. Up to the bijection

$$X \xrightarrow{f} X$$

given by $a \mapsto b$ and $b \mapsto a$, or in other words up to relabelling the elements of X, the topologies of (2) and (3) are the same.

Terminology 1.4.5. The topological space $(X, \mathcal{O})$, where $\mathcal{O}$ is the topology of (2) or (3), is known as the *Sierpiński interval*, or *Sierpiński space*.

Remark 1.4.6. In fact (1) – (4) is a list of every possible set of subsets of X which contains $\emptyset$ and X. In other words, every set of subsets of X which contains $\emptyset$ and X defines a topology on X.

Example 1.4.7. Let $X = \{a, b, c\}$ be a set with three elements. We can equip X with exactly twenty nine topologies! Up to relabelling, there are exactly nine.

(1) The set
$$\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$$
defines a topology on X.

(2) The set $\mathcal{O}_X$ given by
$$\{\emptyset, \{a\}, \{c\}, X\}$$
does not define a topology on X. This is because
$$\{a\} \cup \{c\} = \{a, c\}$$
does not belong to $\mathcal{O}_X$, so (3) of Definition 1.1.1 is not satisfied.

(3) The set $\mathcal{O}_X$ given by
$$\{\emptyset, \{a, b\}, \{a, c\}, X\}$$
does not define a topology on X. This is because
$$\{a, b\} \cap \{a, c\} = \{a\}$$
does not belong to $\mathcal{O}_X$, so (4) of Definition 1.1.1 is not satisfied.

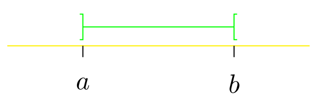
Remark 1.4.8. There are quite a few more 'non-topologies' on X.

1.5. Open, closed, and half open intervals

Notation 1.5.1. Let $\mathbb{R}$ denote the set of real numbers.

Notation 1.5.2. Let $a, b \in \mathbb{R}$.

(1) We denote by $]a, b[$ the set
$$\{x \in \mathbb{R} \mid a < x < b\}.$$



(2) We denote by $]a, \infty[$ the set

$$\{x \in \mathbb{R} \mid x > a\}.$$

(3) We denote by $]-\infty, b[$ the set

$$\{x \in \mathbb{R} \mid x < b\}.$$

(4) We sometimes denote $\mathbb{R}$ by $]-\infty, \infty[$.

Terminology 1.5.3. We shall refer to any of (1) – (4) in Notation 1.5.2 as an *open interval*.

Remark 1.5.4. We shall never use the notation (a, b), (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$ for an open interval. In particular, for us (a, b) will always mean an ordered pair of real numbers a and b.

Notation 1.5.5. Let $a, b \in \mathbb{R}$. We denote by $[a, b]$ the set

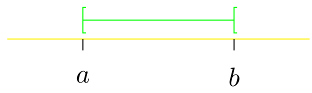
$$\{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Terminology 1.5.6. We shall refer to $[a, b]$ as a *closed interval*.

Notation 1.5.7. Let $a, b \in \mathbb{R}$.

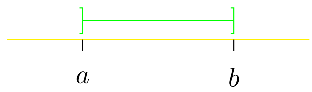
(1) We denote by $[a, b[$ the set

$$\{x \in \mathbb{R} \mid a \leq x < b\}.$$



(2) We denote by $]a, b]$ the set

$$\{x \in \mathbb{R} \mid a < x \leq b\}.$$



(3) We denote by $[a, \infty[$ the set

$$\{x \in \mathbb{R} \mid x \geq a\}.$$

(4) We denote by $]-\infty, b]$ the set

$$\{x \in \mathbb{R} \mid x \leq b\}.$$

Terminology 1.5.8. We shall refer to any of (1) – (4) of Notation 1.5.7 as a *half open interval*.

Terminology 1.5.9. By an *interval* we shall mean a subset of $\mathbb{R}$ which is either an open interval, a closed interval, or a half open interval.

1.6. Standard topology on $\mathbb{R}$

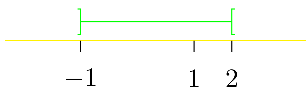
Definition 1.6.1. Let $\mathcal{O}_{\mathbb{R}}$ denote the set of subsets U of $\mathbb{R}$ with the property that, for every $x \in U$, there is an open interval I such that $x \in I$ and $I \subset U$.

Observation 1.6.2. We have that $\mathbb{R}$ belongs to $\mathcal{O}_{\mathbb{R}}$. Moreover $\emptyset$ belongs to $\mathcal{O}_{\mathbb{R}}$, since the required property vacuously holds.

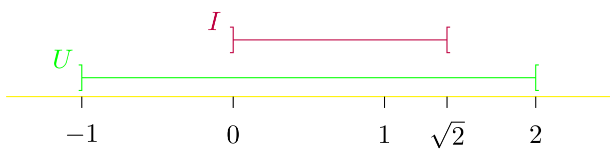
Example 1.6.3. Let U be an open interval $]a, b[$.

Then U belongs to $\mathcal{O}_{\mathbb{R}}$. For every $x \in U$, we can take the corresponding open interval I such that $x \in I$ and $I \subset U$ to be U itself.

There are infinitely many other possibilities for I. For instance, suppose that U is the open interval $]-1, 2[$. Let $x = 1$.



We can take I to be $]-1, 2[$, but also for example $\left]0, \sqrt{2}\right[$.



E1. Exercises for Lecture 1

E1.1. Exam questions

Task E1.1.1. Let $X = \{a, b, c, d\}$. Which of the following defines a topology on X?

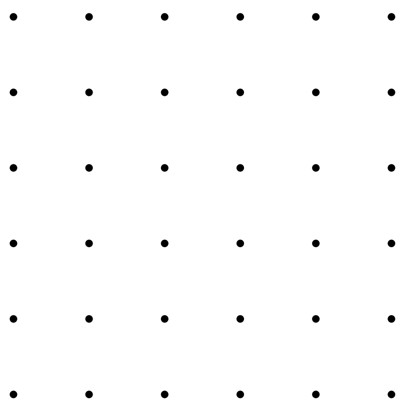
(1) $\{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, d\}, X\}$

(2) $\{\emptyset, \{a, c\}, \{d\}, \{b, d\}, \{a, c, d\}, X\}$

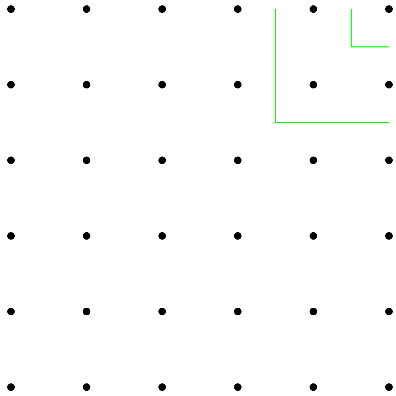
(3) $\{\emptyset, \{a\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, X\}$

Task E1.1.2.

(1) Let X be an $n \times n$ grid of integer points in $\mathbb{R}^2$, where $n \in \mathbb{N}$.

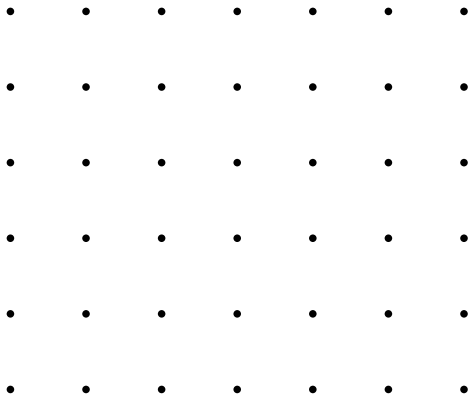


Let $\mathcal{O}$ be the set of subsets of X which are $m \times m$ grids, for $0 \leq m \leq n$, at the top right corner. Think of the case $m = 0$ as the empty set.

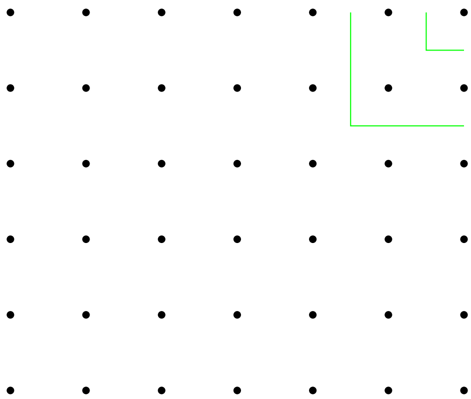


Does $(X, \mathcal{O})$ define a topological space?

(2) Let Y be an $(n+1) \times n$ grid of integer points in $\mathbb{R}^2$.



Let $\mathcal{O}$ be the set of subsets of Y which are $m \times m$ grids, for $0 \leq m \leq n$, at the top right corner. Again, think of the case $m = 0$ as the empty set.

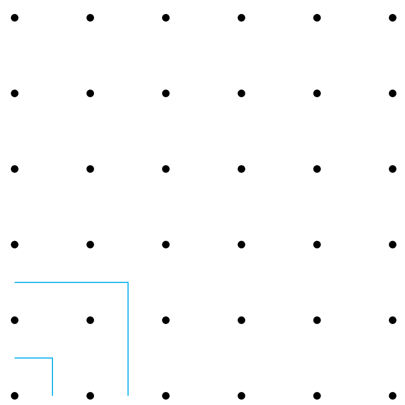


Does $(Y, \mathcal{O})$ define a topological space?

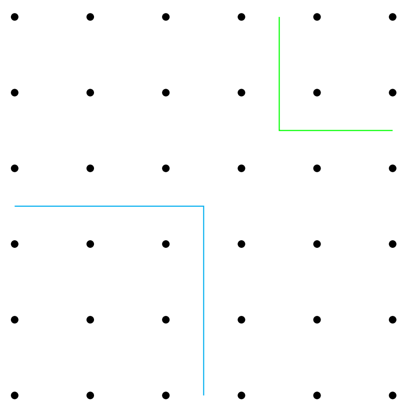
(3) Let X be as in (1). Suppose that $n \geq 3$. Let $\mathcal{O}'$ be the union of the following sets of subsets of X.

(a) $\mathcal{O}$.

(b) The set of subsets of X which are $m \times m$ grids, for $0 \leq m \leq n$, at the bottom left corner.



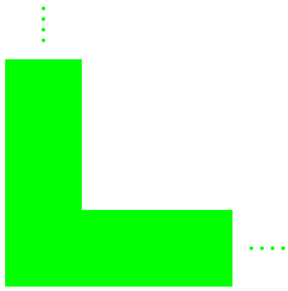
(c) Unions of subsets of X of the kind considered in (a) and (b). For instance, the union of a 3×3 grid at the bottom left corner, and a 2×2 grid at the top right corner.



Does $(X, \mathcal{O}')$ define a topological space?

Task E1.1.3 (Continuation Exam, August 2013)**.** Let X denote the set

$$([0,1] \times]0,\infty]) \cup (]0,\infty] \times [0,1]) \,.$$



Let $\mathcal{O}$ be the union of $\{\emptyset, X\}$, the set

$$\{[0,1] \times [0,n] \mid n \in \mathbb{N}\}$$

and the set

$$\{[0,n] \times [0,1] \mid n \in \mathbb{N}\}.$$

Is $(X, \mathcal{O})$ a topological space?

E1.2. In the lecture notes

Task E1.2.1. Let X be a set.

(1) Verify that conditions (1) – (4) of Definition 1.1.1 are satisfied by the discrete topology on X.

(2) Verify that conditions (1) – (4) of Definition 1.1.1 are satisfied by the indiscrete topology on X.

Task E1.2.2.

(1) Check that you agree that (1) of Example 1.4.3 is the discrete topology.

(2) Verify that (2) and (3) of Example 1.4.3 define topologies.

Task E1.2.3.

(1) Verify that (1) of Example 1.4.7 defines a topology.

(2) Can you find the nine different topologies, up to relabelling, on a set with three elements?

(3) Find four examples of non-topologies on a set with three elements, in addition to (2) and (3) of Example 1.4.7.

E1.3. For a deeper understanding

Task E1.3.1. Let X be a set. Let $\mathcal{C}$ be a set of subsets of X such that the following hold.

(1) The empty set $\emptyset$ belongs to $\mathcal{C}$.

(2) The set X belongs to $\mathcal{C}$.

(3) Let V be an intersection of (possibly infinitely many) subsets of X which belong to $\mathcal{C}$. Then V belongs to $\mathcal{C}$.

(4) Let V and V' be subsets of X which belong to $\mathcal{C}$. Then $V \cup V'$ belongs to $\mathcal{C}$.

Let $\mathcal{O}$ be given by

$$\{X \setminus V \mid V \text{ belongs to } \mathcal{C}\}.$$

Prove that $(X, \mathcal{O})$ is a topological space.

Remark E1.3.2. Conversely, let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{C}$ denote the set of closed subsets of X. Then $\mathcal{C}$ satisfies (1) – (4) of Task E1.3.1.

Task E1.3.3 (Longer)**.** Let I be a subset of $\mathbb{R}$. Prove that I is an interval if and only if it has the following property: if $x < y < x'$ for $x, x' \in I$ and $y \in \mathbb{R}$, then $y \in I$. For proving that I is an interval if this condition is satisfied, you may wish to proceed as follows.

(1) Suppose that I is bounded. Denote the greatest lower bound of I by a, and denote the least upper bound of I by b. Prove that if $a < y < b$, then $y \in I$.

(2) Using this, deduce that I is $]a, b[$, $[a, b]$, $[a, b[$, or $]a, b]$.

(3) Give a proof when I is not bounded.

Remark E1.3.4. Task E1.3.3 relies crucially on the existence of a least upper bound for a subset of $\mathbb{R}$ which is bounded above, and on the existence of a greatest lower bound for a subset of $\mathbb{R}$ which is bounded below. This is known as the *completeness* of $\mathbb{R}$.

We shall demonstrate in later lectures that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ has important properties. To use a couple of terms which we shall define later, it is *connected* and *locally compact*. The proofs ultimately rest upon the completeness of $\mathbb{R}$, via Task E1.3.3.

Task E1.3.5. Let I_0 and I_1 be intervals. Prove that $I_0 \cap I_1$ is an interval. You may wish to appeal to Task E1.3.3.

E1.4. Exploration — Alexandroff topological spaces

Definition E1.4.1. Let X be a set, and let $X \Diamond X$ denote the set of ordered pairs (x_0, x_1) of X such that x_0 is not equal to x_1. A *pre-order* on X is the data of a map

$$X \Diamond X \xrightarrow{\chi} \{0, 1\},$$

or, in other words, for every ordered pair (x_0, x_1) of distinct elements of X, an element of the set $\{0, 1\}$. We require that for any ordered triple (x_0, x_1, x_2) of mutually distinct elements of X, such that $\chi(x_0, x_1) = 1$ and $\chi(x_1, x_2) = 1$, we have that $\chi(x_0, x_2) = 1$.

Terminology E1.4.2. There is an *arrow from x_0 to x_1* if $\chi(x_0, x_1) = 1$. We depict this as follows.

$$x_0 \longrightarrow x_1$$

Example E1.4.3. Let $X = \{0, 1\}$. There are four pre-orders on X, pictured below.

$$0 \longrightarrow 1 \qquad 0 \longleftarrow 1 \qquad 0 \rightleftarrows 1 \qquad 0 \qquad 1$$

The rightmost diagram should be interpreted as: $\chi(0, 1) = 0$ and $\chi(1, 0) = 0$.

Example E1.4.4. Let $X = \{0, 1, 2\}$. There are 29 pre-orders on X. A few are pictured below.

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, shift left] & 1 \arrow[l, shift left] \\ & 2 \arrow[u] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 1 \\ & 2 \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 & 1 \arrow[d] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} 0 & 1 \\ & 2 \end{tikzcd}$$

Example E1.4.5. The following are not examples of pre-orders on X.

$$
\begin{tikzcd}
0 \arrow[r] & 1 \arrow[d] \\
 & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & 1 \arrow[d] \\
 & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, shift left] \arrow[dr] & 1 \arrow[l, shift left] \arrow[d, shift left] \\
 & 2 \arrow[u, shift left]
\end{tikzcd}
$$

Task E1.4.6. Why do the diagrams of Example E1.4.5 not define pre-orders.?

Example E1.4.7. The following defines a pre-order on $\mathbb{N}$.

$$
1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \longrightarrow 6 \longleftarrow 7 \longrightarrow \quad \cdots\cdots
$$

Notation E1.4.8. Let X be a set, and let χ be a pre-order on X. For any pair (x_0, x_1) of elements of X, we write $x_0 < x_1$ if eiher there is an arrow from x_0 to x_1 or $x_0 = x_1$.

Definition E1.4.9. Let $\mathcal{O}_<$ denote the set of subsets U of X with the property that if $x \in U$ and x' has the property that $x < x'$, then $x' \in U$.

Task E1.4.10. Prove that $(X, \mathcal{O}_<)$ is a topological space.

Task E1.4.11. Which of the four pre-orders of Example E1.4.3 corresponds to the topology defining the Sierpiński interval? Which corresponds to the discrete topology? Which to the indiscrete topology?

Task E1.4.12. Find a pre-order on $X = \{a, b, c\}$ which corresponds to the topology $\mathcal{O}$ on X given by

$$
\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\} \,.
$$

Task E1.4.13. List all the subsets of $X = \{a, b, c, d\}$ which belong to the topology $\mathcal{O}$ on X corresponding to the following pre-order.

$$
\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b & & c \\
 & d \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

The topological space $(X, \mathcal{O})$ is sometimes known as the *pseudo-circle*.

Task E1.4.14. Let $(X, <)$ be a set equipped with a pre-order, and let $\mathcal{O}_<$ denote the corresponding topology on X. Prove that, for any set $\{U_j\}_{j \in J}$ of subsets of X belonging to $\mathcal{O}_X$, we have that $\bigcap_{j \in J} U_j$ belongs to $\mathcal{O}_<$. In particular, this holds even if J is infinite.

Remark E1.4.15. In other words, $(X, \mathcal{O}_<)$ is an Alexandroff topological space.

Notation E1.4.16. Let $(X, \mathcal{O})$ be an Alexandroff topological space. For any $x \in X$, let U_x denote the intersection of all subsets of X which contain x and which belong to $\mathcal{O}$.

Definition E1.4.17. Let $(X, \mathcal{O})$ be an Alexandroff topological space. For any $x_0, x_1 \in X$, define $x_0 < x_1$ if $U_{x_1} \subset U_{x_0}$.

Task E1.4.18. Prove that $<$ defines a pre-order on X.

Task E1.4.19. Let $X = \{a, b, c, d, e\}$, and let $\mathcal{O}$ denote the topology on X given by

$$\{\emptyset, \{a, b\}, \{c\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, d, e\}, X\}.$$

Draw the pre-order corresponding to $(X, \mathcal{O})$.

E1.5. Exploration — Zariski topologies

Notation E1.5.1. Let $\mathbb{Z}$ denote the set of integers.

Notation E1.5.2. Let $\mathsf{Spec}(\mathbb{Z})$ denote the set of prime numbers.

Notation E1.5.3. For any integer n, let $V(n)$ denote the set

$$\{p \in \mathbb{Z} \mid p \text{ is prime, and } p \mid n\}.$$

Definition E1.5.4. Let $\mathcal{O}$ denote the set

$$\{\mathsf{Spec}(\mathbb{Z}) \setminus V(n) \mid n \in \mathbb{Z}\}.$$

Task E1.5.5. Prove that $(\mathsf{Spec}(\mathbb{Z}), \mathcal{O})$ is a topological space. You may wish to make use of Task E1.3.1.

Terminology E1.5.6. The topology $\mathcal{O}$ on $\mathsf{Spec}(\mathbb{Z})$ is known as the *Zariski topology*.

Remark E1.5.7 (Ignore if you have not met the notion of a ring before)**.** Generalising this, one can define a topology on the set of prime ideals of any commutative ring. This is a point of departure for *algebraic geometry*.

2. Tuesday 7th January

2.1. Standard topology on $\mathbb{R}$, continued

Example 2.1.1. Let U be a disjoint union of open intervals. For instance, the union of $]-3,-1[$ and $]4,7[$.

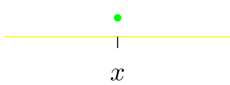
Then U belongs to $\mathcal{O}_{\mathbb{R}}$. There are two cases.

(1) If $-3 < x < -1$, we can, for instance, take I to be $]-3,-1[$

(2) If $4 < x < 7$, we can, for instance, take I to be $]4,7[$.

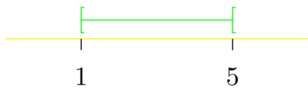
Remark 2.1.2. In fact, *every* subset of U which belongs to $\mathcal{O}_{\mathbb{R}}$ is a disjoint union of (possibly infinitely many) open intervals. To prove this is the topic of Task E2.3.7.

Example 2.1.3. Let $U = \{x\}$ be a subset of $\mathbb{R}$ consisting of a single $x \in \mathbb{R}$.

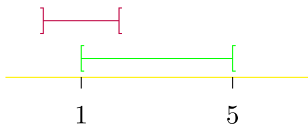


Then U does not belong to $\mathcal{O}_{\mathbb{R}}$. The only subset of $\{x\}$ to which x belongs is $\{x\}$ itself, and $\{x\}$ is not an open interval.

Example 2.1.4. Let U be the half open interval $[1, 5[$.



Then U does not belong to $\mathcal{O}_{\mathbb{R}}$, since there is no open interval I such that $1 \in I$ and $I \subset U$.



Lemma 2.1.5. Let $\{U_j\}_{j \in J}$ be a set of (possibly infinitely many) subsets of $\mathbb{R}$ such that $U_j \in \mathcal{O}_{\mathbb{R}}$ for all $j \in J$. Then $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}_{\mathbb{R}}$.

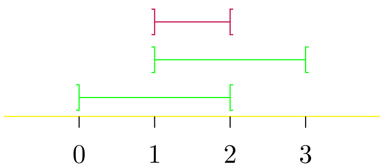
Proof. Let

$$x \in \bigcup_{j \in J} U_j.$$

By definition of $\bigcup_{j \in J} U_j$, we have that $x \in U_j$ for some $j \in J$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval I such that $x \in I$ and $I \subset U_j \subset \bigcup_{j \in J} U_j$. $\square$

Observation 2.1.6. Let I and I' be open intervals. Then $I \cap I'$ is a (possibly empty) open interval. This is the topic of Task E2.2.1.

Example 2.1.7. The intersection of the open intervals $]0, 2[$ and $]1, 3[$ is the open interval $]1, 2[$.



The intersection of the open intervals $]-3, -1[$ and $]4, 7[$ is the empty set.

Lemma 2.1.8. Let U and U' be subsets of $\mathbb{R}$ which belong to $\mathcal{O}_{\mathbb{R}}$. Then $U \cap U'$ belongs to $\mathcal{O}_{\mathbb{R}}$.

Proof. Let $x \in U \cap U'$. By definition of $\mathcal{O}_{\mathbb{R}}$, we have the following.

(1) There is an open interval I_U such that $x \in I_U$ and $I_U \subset U$.

(2) There is an open interval $I_{U'}$ such that $x \in I_{U'}$ and $I_{U'} \subset U'$.

Then $x \in I_U \cap I_{U'}$ and $I_U \cap I_{U'} \subset U \cap U'$. By Observation 2.1.6, we have that $I_U \cap I_{U'}$ is an open interval. □

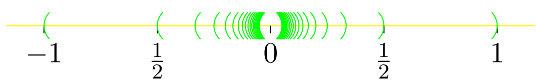
Proposition 2.1.9. The set $\mathcal{O}_{\mathbb{R}}$ defines a topology on $\mathbb{R}$.

Proof. This is exactly established by Observation 1.6.2, Lemma 2.1.5, and Lemma 2.1.8. □

Terminology 2.1.10. We shall refer to $\mathcal{O}_{\mathbb{R}}$ as the *standard topology* on $\mathbb{R}$.

Remark 2.1.11. An infinite intersection of subsets of $\mathbb{R}$ which belong to $\mathcal{O}_{\mathbb{R}}$ does *not* necessarily belong to $\mathcal{O}_{\mathbb{R}}$. For instance, by Example 1.6.3, we have that $\left]-\frac{1}{n}, \frac{1}{n}\right[$ belongs to $\mathcal{O}_{\mathbb{R}}$ for every integer $n \geq 1$. However,

$$\bigcap_{n \in \mathbb{N}} \left]-\tfrac{1}{n}, \tfrac{1}{n}\right[= \{0\}.$$



By Example 2.1.3, the set $\{0\}$ does not belong to $\mathcal{O}_{\mathbb{R}}$.

Remark 2.1.12. The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is fundamental. We shall construct all our geometric examples of topological spaces in various 'canonical ways' from it.

A principal reason that we allow infinite unions in (3) of Definition 1.1.1, but only finite intersections in (4) of Definition 1.1.1, is that these properties hold for $\mathcal{O}_{\mathbb{R}}$.

Remark 2.1.13. An *Alexandroff topological space* is a topological space $(X, \mathcal{O})$ which, unlike $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ and the other geometric examples of topological spaces that we shall meet, has the property that if U is an intersection of (possibly infinitely many) subsets of X which belong to $\mathcal{O}$, then U belongs to $\mathcal{O}$. Alexandroff topological spaces are the topic of Exploration E1.4.

2.2. Subspace topologies

Remark 2.2.1. We shall explore several 'canonical ways' to construct topological spaces. In this section, we discuss the first of these.

Definition 2.2.2. Let $(Y, \mathcal{O}_Y)$ be a topological space, and let X be a subset of Y. Let $\mathcal{O}_X$ denote the set

$$\{X \cap U \mid U \in \mathcal{O}_Y\}.$$

Proposition 2.2.3. Let $(Y, \mathcal{O}_Y)$ be a topological space, and let X be a subset of Y. Then $(X, \mathcal{O}_X)$ is a topological space.

Proof. We verify that each of the conditions of Definition 1.1.1 holds.

(1) Since $\mathcal{O}_Y$ is a topology on Y, we have that $\emptyset$ belongs to $\mathcal{O}_Y$. We also have that $\emptyset = X \cap \emptyset$. Thus $\emptyset$ belongs to $\mathcal{O}_X$.

(2) Since $\mathcal{O}_Y$ is a topology on Y, we have that Y belongs to $\mathcal{O}_Y$. We also have that $X = X \cap Y$. Thus X belongs to $\mathcal{O}_X$.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of X which belong to $\mathcal{O}_X$. By definition of $\mathcal{O}_X$, we have, for every $j \in J$, that

$$U_j = X \cap U'_j,$$

for a subset U'_j of Y which belongs to $\mathcal{O}_Y$. Now

$$\begin{aligned} \bigcup_{j \in J} U_j &= \bigcup_{j \in J} \left(X \cap U'_j\right) \\ &= X \cap \left(\bigcup_{j \in J} U'_j\right). \end{aligned}$$

Since $\mathcal{O}_Y$ is a topology on Y, we have that $\bigcup_{j \in J} U'_j$ belongs to $\mathcal{O}_Y$. We deduce that $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}_X$.

(4) Suppose that U_0 and U_1 are subsets of X which belong to $\mathcal{O}_X$. By definition of $\mathcal{O}_X$, we have that

$$U_0 = X \cap U'_0$$

and

$$U_1 = X \cap U'_1,$$

for a pair of subsets U'_0 and U'_1 of Y which belong to $\mathcal{O}_Y$. Now

$$\begin{aligned} U_0 \cap U_1 &= \left(X \cap U'_0\right) \cap \left(X \cap U'_1\right) \\ &= X \cap \left(U'_0 \cap U'_1\right). \end{aligned}$$

Since $\mathcal{O}_Y$ is a topology on Y, we have that $U'_0 \cap U'_1$ belongs to $\mathcal{O}_Y$. We deduce that $U_0 \cap U_1$ belongs to $\mathcal{O}_X$.

$\square$

Remark 2.2.4. The flavour of this proof is very similar to many others in the early part of the course. It is a very good idea to work on it until you thoroughly understand it. This is the topic of Task E2.2.2.

Terminology 2.2.5. We refer to $\mathcal{O}_X$ as the *subspace topology* on X with respect to $(Y, \mathcal{O}_Y)$.

2.3. Example of a subspace topology — the unit interval

Definition 2.3.1. Let I denote the closed interval $[0,1]$. Let $\mathcal{O}_I$ denote the subspace topology on I with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Terminology 2.3.2. We refer to $(I, \mathcal{O}_I)$ as the *unit interval*.

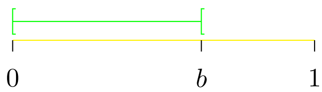
Example 2.3.3. Let $]a,b[$ be an open interval such that $0 < a < b < 1$.

As we observed in Example 1.6.3, the open interval $]a,b[$ belongs to $\mathcal{O}_{\mathbb{R}}$, We also have that

$$]a,b[= I \cap]a,b[.$$

Thus $]a,b[$ belongs to $\mathcal{O}_I$.

Example 2.3.4. Let $[0,b[$ be an half open interval such that $0 < b < 1$.

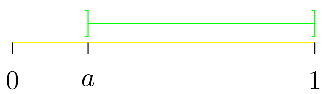


Let a be any real number such that $a < 0$. As we observed in Example 1.6.3, the open interval $]a,b[$ belongs to $\mathcal{O}_{\mathbb{R}}$, We have that

$$[0,b[= I \cap]a,b[.$$

Thus $[0,b[$ belongs to $\mathcal{O}_I$.

Example 2.3.5. Let $]a,1]$ be an half open interval such that $0 < a < 1$.

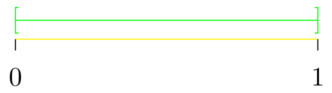


Let b be any real number such that $b > 1$. As we observed in Example 1.6.3, the open interval $]a, b[$ belongs to $\mathcal{O}_{\mathbb{R}}$, We have that

$$]a, 1] = I \cap]a, b[\,.$$

Thus $]a, 1]$ belongs to $\mathcal{O}_I$.

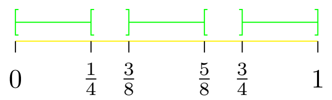
Example 2.3.6. As we proved in Proposition 2.2.3, the set I belongs to $\mathcal{O}_I$.



Example 2.3.7. Disjoint unions of subsets of I of the kind discussed in Example 2.3.3, Example 2.3.4, and Example 2.3.5, belong to $\mathcal{O}_I$. This is a consequence of Proposition 2.2.3, but could also be demonstrated directly. For instance, the set

$$\left[0, \tfrac{1}{4}\right[\cup \left]\tfrac{3}{8}, \tfrac{5}{8}\right[\cup \left]\tfrac{1}{4}, 1\right]$$

belongs to $\mathcal{O}_I$.



E2. Exercises for Lecture 2

E2.1. Exam questions

Task E2.1.1. Decide whether the following subsets of $\mathbb{R}$ are open, closed, both, or neither with respect to $\mathcal{O}_{\mathbb{R}}$.

(1) $]-23, 150[$

(2) $\mathbb{R}$

(3) $[2, 3]$

(4) $\bigcup_{n \in \mathbb{Z}} \left] n - \frac{1}{2}, n + \frac{1}{2} \right[$

(5) $]-\infty, 2]$.

(6) $\bigcup_{n \in \mathbb{N}} \left] \frac{1}{n}, 10 \right[$.

(7) $]5, 8[\cup]47, 60]$

(8) $\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 1 - \frac{1}{n} \right[$

Task E2.1.2. Give an example to demonstrate that an infinite union of closed subsets of $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ need not be closed.

Task E2.1.3. Let X be the subset $[1, 2] \cup [4, 5[$ of $\mathbb{R}$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. For each of the following, give an example of a subset U of X which has the required property, and which belongs to $\mathcal{O}_X$.

(1) We have that $U \cap [4, 5[= \emptyset$, and neither 1 nor 2 belongs to U.

(2) We have that $U \cap [1, 2] = \emptyset$, and 4 does not belong to U.

(3) We have that $U \cap [4, 5[= \emptyset$, and 1 belongs to U.

(4) We have that $U \cap [1, 2] = \emptyset$, and 4 belongs to U.

(5) Both 2 and 4 belong to U.

(6) We have that $U \cap [1, 2]$ is not empty, that $U \cap [4, 5[$ is not empty, and that neither 1, 2, nor 4 belongs to U.

E2.2. In the lecture notes

Task E2.2.1. Prove Observation 2.1.6.

Since this task is appealed to in the proof of Proposition 2.1.9, you are not permitted to use that $\mathcal{O}_{\mathbb{R}}$ is a topology on $\mathbb{R}$!

Task E2.2.2. Take a look at the proof of Proposition 2.2.3. Afterwards, cover it up, and try to prove Proposition 2.2.3 for yourself. There is esssentially only one way to do it. Keep working on this until you can manage it.

E2.3. For a deeper understanding

Task E2.3.1. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y, and let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(Y, \mathcal{O}_Y)$. Let A be a subset of X. Let $\mathcal{O}_A^X$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Let $\mathcal{O}_A^Y$ denote the subspace topology on A with respect to $(Y, \mathcal{O}_Y)$. Prove that $\mathcal{O}_A^X = \mathcal{O}_A^Y$.

Task E2.3.2. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y, and let $\mathcal{O}_X$ be the subspace topology on X with respect to $(Y, \mathcal{O}_Y)$. Prove that a subset V of X is closed with respect to $\mathcal{O}_X$ if and only if there is a subset V' of Y with the following properties.

(1) We have that V' is closed with respect to $(Y, \mathcal{O}_Y)$.

(2) We have that $V = X \cap V'$.

Task E2.3.3. Let $(Y, \mathcal{O}_Y)$ be a topological space. Let X be a subset of Y, and let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(Y, \mathcal{O}_Y)$.

(1) Suppose that X belongs to $\mathcal{O}_Y$. Prove that if U belongs to $\mathcal{O}_X$, then U belongs to $\mathcal{O}_Y$.

(2) Does the conclusion of (1) necessarily hold if X does not belong to $\mathcal{O}_Y$?

(3) Suppose that X is closed with respect to $\mathcal{O}_Y$. Let V be a subset of X which is closed with respect to $\mathcal{O}_X$. Prove that V, when viewed as a subset of Y, is closed with respect to $\mathcal{O}_Y$. You may wish to appeal to Task E2.3.2.

(4) Does the conclusion of (3) necessarily hold if X is not closed with respect to $\mathcal{O}_Y$?

Task E2.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{U_j\}_{j \in J}$ be a set of subsets of X with the property that $X = \bigcup_{j \in J} U_j$. For every $j \in J$, let $\mathcal{O}_{U_j}$ denote the subspace topology on U_j with respect to $(X, \mathcal{O}_X)$. Suppose that U_j belongs to $\mathcal{O}_X$ for every $j \in J$. Let U be a subset of X such that $U \cap U_j$ belongs to $\mathcal{O}_{U_j}$ for every $j \in J$. Prove that U belongs to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Appealing to Task E2.3.3 (1), observe that $U \cap A_j$ belongs to $\mathcal{O}_X$.

(2) Prove that
$$U = \bigcup_{j \in J} U \cap A_j.$$
For this, you may wish to begin by observing that $U = U \cap X$, and then appeal to one of the assumptions.

Remark E2.3.5. There is an analogous result for closed sets, but an additional hypothesis is required. This is the topic of Task E8.3.8.

Task E2.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{U_j\}_{j \in J}$ be a set of subsets of X with the property that $X = \bigcup_{j \in J} U_j$. For every $j \in J$, let $\mathcal{O}_{U_j}$ denote the subspace topology on U_j with respect to $(X, \mathcal{O}_X)$. Suppose that U_j belongs to $\mathcal{O}_X$ for every $j \in J$. Let V be a subset of X such that $V \cap U_j$ is closed with respect to $\mathcal{O}_{U_j}$ for every $j \in J$. Prove that V is closed with respect to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Observe that, since $V \cap U_j$ is closed with respect to $\mathcal{O}_{U_j}$, for every $j \in J$, we have that $U_j \setminus (V \cap U_j)$ belongs to $\mathcal{O}_{U_j}$, for every $j \in J$.

(2) Observe that $U_j \setminus (V \cap U_j) = U_j \cap (X \setminus V)$.

(3) By Task E2.3.4, deduce that $X \setminus V$ belongs to $\mathcal{O}_X$.

Task E2.3.7 (More difficult)**.** Prove that a subset of $\mathbb{R}$ belongs to $\mathcal{O}_{\mathbb{R}}$ if and only if it is a disjoint union of open intervals. For proving that if U belongs to $\mathcal{O}_{\mathbb{R}}$, then it is a disjoint union of open intervals, you may wish to proceed as follows.

(1) Define a relation $\sim$ on U by $a \sim b$ if
$$[\min\{a, b\}, \max\{a, b\}] \subset U.$$
Verify that $\sim$ defines an equivalence relation.

(2) Let
$$U \xrightarrow{q} U/{\sim}$$
denote the map given by $x \mapsto \langle x \rangle$, where $\langle x \rangle$ denotes the equivalence class of x with respect to $\sim$. By means of Task E1.3.3, prove that, for every $y \in U/{\sim}$, the subset $q^{-1}(y)$ of U is an interval.

(3) Moreover, appealing to the fact that U belongs to $\mathcal{O}_{\mathbb{R}}$, prove that, for every $y \in U/{\sim}$, the interval $q^{-1}(y)$ is open.

(4) Verify that, for distinct $y, y' \in U/{\sim}$, the set

$$q^{-1}(y) \cap q^{-1}(y')$$

is empty. Verify that

$$U = \bigcup_{y \in U/{\sim}} q^{-1}(y).$$

Remark E2.3.8. In fact, a subset of $\mathbb{R}$ is open in the standard topology on $\mathbb{R}$ if and only if it is a disjoint union of *countably many* open intervals. This will follow from Task E2.3.7 by a later task.

3. Monday 13th January

3.1. Product topologies

Remark 3.1.1. In this section, we discuss our second 'canonical way' to construct topological spaces.

Definition 3.1.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{O}_{X \times Y}$ denote the set of subsets U of $X \times Y$ with the property that, for every $(x, y) \in U$, there is a subset U_X of X and a subset U_Y of Y with the following properties.

(1) We have that $x \in U_X$, and that U_X belongs to $\mathcal{O}_X$.

(2) We have that $y \in U_Y$, and that U_Y belongs to $\mathcal{O}_Y$.

(3) We have that $U_X \times U_Y \subset U$.

Proposition 3.1.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is a topological space.

Proof. We verify that each of the conditions of Definition 1.1.1 holds.

(1) The empty set $\emptyset$ belongs to $\mathcal{O}_{X \times Y}$, since the required property vacuously holds.

(2) Let $(x, y) \in X \times Y$. We have the following.

   (a) Since $\mathcal{O}_X$ is a topology on X, we have that X belongs to $\mathcal{O}_X$. Evidently, $x \in X$.

   (b) Since $\mathcal{O}_Y$ is a topology on Y, we have that Y belongs to $\mathcal{O}_Y$. Evidently, $y \in Y$.

   (c) We have that $X \times Y \subset X \times Y$.

   Taking U_X to be X, and taking U_Y to be Y, we deduce that $X \times Y$ belongs to $\mathcal{O}_{X \times Y}$.

(3) Let $\{U_j\}_{j \in J}$ be a set of subsets of $X \times Y$ which belong to $\mathcal{O}_{X \times Y}$. Let $(x, y) \in \bigcup_{j \in J} U_j$. By definition of $\bigcup_{j \in J} U_j$, there is a $j \in J$ such that $(x, y) \in U_j$.

   By definition of $\mathcal{O}_{X \times Y}$, there is a subset U_X of X and a subset U_Y of Y with the following properties.

   (a) We have that $x \in U_X$, and that U_X belongs to $\mathcal{O}_X$.

   (b) We have that $y \in U_Y$, and that U_Y belongs to $\mathcal{O}_Y$.

(c) We have that $U_X \times U_Y \subset U_j$.

We have that $U_j \subset \bigcup_{j \in J} U_j$. By (c), we deduce that $U_X \times U_Y \subset \bigcup_{j \in J} U_j$. We conclude from the latter, (a), and (b), that $\bigcup_{j \in J} U_j$ belongs to $\mathcal{O}_{X \times Y}$.

(4) Let U_0 and U_1 be subsets of $X \times Y$ which belong to $\mathcal{O}_{X \times Y}$. Let $(x, y) \in U_0 \cap U_1$. By definition of $\mathcal{O}_{X \times Y}$, there is a subset U_0^X of X and a subset U_0^Y of Y with the following properties.

(a) We have that $x \in U_0^X$, and that U_0^X belongs to $\mathcal{O}_X$.

(b) We have that $y \in U_0^Y$, and that U_0^Y belongs to $\mathcal{O}_Y$.

(c) We have that $U_0^X \times U_0^Y \subset U_0$.

Moreover, by definition of $\mathcal{O}_{X \times Y}$, there is a subset U_1^X of X and a subset U_1^Y of Y with the following properties.

(d) We have that $x \in U_1^X$, and that U_1^X belongs to $\mathcal{O}_X$.

(e) We have that $y \in U_1^Y$, and that U_1^Y belongs to $\mathcal{O}_Y$.

(f) We have that $U_1^X \times U_1^Y \subset U_1$.

We deduce the following.

(i) By (a) and (d), we have that $x \in U_0^X \cap U_1^X$. Moreover, since $\mathcal{O}_X$ defines a topology on X, we have by (a) and (d) that $U_0^X \cap U_1^X$ belongs to $\mathcal{O}_X$.

(ii) By (b) and (e), we have that $y \in U_0^Y \cap U_1^Y$. Moreover, since $\mathcal{O}_Y$ defines a topology on Y, we have by (b) and (e) that $U_0^Y \cap U_1^Y$ belongs to $\mathcal{O}_Y$.

(iii) We have that

$$\left(U_0^X \cap U_1^X\right) \times \left(U_0^Y \cap U_1^Y\right) = \left(U_0^X \times U_0^Y\right) \cap \left(U_1^X \times U_1^Y\right).$$

By (c) and (f), we have that

$$\left(U_0^X \times U_0^Y\right) \cap \left(U_1^X \times U_1^Y\right) \subset U_0 \cap U_1.$$

Hence

$$\left(U_0^X \cap U_1^X\right) \times \left(U_0^Y \cap U_1^Y\right) \subset U_0 \cap U_1.$$

Taking U_X to be $U_0^X \cap U_1^X$, and taking U_Y to be $U_0^Y \cap U_1^Y$, we conclude that $U_0 \cap U_1$ belongs to $\mathcal{O}_{X \times Y}$.

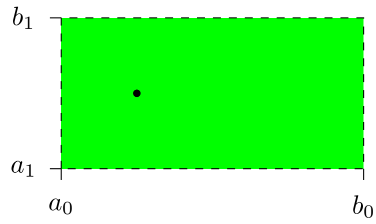
□

Remark 3.1.4. This proof has much in common with the proof of Proposition 2.1.9 and the proof of Proposition 2.2.3. Perhaps you can begin to see how to approach a proof of this kind? Again, it is a very good idea to work on the proof until you thoroughly understand it. This is the topic of Task E3.2.1.

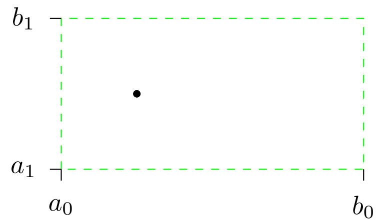
3.2. The product topology on $\mathbb{R}^2$

Definition 3.2.1. Let $\mathcal{O}_{\mathbb{R}^2}$ denote the product topology on $\mathbb{R}^2$ with respect to two copies of $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Example 3.2.2. Let $U_0 =]a_0, b_0[$, and let $U_1 =]a_1, b_1[$ be open intervals. Let $(x, y) \in U_0 \times U_1$.



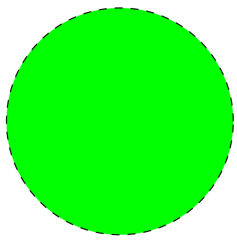
By Example 1.6.3, both U_0 and U_1 belong to $\mathcal{O}_{\mathbb{R}}$. We deduce that $U_0 \times U_1$ belongs to $\mathcal{O}_{\mathbb{R}^2}$, since we can take U_X to be U_0, and can take U_Y to be U_1.



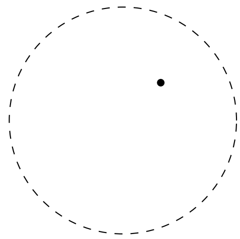
In the figures, the dashed boundary does *not* belong to $U_0 \times U_1$. We shall adopt the same convention in all our figures.

Example 3.2.3. Let U denote the disc

$$\left\{ (x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1 \right\}.$$



Let (x, y) be a point of U.



Let ϵ be a real number such that

$$0 < \epsilon < 1 - \|(x, y)\|.$$

Let U_X denote the open interval

$$\left]x - \tfrac{\epsilon\sqrt{2}}{2}, x + \tfrac{\epsilon\sqrt{2}}{2}\right[.$$

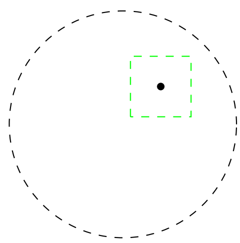
Let U_Y denote the open interval

$$\left]y - \tfrac{\epsilon\sqrt{2}}{2}, y + \tfrac{\epsilon\sqrt{2}}{2}\right[.$$

We have that $x \in U_X$, and that $y \in U_Y$. Let (x', y') be a point of $U_X \times U_Y$. Then

$$\begin{aligned}
\left\|(x', y')\right\| &= \left\|\left(|x'|, |y'|\right)\right\| \\
&< \left\|\left(|x| + \tfrac{\epsilon\sqrt{2}}{2}, |y| + \tfrac{\epsilon\sqrt{2}}{2}\right)\right\| \\
&\leq \|(|x|, |y|)\| + \left\|\left(\tfrac{\epsilon\sqrt{2}}{2}, \tfrac{\epsilon\sqrt{2}}{2}\right)\right\| \\
&= \|(x, y)\| + \epsilon \\
&< \|(x, y)\| + (1 - \|(x, y)\|) \\
&= 1.
\end{aligned}$$

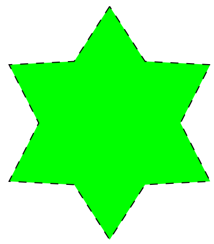
Thus $U_X \times U_Y \subset U$.



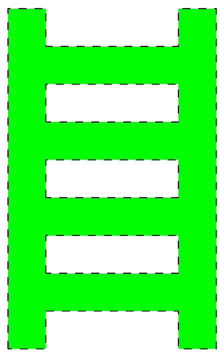
By Example 1.6.3, both U_0 and U_1 belong to $\mathcal{O}_{\mathbb{R}}$. We conclude that U belongs to $\mathcal{O}_{\mathbb{R}^2}$.

Remark 3.2.4. There are very many subsets U of $\mathbb{R}^2$ which belong to $\mathcal{O}_{\mathbb{R}^2}$. We just have to be able to find a small enough 'open rectangle' around every point of U which is contained in U.

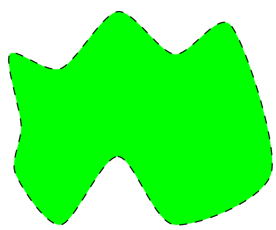
Example 3.2.5. An 'open star' belongs to $\mathcal{O}_{\mathbb{R}^2}$.



Example 3.2.6. An 'open ladder' belongs to $\mathcal{O}_{\mathbb{R}^2}$.



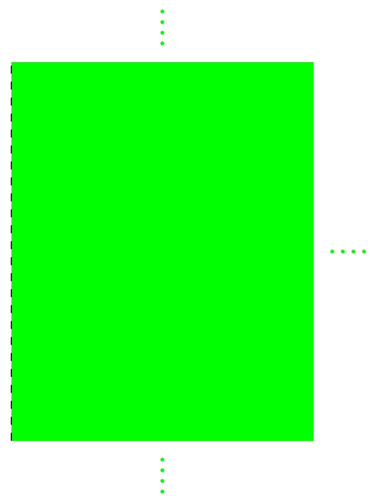
Example 3.2.7. An 'open blob' belongs to $\mathcal{O}_{\mathbb{R}^2}$.



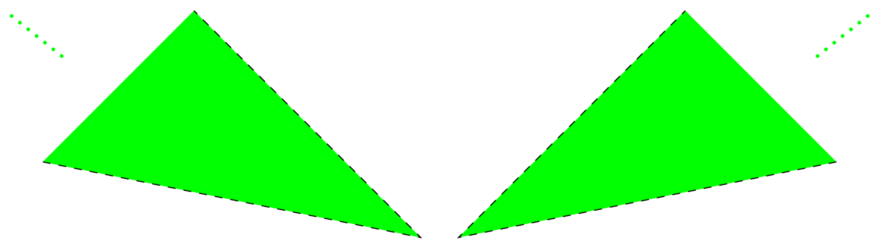
Example 3.2.8. The open half plane given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid x > 0\right\}$$

belongs to $\mathcal{O}_{\mathbb{R}^2}$.



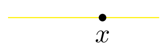
Example 3.2.9. The union of two 'open infinite wedges' belongs to $\mathcal{O}_{\mathbb{R}^2}$.



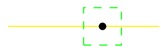
Example 3.2.10. Let X denote the subset of $\mathbb{R}^2$ given by

$$\left\{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1 \text{ and } y = 0\right\}.$$

Let x be any point of X.



No matter how small a rectangle we take around x, there will always be a point of $\mathbb{R}^2$ inside this rectangle which does not belong to X. Thus X does not belong to $\mathcal{O}_{\mathbb{R}^2}$.



Example 3.2.11. Let X denote the 'half open strip' given by $[0,1[\times]0,1[$.

The solid part of the boundary of this figure belongs to X. Let (x,y) belong to either of the vertical boundary lines. For example, we can take (x,y) to be $(0,\frac{1}{2})$.

No matter how small a rectangle we take around (x,y), there will always be a point inside this rectangle which does not belong to X. For example, if (x,y) is $(0,\frac{1}{2})$, there will always be a point (x',y') inside this rectangle such that $x'<0$. Thus X does not belong to $\mathcal{O}_{\mathbb{R}^2}$.

E3. Exercises for Lecture 3

E3.1. Exam questions

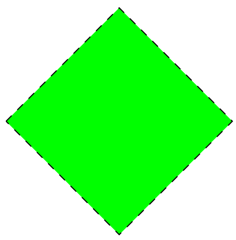
Task E3.1.1. Are the following subsets of $\mathbb{R}^2$ open, closed, both, or neither with respect to the topology $\mathcal{O}_{\mathbb{R}^2}$?

(1) The union of the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < 1 \text{ and } |y| < 1 - x\right\}$$

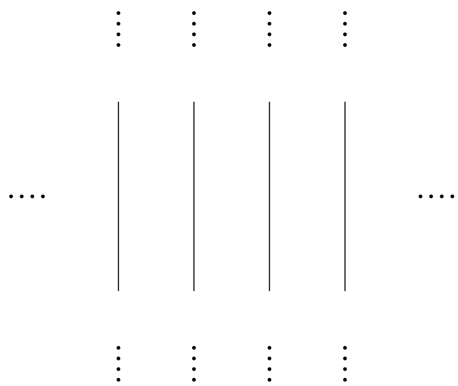
and the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid -1 < x \leq 0 \text{ and } |y| < x + 1\right\}.$$

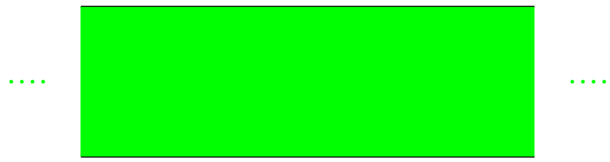


(2) $\bigcup_{n \in \mathbb{Z}} X_n$, where

$$X_n = \left\{(n, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\right\}.$$



(3) $\mathbb{R} \times [0, 1]$.



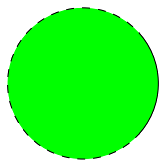
(4) The set consisting of the single point $\{(123, \pi)\}$.

(5) The union of the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid -1 < x < \tfrac{3}{4} \text{ and } \|(x, y)\| < 1\right\}$$

and the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid \tfrac{3}{4} \leq x < 1 \text{ and } \|(x, y)\| \leq 1\right\}.$$



(6) The union of the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid y > 0 \text{ and } \|(x, y)\| < 1\right\}$$

and the set $[3, 5] \times [0, 1]$.

(7) $\bigcup_{n \in \mathbb{N}} \left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \leq 3 - \frac{1}{n}\right\}$.

(8) The set

$$\left\{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\right\}.$$

E3.2. In the lecture notes

Task E3.2.1. Do the same as in Task E2.2.2 for the proof of Proposition 3.1.3.

E3.3. For a deeper understanding

Task E3.3.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let V_X be a subset of X which is closed with respect to $\mathcal{O}_X$, and let V_Y be a subset of Y which is closed with respect to $(Y, \mathcal{O}_Y)$. Prove that $V_X \times V_Y$ is closed with respect to $(X \times Y, \mathcal{O}_{X \times Y})$.

Task E3.3.2. Let $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ be topological spaces. Let Y_0 be a subset of X_0, and let Y_1 be a subset of X_1.

Let $\mathcal{O}_{Y_0}$ denote the subspace topology on Y_0 with respect to $(X_0, \mathcal{O}_{X_0})$. Let $\mathcal{O}_{Y_1}$ denote the subspace topology on Y_1 with respect to $(X_1, \mathcal{O}_{X_1})$.

Let $\mathcal{O}_{Y_0 \times Y_1}$ denote the product topology on $Y_0 \times Y_1$ with respect to $(Y_0, \mathcal{O}_{Y_0})$ and $(Y_1, \mathcal{O}_{Y_1})$. Let $\mathcal{O}'_{Y_0 \times Y_1}$ denote the subspace topology on $Y_0 \times Y_1$ with respect to $(X_0 \times X_1, \mathcal{O}_{X_0 \times X_1})$.

Prove that $\mathcal{O}_{Y_0 \times Y_1} = \mathcal{O}'_{Y_0 \times Y_1}$.

Task E3.3.3. Let $(X_0, \mathcal{O}_{X_0})$, $(X_1, \mathcal{O}_{X_1})$, and $(X_2, \mathcal{O}_{X_2})$ be topological spaces. Let $\mathcal{O}_{X_0 \times (X_1 \times X_2)}$ denote the product topology on $X_0 \times X_1 \times X_2$ with respect to $(X_0, \mathcal{O}_{X_0})$ and $(X_1 \times X_2, \mathcal{O}_{X_1 \times X_2})$. Let $\mathcal{O}_{(X_0 \times X_1) \times X_2}$ denote the product topology on $X_0 \times X_1 \times X_2$ with respect to $(X_0 \times X_1, \mathcal{O}_{X_0 \times X_1})$ and $(X_2, \mathcal{O}_{X_2})$.

Prove that $\mathcal{O}_{X_0 \times (X_1 \times X_2)} = \mathcal{O}_{(X_0 \times X_1) \times X_2}$.

Notation E3.3.4. We shall denote the topology $\mathcal{O}_{X_0 \times (X_1 \times X_2)} = \mathcal{O}_{(X_0 \times X_1) \times X_2}$ on $X_0 \times X_1 \times X_2$ by $\mathcal{O}_{X_0 \times X_1 \times X_2}$.

Notation E3.3.5. We shall denote by $\mathcal{O}_{\mathbb{R}^3}$ the topology $\mathcal{O}_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}}$ on $\mathbb{R}^3$.

Remark E3.3.6. Let $n \in \mathbb{N}$. For every $1 \leq i \leq n$, let $(X_i, \mathcal{O}_{X_i})$ be a topological space. By induction, it follows from Task E3.3.3 that all the possible ways of equipping $X_1 \times \ldots \times X_n$ with a topology, using only the topologies $\mathcal{O}_{X_i}$, for $1 \leq i \leq n$, and product topologies built from these, coincide.

Notation E3.3.7. We shall denote this topology on $X_1 \times \ldots X_n$ by $\mathcal{O}_{X_1 \times \ldots \times X_n}$.

Notation E3.3.8. We shall denote by $\mathcal{O}_{\mathbb{R}^n}$ the topology $\mathcal{O}_{\underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n}}$ on $\mathbb{R}^n$.

E3.4. Exploration — metric spaces

Remark E3.4.1. Some of you may have met the notion of a metric before, for instance in TMA4145 Lineære Metoder. Don't worry if not, all material on metric spaces below, and in future exercises, will not be examined. Certainly I recommend to focus on the topics covered in the lectures, before looking into any of the exercises on metric spaces.

Nevertheless, those of you who are comfortable with the lectures may find the exercises on metric spaces interesting, and useful in future courses. Though it will not be necessary, you are welcome to make use of any of the exercises on metric spaces in the exam wherever there is an opportunity for this.

Definition E3.4.2. Let X be a set. A *metric* on X is a map

$$X \times X \xrightarrow{d} [0, \infty[$$

such that the following hold.

(1) For every x which belongs to X, we have that $d(x, x) = 0$.

(2) For all x_0, x_1, and x_2 which belong to X, we have that

$$d(x_0, x_1) + d(x_1, x_2) \geq d(x_0, x_2).$$

Remark E3.4.3. The condition of Definition E3.4.2 is known as the *triangle inequality*.

Remark E3.4.4. If you have seen the definition of a metric in a previous course, a couple of additional conditions were probably required to be satisfied. For many purposes, these are not needed. In particular, we shall not need them in this section.

Definition E3.4.5. A *metric space* is a pair (X, d) of a set X and a metric d on X.

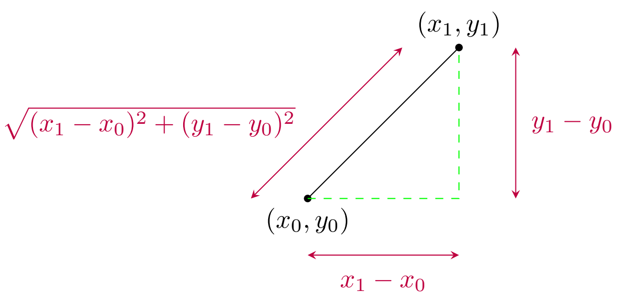
Notation E3.4.6. Let

$$\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{d_{\mathbb{R}^n}} \mathbb{R}^n$$

denote the map given by

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \sqrt{(y_1 - x_1)^2 + \ldots + (y_n - x_n)^2}.$$

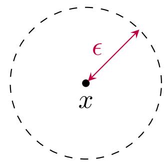
In other words, $d_{\mathbb{R}^n}$ is the usual notion of distance between a pair of points in $\mathbb{R}^n$.



Task E3.4.7. Prove that $d_{\mathbb{R}^n}$ defines a metric on $\mathbb{R}^n$, or look up a proof from an earlier course.

Definition E3.4.8. Let (X, d) be a metric space. Let $x \in X$, and let $\epsilon > 0$ be a real number. The *open ball of radius ϵ around x* is the set $B_\epsilon(x)$ given by

$$\left\{ x' \in X \mid d(x, x') < \epsilon \right\}.$$



Task E3.4.9. Let (X, d) be a metric space. Let $\mathcal{O}_d$ denote the set of subsets U of X with the property that, for every $x \in U$, there is a real number $\epsilon > 0$ such that $B_\epsilon(x)$ is a subset of U. Prove that $\mathcal{O}_d$ defines a topology on X.

Remark E3.4.10. We shall take the point of view that a metric is a way to construct a topology. Once we have constructed this topology, we can forget about the metric from whence it came!

All the topological spaces that we shall be interested in can be constructed without using a metric. For this reason, metrics will never appear in the lectures.

A characteristic feature of topology, as opposed to *geometry*, is that we shall often be manipulating topological spaces in ways which change the distance between pairs of points: squashing and stretching, for instance.

Nevertheless, there are many important areas of mathematics, such as *differential geometry*, which merge both topological and geometrical ideas. Here one sometimes emphasises a construction which relies on a metric, sometimes emphasises a purely topological construction, and often investigates the interplay between both worlds. The courses TMA4190 Mangfoldigheter and MA8402 Lie-Grupper og Lie-Algebraer can lead in this direction.

Remark E3.4.11. Though metrics will never appear in the lectures, many of the concepts that we shall look at for arbitrary topological spaces can be thought of in other, equivalent, ways for topologies coming from a metric. We shall explore this in future exercises.

Task E3.4.12. Let $n \geq 1$. Prove that $\mathcal{O}_{d_{\mathbb{R}^n}} = \mathcal{O}_{\mathbb{R}^n}$.

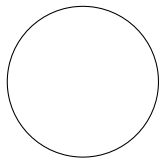
4. Tuesday 14th January

4.1. Examples of product and subspace topologies

Remark 4.1.1. We can combine our two 'canonical' ways of constructing new topological spaces from old ones to obtain many interesting examples of topological spaces.

Notation 4.1.2. We denote by S^1 the set

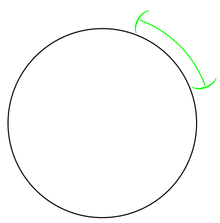
$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1\right\}.$$



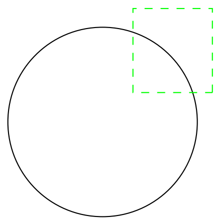
We denote by $\mathcal{O}_{S^1}$ the subspace topology on S^1 with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Terminology 4.1.3. We refer to S^1 as the *circle*.

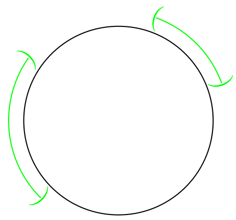
Example 4.1.4. By definition, a subset of S^1 belongs to $\mathcal{O}_{S^1}$ if and only if it is the intersection with S^1 of a subset of $\mathbb{R}^2$ which belongs to $\mathcal{O}_{\mathbb{R}^2}$. The generic example is an 'open arc'.



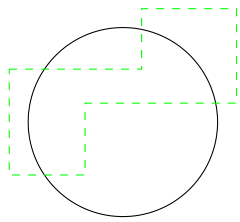
This is, for instance, the intersection with S^1 of an 'open rectangle' in $\mathbb{R}^2$, which belongs to $\mathcal{O}_{\mathbb{R}^2}$ by Example 3.2.2.



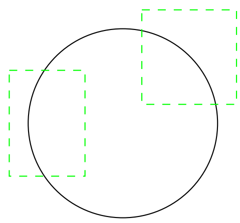
Since $\mathcal{O}_{S^1}$ defines a topology on S^1, we also have that disjoint unions of (possibly infinitely many) 'open arcs' belong to $\mathcal{O}_{S^1}$.



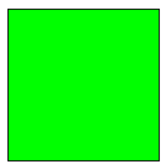
This can also be demonstrated directly. The subset of S^1 given by the two 'open arcs' in the previous picture is, for instance, the intersection with S^1 with the subset of $\mathbb{R}^2$ depicted below, which belongs to $\mathcal{O}_{\mathbb{R}^2}$.



Alternatively, it is the intersection with S^1 with the subset of $\mathbb{R}^2$ consisting of two disjoint 'open rectangles', depicted below.



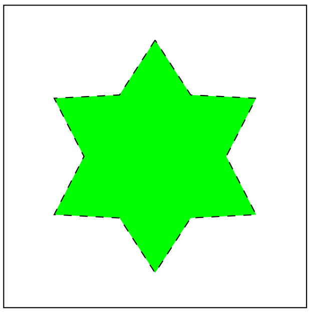
Notation 4.1.5. We denote by $\mathcal{O}_{I^2}$ the product topology on I^2 with respect to two copies of $(I, \mathcal{O}_I)$.



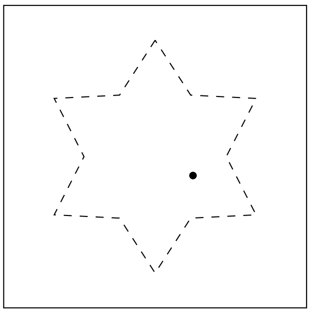
Terminology 4.1.6. We refer to I^2 as the *unit square*.

Remark 4.1.7. The topology $\mathcal{O}_{I^2}$ coincides with the subspace topology on I^2 with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. To prove this is the topic of Task E3.3.2.

Example 4.1.8. Any of the open sets pictured in Examples 3.2.2 – 3.2.3 and 3.2.5 – 3.2.7 which 'fit inside I^2' belong to I^2. For instance, an 'open star'.

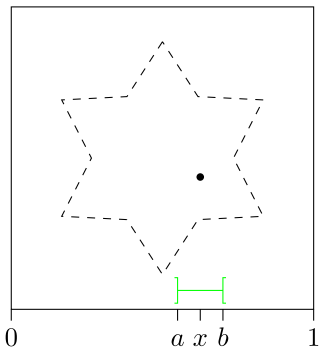


To see this, let (x, y) be a point of a subset U of I^2 of this kind.

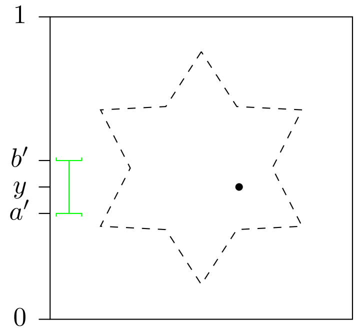


We have the following.

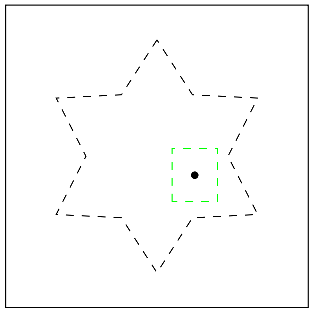
(1) We can find an open interval $U_X =]a, b[$ such that $0 < a < b < 1$ and $x \in U_X$.



(2) We can find an open interval $U_Y =]a', b'[$ such that $0 < a' < b' < 1$ and $y \in U_Y$.



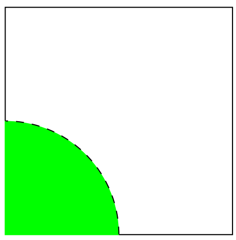
(3) We have that $U_X \times U_Y \subset U$.



As we observed in Example 2.3.3, both U_X and U_Y belong to $\mathcal{O}_I$. Thus (1) – (3) together demonstrate that U belongs to $\mathcal{O}_{I^2}$.

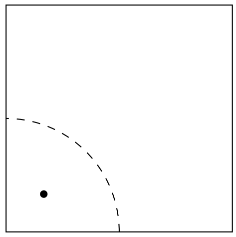
Example 4.1.9. Let U be the subset of I^2 given by

$$\left\{(x, y) \in I^2 \mid |(x, y)| < \tfrac{1}{2}\right\}.$$

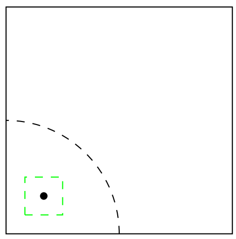


We have the following.

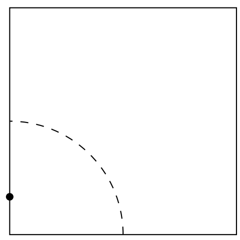
(1) Let (x, y) be a point of U which does not lie on the boundary of I^2.



As in Example 4.1.8, we can find an 'open rectangle' around (x, y) which is a subset of U.



(2) Let (x, y) be a point of U with $x = 0$.



Let ϵ be a real number such that

$$0 < \epsilon < \tfrac{1}{2} - y.$$

Let U_X denote the half open interval

$$\left[0, \tfrac{\epsilon\sqrt{2}}{2}\right[.$$

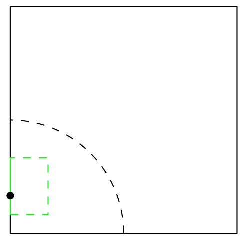
Let U_Y denote the open interval

$$\left]y - \tfrac{\epsilon\sqrt{2}}{2}, y + \tfrac{\epsilon\sqrt{2}}{2}\right[.$$

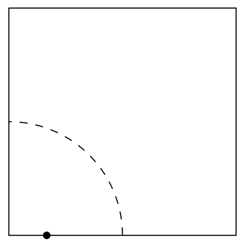
We have that $(0, y) \in U_X \times U_Y$. As we saw in Example 2.3.4, we have that U_X belongs to $\mathcal{O}_I$. As we saw in Example 2.3.3, we have that U_Y belongs to $\mathcal{O}_I$. Moreover, let (x', y') be a point of $U_X \times U_Y$. Arguing as in Example 3.2.3, we have that

$$\left\|(x', y')\right\| < \tfrac{1}{2}.$$

Thus $U_X \times U_Y \subset U$.



(3) Let (x, y) be a point of U with $y = 0$.



Let ϵ be a real number such that

$$0 < \epsilon < \tfrac{1}{2} - x.$$

Let U_X denote the open interval

$$\left]x - \tfrac{\epsilon\sqrt{2}}{2}, x + \tfrac{\epsilon\sqrt{2}}{2}\right[.$$

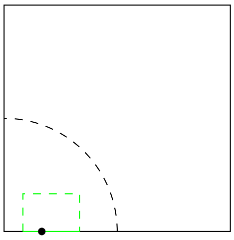
Let U_Y denote the half open interval

$$\left[0, \tfrac{\epsilon\sqrt{2}}{2}\right[.$$

We have that $(x, 0) \in U_X \times U_Y$. As we saw in Example 2.3.3, U_X belongs to $\mathcal{O}_I$. As we saw in Example 2.3.4, U_Y belongs to $\mathcal{O}_I$. Moreover, let (x', y') be a point of $U_X \times U_Y$. Arguing as in Example 3.2.3, we have that

$$\left\|(x', y')\right\| < \tfrac{1}{2}.$$

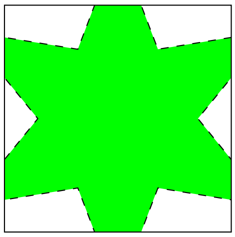
Thus $U_X \times U_Y \subset U$.



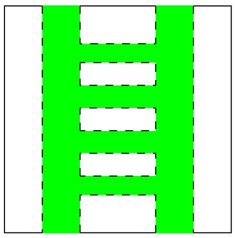
We conclude that U belongs to $\mathcal{O}_{I^2}$.

Remark 4.1.10. Many more subsets of I^2 with 'segments on the boundary' belong to $\mathcal{O}_{I^2}$.

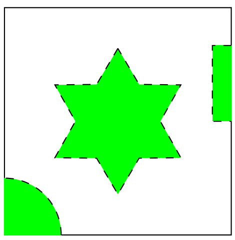
Example 4.1.11. A 'truncated star' belongs to $\mathcal{O}_{I^2}$.



Example 4.1.12. A 'half open ladder' belongs to $\mathcal{O}_{I^2}$.



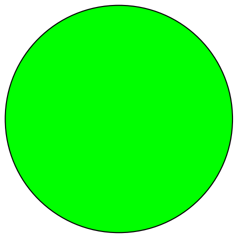
Example 4.1.13. The following subset of I^2 belongs to $\mathcal{O}_{I^2}$.



Remark 4.1.14. We now introduce a few more important examples of product and subspace topologies. Exploring them is the topic of Tasks E4.1.3 – E4.1.6.

Notation 4.1.15. Let D^2 denote the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| \leq 1\right\}.$$

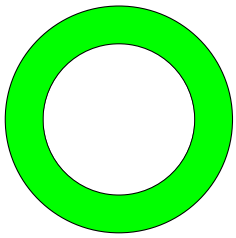


We denote by $\mathcal{O}_{D^2}$ the subspace topology on D^2 with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Terminology 4.1.16. We refer to $(D^2, \mathcal{O}_{D^2})$ as the *unit disc*.

Notation 4.1.17. Let k be a real number such that $0 < k < 1$. Let A_k denote the set

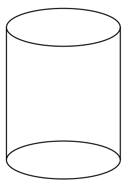
$$\left\{(x,y) \in \mathbb{R}^2 \mid k \leq \|(x,y)\| \leq 1\right\}.$$



We denote by $\mathcal{O}_{A_k}$ the subspace topology on A_k with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Terminology 4.1.18. We refer to $(A_k, \mathcal{O}_{A_k})$ as an *annulus*.

Notation 4.1.19. We denote by $\mathcal{O}_{S^1 \times I}$ the product topology on $S^1 \times I$ with respect to $(S^1, \mathcal{O}_{S^1})$ and $(I, \mathcal{O}_I)$.



This cylinder is hollow!

Terminology 4.1.20. We refer to $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ as the *cylinder*.

4.2. Definition of a continuous map

Notation 4.2.1. Let X and Y be sets. Let

$$X \xrightarrow{f} Y$$

be a map. Let U be a subset of Y. We denote by $f^{-1}(U)$ the set

$$\{x \in X \mid f(x) \in U\}.$$

Terminology 4.2.2. We refer to $f^{-1}(U)$ as the *inverse image of* U *under* f.

Definition 4.2.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *continuous* if, for every $U \in \mathcal{O}_Y$, the subset $f^{-1}(U)$ of X belongs to $\mathcal{O}_X$.

Remark 4.2.4. A map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous with respect to the standard topology on both copies of $\mathbb{R}$ if and only if it is continuous in the $\epsilon - \delta$ sense that you have met in earlier courses. To prove this is the topic of Task E4.2.9.

4.3. Examples of continuous maps between finite topological spaces

Example 4.3.1. Let X be a set with two elements $\{a, b\}$. Let $\mathcal{O}_X$ denote the topology on X given by

$$\{\emptyset, \{b\}, X\}.$$

In other words, $(X, \mathcal{O}_X)$ is the Sierpiński interval. Let Y denote the set with three elements $\{a', b', c'\}$. Let $\mathcal{O}_Y$ denote the topology on Y given by

$$\big\{\emptyset, \{a'\}, \{c'\}, \{a', c'\}, \{b', c'\}, Y\big\}.$$

Let

$$X \xrightarrow{f} Y$$

denote the map given by $a \mapsto b'$ and $b \mapsto c'$. We have the following.

(1) $f^{-1}(\emptyset) = \emptyset$.

(2) $f^{-1}(\{a'\}) = \emptyset$.

(3) $f^{-1}(\{c'\}) = \{b\}$.

(4) $f^{-1}(\{a', c'\}) = \{b\}$.

(5) $f^{-1}(\{b', c'\}) = X$.

(6) $f^{-1}(Y) = X$.

We see that $f^{-1}(U) \in \mathcal{O}_X$ for every $U \in \mathcal{O}_Y$. Thus f is continuous.

Example 4.3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be as in Example 4.3.1. Let

$$Y \xrightarrow{g} X$$

denote the map given by $a' \mapsto a$, $b' \mapsto b$, and $c' \mapsto a$. We have that

$$g^{-1}(\{b\}) = \{b'\}.$$

Thus g is not continuous, since $\{b\}$ belongs to $\mathcal{O}_X$, but $\{b'\}$ does not belong to $\mathcal{O}_Y$.

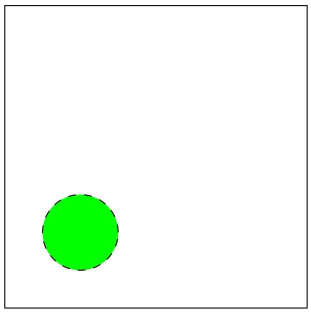
E4. Exercises for Lecture 4

E4.1. Exam questions

Task E4.1.1. Are the following subsets of I^2 open, closed, both, or neither, with respect to $\mathcal{O}_{I^2}$?

(1) The disc given by

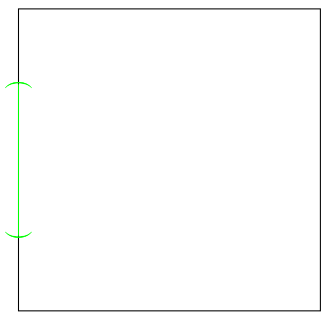
$$\left\{(x, y) \in \mathbb{R}^2 \mid \left\|(x - \tfrac{1}{4}, y - \tfrac{1}{4})\right\| < \tfrac{1}{8}\right\}.$$



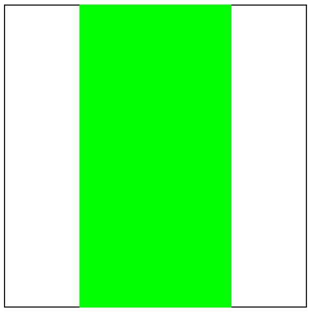
Can you justify your answer rigorously?

(2) The set

$$\left\{(0, y) \in I^2 \mid \tfrac{1}{4} < y < \tfrac{3}{4}\right\}.$$



(3) $\left[\frac{1}{4}, \frac{3}{4}\right] \times I$.

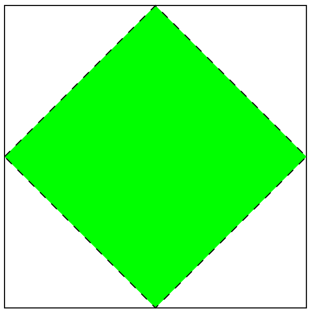


(4) The union of the set

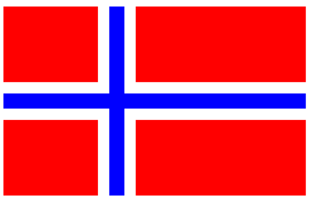
$$\left\{(x, y) \in I^2 \mid 0 < x \leq \tfrac{1}{2} \text{ and } |y| < 2x\right\}$$

and the set

$$\left\{(x, y) \in I^2 \mid \tfrac{1}{2} \leq x < 1 \text{ and } |y| < 2 - 2x\right\}.$$



Task E4.1.2. Let X denote the subset of $\mathbb{R}^2$ consisting of the red and blue parts of the flag below.



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. For each of the following, draw an example of a subset U of X which has the required property, and which belongs to $\mathcal{O}_X$. Use dashes to indicate which parts of the boundary U in your picture are not to be thought of as belonging to U.

(1) U intersects none of the rectangles except the upper right red rectangle; and U does not intersect the boundary of this rectangle.

(2) U intersects all four red rectangles and both of the blue rectangles; but U does not intersect the boundary of X.

(3) U intersects both of the blue rectangles; but U does not intersect any of the red rectangles.

(4) U intersects only the horizontal blue rectangle, the upper left red rectangle, and the lower left red rectangle; U contains a segment of the border of both the upper left red rectangle and the lower left red rectangle; but U does not contain the entirety of either of the upper left red rectangle or the lower left red rectangle.

(5) U intersects only the vertical blue rectangle and the two upper red rectangles; U contains a segment on all four sides of both of the two upper red rectangles; but U does not contain the entirety of either of the upper red rectangles.

Task E4.1.3. For each of the following, give an example of a subset U of the unit disc D^2 which has the required property.

(1) U belongs to $\mathcal{O}_{D^2}$ but, when viewed as a subset of $\mathbb{R}^2$, does not belong to $\mathcal{O}_{\mathbb{R}^2}$.

(2) U belongs to $\mathcal{O}_{D^2}$ and, when viewed as a subset of $\mathbb{R}^2$, also belongs to $\mathcal{O}_{\mathbb{R}^2}$.

(3) U does not belong to $\mathcal{O}_{D^2}$ and, when viewed as a subset of $\mathbb{R}^2$, also does not belong to $\mathcal{O}_{\mathbb{R}^2}$.

(4) U is closed with respect to $\mathcal{O}_{D^2}$ but, when viewed as a subset of $\mathbb{R}^2$, is not closed with respect to $\mathcal{O}_{\mathbb{R}^2}$.

(5) U belongs to $\mathcal{O}_{D^2}$ and, when viewed as a subset of $\mathbb{R}^2$, is closed with respect to $\mathcal{O}_{\mathbb{R}^2}$.

Task E4.1.4. Draw the subset U of the annulus $A_{\frac{1}{2}}$ given by

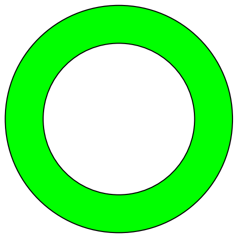
$$\left\{(x,y) \in A_{\frac{1}{2}} \mid \tfrac{1}{4} < x \leq 1\right\}.$$

Does U belong to $\mathcal{O}_{A_{\frac{1}{2}}}$? Does the set V given by

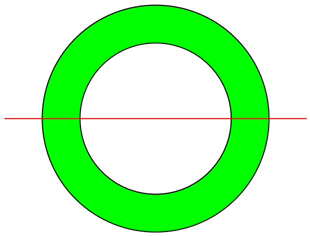
$$\left\{(x,y) \in A_{\frac{1}{2}} \mid \tfrac{1}{4} \leq x \leq 1\right\}$$

belong to $\mathcal{O}_{A_{\frac{1}{2}}}$?

Task E4.1.5. Let $(A_k, \mathcal{O}_{A_k})$ be an annulus.



The horizontal line depicted below is a segment of the x-axis in $\mathbb{R}^2$.



For each of the following, give and draw an example of a subset U of A_k which has the required property, and which belongs to $\mathcal{O}_{A_k}$.

(1) U contains a segment of the inner circle which is above the horizontal line, and does not contain a segment of the inner circle which is below the horizontal line; U contains a segment of the outer circle which is below the horizontal line, and does not contain a segment of the outer circle which is above the horizontal line.

(2) U contains a segment of the inner circle which is above the horizontal line, and its reflection in the horizontal line; U does not contain any segment of the outer circle.

(3) U contains the entire outer circle, but does not contain any point of the inner circle.

(4) U contains neither a segment of the inner circle, nor a segment of the outer circle.

Task E4.1.6. Draw the following subsets U of the cylinder $S^1 \times I$, and decide whether or not they belong to $\mathcal{O}_{S^1 \times I}$.

(1) $S^1 \times \{1\}$.

(2) $U \times \{0\}$, where U is the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid -\tfrac{1}{4} < y < \tfrac{1}{4}\right\}.$$

(3) $U \times \left]\tfrac{1}{4}, \tfrac{1}{2}\right[$, where U is the subset of S^1 given in (2).

(4) $\{(0,1)\} \times I$.

(5) $S^1 \times \left]\frac{3}{4}, 1\right]$.

(6) $\left(U_0 \times \left]\frac{1}{4}, \frac{1}{2}\right[\right) \cup \left(U_1 \times \left]\frac{1}{2}, \frac{3}{4}\right[\right)$, where U_0 is the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid \tfrac{1}{4} \leq x < \tfrac{1}{2}\right\},$$

and U_1 is the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid -\tfrac{1}{2} < x < -\tfrac{1}{4}\right\}.$$

(7) $(U_0 \times I) \cup \left(U_1 \times \left]\frac{1}{2}, \frac{3}{4}\right[\right)$, where U_0 is the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid \tfrac{1}{8} < x < \tfrac{1}{4}\right\}$$

and U_1 is the subset of S^1 given in (6).

Task E4.1.7. Let X be the set $\{a, b, c\}$. Let $\mathcal{O}_X$ denote the topology on X given by

$$\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}.$$

Let Y be the set $\{a', b', c', d', e'\}$. Let $\mathcal{O}_Y$ denote the topology on Y given by

$$\left\{\emptyset, \{a'\}, \{e'\}, \{a', e'\}, \{b', c'\}, \{a', b', c'\}, \{b', c', e'\}, \{a', b', c', e'\}, \{b', c', d', e'\}, Y\right\}.$$

Which of the following maps

$$X \xrightarrow{f} Y$$

are continuous?

(1) $a \mapsto d'$, $b \mapsto e'$, $c \mapsto d'$.

(2) $a \mapsto e'$, $b \mapsto e'$, $c \mapsto c'$.

(3) $a \mapsto c'$, $b \mapsto a'$, $c \mapsto d'$.

(4) $a \mapsto b'$, $b \mapsto c'$, $c \mapsto d'$.

Remark E4.1.8. It may save you some work to appeal to Task E4.2.5.

E4.2. For a deeper understanding

Definition E4.2.1. Let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{B}$ be a set of subsets of X which belong to $\mathcal{O}$. Then $\mathcal{B}$ is a *basis* for $(X, \mathcal{O})$ if, for every subset of U of X which belongs to $\mathcal{O}$, there is a set $\{U_j\}_{j \in J}$ of (possibly infinitely many) subsets of X which belong to $\mathcal{B}$ such that $U = \bigcup_{j \in J} U_j$.

Task E4.2.2. Let $\mathcal{B}$ denote the set of open intervals. Prove that $\mathcal{B}$ is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Task E4.2.3. Let

$$\mathcal{B} = \{]x - \epsilon, x + \epsilon[\mid x, \epsilon \in \mathbb{R} \text{ and } \epsilon > 0\}.$$

Prove that $\mathcal{B}$ is a basis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Remark E4.2.4. You may find it a little difficult at first to find the idea needed to accomplish Tasks E4.2.2 and E4.2.3. Don't worry if so, feel free to ask me about it. The idea will be used in different forms several times in the course.

Task E4.2.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{B}$ be a basis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(U)$ belongs to $\mathcal{O}_X$ for every subset U of Y which belongs to $\mathcal{B}$.

Corollary E4.2.6. Let $(X, \mathcal{O}_X)$ be a topological space. A map

$$X \xrightarrow{f} \mathbb{R}$$

is continuous with respect to the standard topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$ if and only if $f^{-1}(]a, b[)$ belongs to $\mathcal{O}_X$, for every open interval $]a, b[$.

Proof. Follows immediately from Task E4.2.2 and Task E4.2.5. □

Definition E4.2.7. A map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is *continuous in the* ϵ-δ *sense* if, for all $x, c, \epsilon \in \mathbb{R}$ with $\epsilon > 0$, there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Remark E4.2.8. This is the notion of a continuous map that you have met in earlier courses.

Task E4.2.9. Prove that a map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous with respect to the standard topology $\mathcal{O}_{\mathbb{R}}$ on both copies of $\mathbb{R}$ if and only if it is continuous in the $\epsilon - \delta$ sense. You may find it helpful to appeal to Task E4.2.3 and to Task E4.2.5.

Definition E4.2.10. Let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{S}$ be a set of subsets of X which belong to $\mathcal{O}$. Let $\mathcal{B}$ denote the set of subsets U of X such that

$$U = \bigcap_{j \in J} U_j,$$

for a set $\{U_j\}_{j \in J}$ of subsets of X which belong to $\mathcal{S}$, where J is finite. Then $\mathcal{S}$ is a *subbasis* for $(X, \mathcal{O})$ if $\mathcal{B}$ is a basis for $(X, \mathcal{O})$.

Task E4.2.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{S}$ be a subbasis for $(Y, \mathcal{O}_Y)$. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(U)$ belongs to $\mathcal{O}_X$ for every subset U of Y which belongs to $\mathcal{S}$. You may wish to appeal to Task E4.2.5.

Task E4.2.12. Let $\mathcal{S}$ denote the union of the set

$$\{]-\infty, x[\mid x \in \mathbb{R}\}$$

and the set

$$\{]x, \infty[\mid x \in \mathbb{R}\}.$$

Prove that $\mathcal{S}$ is a subbasis for $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. You may wish to appeal to Task E4.2.2.

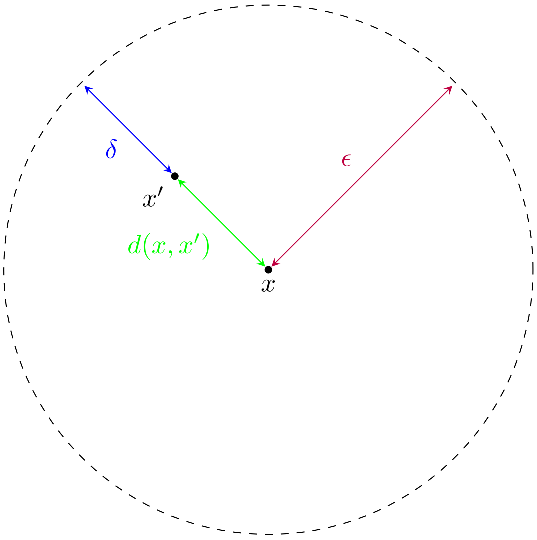
E4.3. Exploration — continuity for metric spaces

Definition E4.3.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A map

$$X \xrightarrow{f} Y$$

is *continuous in the metric sense* if, for all $x \in X$, and all $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that $f(B_\delta(x))$ is a subset of $B_\epsilon(f(x))$.

Task E4.3.2. Let (X, d) be a metric space. Prove that for any x which belongs to X, any $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, and any x' which belongs to $B_\epsilon(x)$, there is a $\delta \in \mathbb{R}$ such that $\delta > 0$, and such that $B_\delta(x')$ is a subset of $B_\epsilon(x)$. You may wish to let δ be $\epsilon - d(x, x')$.



You may then wish to observe that, for every x'' which belongs to $B_\delta(x)$, the following holds, by definition of d.

$$\begin{aligned} d(x, x'') &\leq d(x, x') + d(x', x'') \\ &< d(x, x') + \delta \\ &= d(x, x') + \epsilon - d(x, x') \\ &= \epsilon. \end{aligned}$$

Task E4.3.3. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{O}_{d_X}$ be the topology on X corresponding to d_X of Task E3.4.9, and let $\mathcal{O}_{d_Y}$ be the topology on Y corresponding to d_Y. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if it is continuous in the metric sense. You may wish to proceed as follows.

(1) Suppose that f is continuous in the metric sense. Suppose that U belongs to $\mathcal{O}_{d_Y}$. Suppose that x belongs to $f^{-1}(U)$. By definition of $\mathcal{O}_{d_Y}$, observe that there is an $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ such that $B_\epsilon(f(x))$ is a subset of U.

(2) Since f is continuous in the metric sense, there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that $f(B_\delta(x))$ is a subset of $B_\epsilon(f(x))$. Deduce that $f(B_\delta(x))$ is a subset of U, and thus that $B_\delta(x)$ is a subset of $f^{-1}(U)$.

(3) By definition of $\mathcal{O}_{d_X}$, we have that $B_\delta(x)$ belongs to $\mathcal{O}_{d_X}$.

(4) By Task E8.3.1, deduce from (2) and (3) that f is continuous.

(5) Suppose instead that f is continuous. Let $\epsilon \in \mathbb{R}$ be such that $\epsilon > 0$. Suppose that x belongs to X. By Task E4.3.2, for every y which belongs to $B_\epsilon(f(x))$, there is a $\zeta \in \mathbb{R}$ with $\zeta > 0$ such that $B_\zeta(y)$ is a subset of $B_\epsilon(f(x))$. By definition of $\mathcal{O}_{d_Y}$, we have that $B_\zeta(y)$ belongs to $\mathcal{O}_{d_Y}$. By Task E8.3.1, deduce that $B_\epsilon(f(x))$ belongs to $\mathcal{O}_{d_Y}$.

(6) Since f is continuous, deduce that $f^{-1}(B_\epsilon(f(x)))$ belongs to $\mathcal{O}_{d_X}$.

(7) By definition of $\mathcal{O}_{d_X}$, deduce that there is a $\delta \in \mathbb{R}$ with $\delta > 0$ such that $B_\delta(x)$ is a subset of $f^{-1}(B_\epsilon(f(x)))$.

(8) Deduce that $f(B_\delta(x))$ is a subset of $B_\epsilon(f(x))$. Conclude that f is continuous in the metric sense.

Definition E4.3.4. Let X be a set. A metric d on X is *symmetric* if, for all x_0 and x_1 which belong to X, we have that $d(x_0, x_1) = d(x_1, x_0)$.

Definition E4.3.5. A metric space (X, d) is *symmetric* if d is symmetric.

Definition E4.3.6. Let (X, d) be a metric space. Let A_0 and A_1 be subsets of X. The *distance from* A_0 *to* A_1 with respect to d is

$$\inf \{d(x_0, x_1) \mid x_0 \in A_0 \text{ and } x_1 \in A_1\}.$$

Notation E4.3.7. Let (X, d) be a metric space. Let A_0 and A_1 be subsets of X. We denote the distance from A_0 to A_1 with respect to d by $d(A_0, A_1)$. Suppose that x belongs to X, and that A is a subset of X. We shall denote $d(\{x\}, A)$ simply by $d(x, A)$.

Remark E4.3.1. Let (X, d) be a symmetric metric space. Let A be a subset of X. Suppose that a belongs to A. By (1) of Definition E3.4.2, we have that $d(a, A) = 0$.

Task E4.3.8. Let (X, d) be a symmetric metric space. Let A be a subset of X. Suppose that x belongs to X. Let X be equipped with the topology $\mathcal{O}_d$ corresponding to d of Task E3.4.9. Prove that the map

$$X \xrightarrow{d(-, A)} \mathbb{R}$$

given by $x \mapsto d(x, A)$ is continuous. You may wish to proceed as follows.

(1) By Task E3.4.12, we have that $\mathcal{O}_{\mathbb{R}} = \mathcal{O}_{d_{\mathbb{R}}}$. By Task E4.3.3, it therefore suffices to demonstrate that $d(-, A)$ is continuous in the metric sense.

(2) Suppose that a belongs to A. By definition of d, we have that $d(y, a) \leq d(y, x) + d(x, a)$. Since $d(y, A) \leq d(y, a)$, we deduce that $d(y, A) \leq d(y, x) + d(x, a)$.

(3) Deduce that $d(x,a) \geq d(y,A) - d(y,x)$. Since this inequality holds for all a which belong to A, deduce that $d(x,A) \geq d(y,A) - d(y,x)$. Deduce that $d(y,A) \leq d(x,A) + d(y,x)$.

(4) Carrying out exactly the same argument, but swapping x and y, observe that $d(y,A) \geq d(x,A) - d(x,y)$.

(5) Let $\epsilon \in \mathbb{R}$ be such that $\epsilon > 0$. Suppose that $d(x,y) < \epsilon$. Deduce from (3), (4), and the fact that d is symmetric, that

$$d(x,A) - \epsilon \leq d(y,A) \leq d(x,A) + \epsilon.$$

(6) Deduce from (5) that $d\left(B_\epsilon(x), A\right)$ is a subset of $B_\epsilon\left(d(x,A)\right)$. Conclude that $d(-,A)$ is continuous in the metric sense, as required.

5. Monday 20th January

5.1. Geometric examples of continuous maps

Remark 5.1.1. Most of our continuous maps between geometric examples of topological spaces will be constructed from polynomial maps

$$\mathbb{R} \longrightarrow \mathbb{R}$$

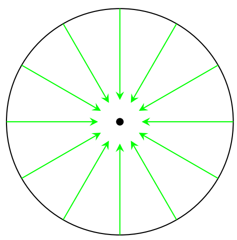
in 'canonical' ways: by restrictions, products, and quotients. Don't worry about this for now. We shall take it for granted, leaving details for the exercises, and instead focus on developing a geometric feeling for continuity.

Example 5.1.2. Let

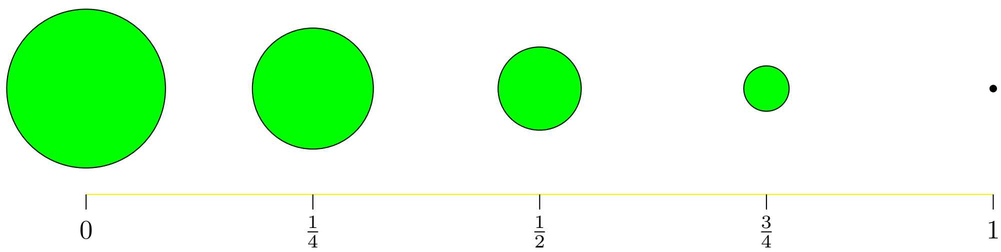
$$D^2 \times I \xrightarrow{f} D^2$$

be given by $(x, y, t) \mapsto ((1-t)x, (1-t)y)$. Then f is continuous. To prove this is Task E5.2.6.

Remark 5.1.3. We may think of f as 'shrinking D^2 onto its centre', as t moves from 0 to 1.



We can picture the image of $D^2 \times \{t\}$ under f as follows, as t moves from 0 to 1.



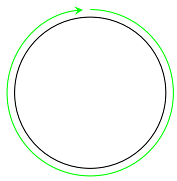
Example 5.1.4. Let $k \in \mathbb{R}$. There is a continuous map

$$I \xrightarrow{f} S^1$$

which can be thought of as travelling k times around a circle, starting at $(0,1)$. To construct f rigorously is the topic of Task E5.2.7.

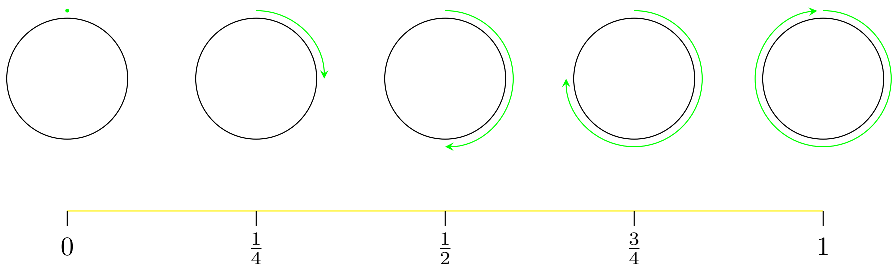
Remark 5.1.5. Let us picture f for a few values of k.

(1) Let $k = 1$. Then we travel exactly once around S^1.

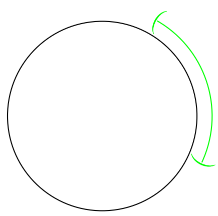


Don't be misled by the picture. The path really travels around the circle, not slightly outside it.

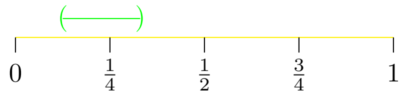
We may picture $f([0,t])$ as t moves from 0 to 1 as follows.



Recall from Examples 4.1.4 that a typical open subset U of S^1 is an 'open arc'.

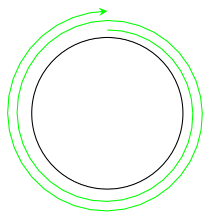


We have that $f^{-1}(U)$ is an open interval as follows.



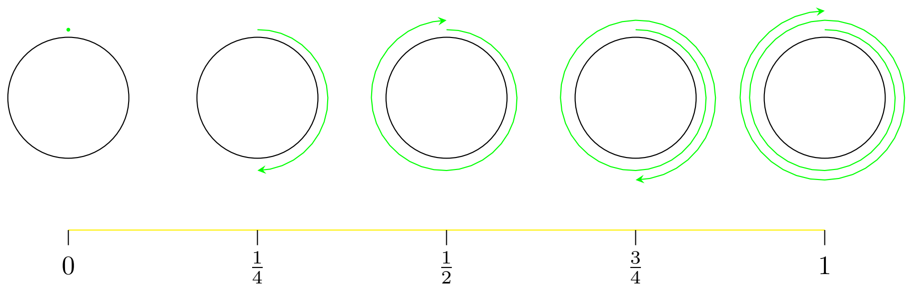
In particular, $f^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus, even though we have not yet rigorously constructed f, we can intuitively believe that it is continuous.

(2) Let $k = 2$. Then we travel exactly twice around S^1.

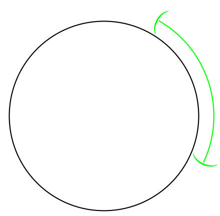


Again, don't be misled by the picture. The path really travels twice around the circle, thus passing through every point on the circle twice, not in a spiral outside the circle.

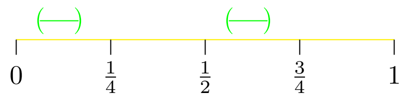
We may picture $f([0, t])$ as t moves from 0 to 1 as follows.



Let U denote the subset of S^1 given by the 'open arc' depicted below.

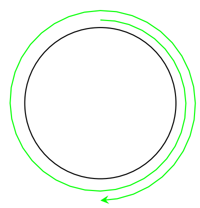


Then $f^{-1}(U)$ is a disjoint union of open intervals as follows.

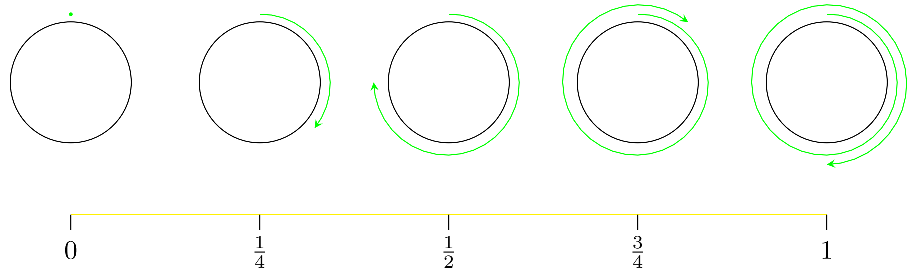


In particular, $f^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus, again, even though we have not rigorously constructed f, we can believe intuitively that it is continuous.

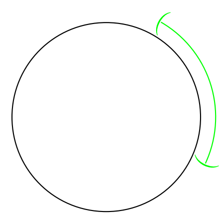
(3 Let $k = \frac{3}{2}$. Then we travel exactly one and a half times around S^1.



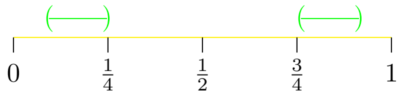
We may picture $f([0, t])$ as t moves from 0 to 1 as follows.



Let U denote the subset of S^1 given by the 'open arc' depicted below.



Then $f^{-1}(U)$ is a disjoint union of open intervals as follows.



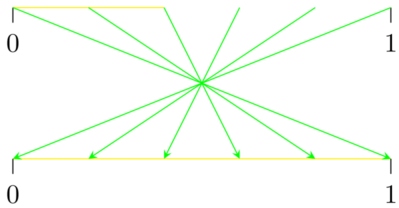
In particular, $f^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus, once more, even though we have not rigorously constructed f, we can believe intuitively that it is continuous.

Example 5.1.6. Let

$$I \xrightarrow{f} I$$

be given by $t \mapsto 1 - t$. Then f is continuous, by Task E5.3.14.

Remark 5.1.7. We may picture f as follows.



Let U denote the subset of I given by the following open interval.

Then $f^{-1}(U)$ is the following open interval.

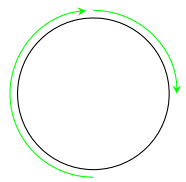
In particular, $f^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus, even though this is not quite a proof yet, we can intuitively believe that f is continuous.

Example 5.1.8. There is a map

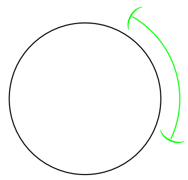
$$I \xrightarrow{f} S^1$$

travels around the circle at half speed from $(0,1)$ to $(1,0)$ for $0 \leq t \leq \frac{1}{2}$, and at normal speed from $(0,-1)$ to $(0,1)$ for $\frac{1}{2} < t \leq 1$. It is not continuous. To construct f rigorously, and to prove that it is not continuous, is the topic of Task E5.2.8.

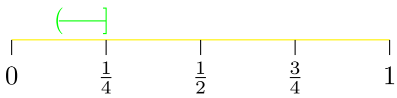
Remark 5.1.9. We may picture f as follows.



Let U denote the subset of S^1 given by the 'open arc' depicted below.



Then $f^{-1}(U)$ is a half open interval as follows.



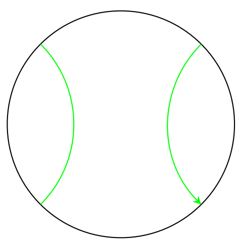
In particular, $f^{-1}(U)$ does not belong to $\mathcal{O}_I$. Thus we can see intuitively that f is not continuous.

Example 5.1.10. There is a map

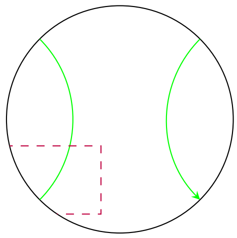
$$I \xrightarrow{f} D^2$$

which begins at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, travels around an arc of radius $\frac{1}{4}$ centred at $\left(-\frac{3}{4}, 0\right)$ to $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, jumps to $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and then travels around an arc of radius $\frac{1}{4}$ centred at $\left(\frac{3}{4}, 0\right)$ to $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. It is not continuous. To construct f rigorously, and to prove that it is not continuous, is the topic of Task E5.2.9.

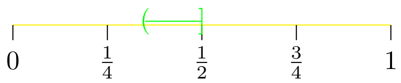
Remark 5.1.11. We may picture f as follows.



Let U denote the subset of D^2 given by the 'open rectangle' depicted below.



Then $f^{-1}(U)$ is a half open interval as follows.



In particular, $f^{-1}(U)$ does not belong to $\mathcal{O}_I$. Thus we can see intuitively that f is not continuous.

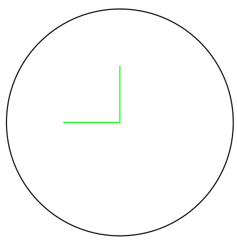
Remark 5.1.12. Intuitively, continuous maps cannot 'jump'!

Example 5.1.13. Let

$$\mathbb{R} \xrightarrow{f} D^2$$

be the map given by

$$x \mapsto \begin{cases} (-\frac{1}{2}, 0) & \text{for } x \leq -\frac{1}{2}, \\ (x, 0) & \text{for } -\frac{1}{2} \leq x \leq 0, \\ (0, x) & \text{for } 0 \leq x \leq \frac{1}{2}, \\ (0, \frac{1}{2}) & \text{for } x \geq \frac{1}{2}. \end{cases}$$



Then f is continuous. To prove this is the topic of Task E5.2.10.

Remark 5.1.14. In particular, continuous maps can have 'sharp edges'. In *differential topology*, maps are required to moreover be *smooth*: sharp edges are disallowed! The courses MA3402 Analyse på Mangfoldigheter and TMA4190 Mangfoldigheter both lead towards differential topology.

5.2. Inclusion maps are continuous

Terminology 5.2.1. Let X be a set, and let A be a subset of X. We refer to the map

$$A \xrightarrow{i} X$$

given by $x \mapsto x$ as an *inclusion map*.

Proposition 5.2.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. The inclusion map

$$A \xrightarrow{i} X$$

is continuous.

Proof. Let U be a subset of X which belongs to $\mathcal{O}_X$. Then $i^{-1}(U) = A \cap U$. By definition of $\mathcal{O}_A$, we have that $A \cap U$ belongs to $\mathcal{O}_A$. We conclude that $i^{-1}(U)$ belongs to $\mathcal{O}_A$. $\square$

Notation 5.2.3. Let X, Y, and Z be sets. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be maps. We denote by

$$X \xrightarrow{g \circ f} Z$$

the *composition* of f and g, given by $x \mapsto g(f(x))$.

5.3. Compositions of continuous maps are continuous

Proposition 5.3.1. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be continuous maps. The map

$$X \xrightarrow{g \circ f} Z$$

is continuous.

Proof. Let U be a subset of Z which belongs to $\mathcal{O}_Z$. Then

$$\begin{aligned}(g \circ f)^{-1}(U) &= \{x \in X \mid g(f(x)) \in U\} \\ &= \left\{x \in X \mid f(x) \in g^{-1}(U)\right\} \\ &= f^{-1}\left(g^{-1}(U)\right).\end{aligned}$$

Since g is continuous, we have that $g^{-1}(U)$ belongs to $\mathcal{O}_Y$. We deduce, since f is continuous, that $f^{-1}\left(g^{-1}(U)\right)$ belongs to $\mathcal{O}_X$. Thus $(g \circ f)^{-1}(U)$ belongs to $\mathcal{O}_X$. □

5.4. Projection maps are continuous

Notation 5.4.1. Let X and Y be sets. We denote by

$$X \times Y \xrightarrow{p_1} X$$

the map given by $(x, y) \mapsto x$. We denote by

$$X \times Y \xrightarrow{p_2} Y$$

the map given by $(x, y) \mapsto y$.

Terminology 5.4.2. We refer to p_1 and p_2 as *projection maps*.

Proposition 5.4.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X \times Y}$. Then

$$X \times Y \xrightarrow{p_1} X$$

and

$$X \times Y \xrightarrow{p_2} Y$$

are continuous.

Proof. Suppose that U_X is a subset of X which belongs to $\mathcal{O}_X$. Then

$$p_1^{-1}(U_X) = U_X \times Y.$$

We have that $U_X \times Y$ belongs to $\mathcal{O}_{X \times Y}$. Thus p_1 is continuous.

Suppose now that U_Y is a subset of Y which belongs to $\mathcal{O}_Y$. Then

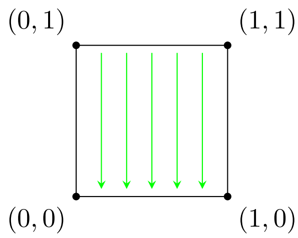
$$p_2^{-1}(U_Y) = X \times U_Y.$$

We have that $X \times U_Y$ belongs to $\mathcal{O}_{X \times Y}$. Thus p_2 is continuous. □

Remark 5.4.4. We can think of

$$I \times I \xrightarrow{p_1} I$$

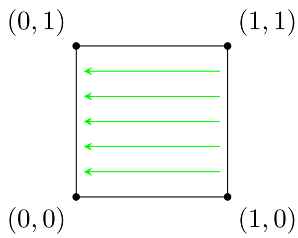
as the map $(x, y) \mapsto (x, 0)$. We can picture this as follows.



We can think of

$$I \times I \xrightarrow{p_2} I.$$

as the map $(x, y) \mapsto (0, y)$. We can picture this as follows.



E5. Exercises for Lecture 5

E5.1. Exam questions

Remark E5.1.1. You may find it helpful to carry out Tasks E5.2.3 – E5.2.5 before attempting the tasks in this section.

Terminology E5.1.2. Let X be a set. We refer to the map

$$X \longrightarrow X$$

given by $x \mapsto x$ as the *identity map* from X to itself.

Task E5.1.3. Let $(X, \mathcal{O}_X)$ be a topological space. Prove that the identity map

$$X \xrightarrow{id} X$$

is continuous.

Terminology E5.1.4. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is *constant* if $f(x) = f(x')$ for all $x, x' \in X$.

Task E5.1.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a constant map. Prove that f is continuous. You may wish to proceed as follows.

(1) Observe that if f is constant, then there is a $y \in Y$ such that $f(x) = y$ for all $x \in X$.

(2) Let U be a subset of Y which belongs to $\mathcal{O}_Y$. Determine $f^{-1}(U)$ in the cases that $y \in U$, and in the case that $y \notin U$.

Terminology E5.1.6. Let X and Y be sets, and let A be a subset of X. Let

$$X \xrightarrow{f} Y$$

be a map. The *restriction* of f to A is the map

$$A \longrightarrow Y$$

given by $x \mapsto f(x)$.

Remark E5.1.7. In other words, the restriction of f to A is the map

$$A \xrightarrow{f \circ i} Y,$$

where

$$A \xrightarrow{i} X$$

is the inclusion map.

Task E5.1.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Let A be a subset of X, and let A be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Prove that the restriction of f to A defines a continuous map

$$A \longrightarrow Y.$$

Task E5.1.9. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of Y, and let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(Y, \mathcal{O}_Y)$. Prove that if

$$X \xrightarrow{f} Y$$

is a continuous map such that $f(X) \subset A$, then the map

$$X \longrightarrow A$$

given by $x \mapsto f(x)$ is continuous.

Task E5.1.10. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of Y, and let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(Y, \mathcal{O}_Y)$. Prove that if

$$X \xrightarrow{f} A$$

is a continuous map, then the map

$$X \longrightarrow Y$$

given by $x \mapsto f(x)$ is continuous.

Terminology E5.1.11. Let X be a set. We refer to the map

$$X \xrightarrow{\Delta} X \times X$$

given by $x \mapsto (x, x)$ as the *diagonal map*.

Task E5.1.12. Let $(X, \mathcal{O}_X)$ be a topological space. Let $X \times X$ be equipped with the product topology $\mathcal{O}_{X \times X}$ with respect to two copies of $(X, \mathcal{O}_X)$. Prove that

$$X \xrightarrow{\Delta} X \times X$$

is continuous. You may wish to proceed as follows.

(1) Let U_0 and U_1 be subsets of X which belong to $\mathcal{O}_X$. Prove that $\Delta^{-1}(U_0 \times U_1)$ belongs to $\mathcal{O}_X$.

(2) Let U be a subset of $X \times X$ which belongs to $\mathcal{O}_{X \times X}$. Prove that $\Delta^{-1}(U)$ belongs to $\mathcal{O}_X$, by appealing to (1) and to Task E8.3.1.

Task E5.1.13. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that a map

$$X \xrightarrow{f} Y$$

is continuous if and only if $f^{-1}(V)$ is closed with respect $\mathcal{O}_X$, for every subset V of Y which is closed with respect to $\mathcal{O}_Y$.

Task E5.1.14. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. Let $(Y, \mathcal{O}_Y)$ be a topological space. Prove that any map

$$X \xrightarrow{f} Y$$

is continuous.

Task E5.1.15. Let $(X, \mathcal{O}_X)$ be a topological space. Let Y be a set. Let $\mathcal{O}_Y$ be the indiscrete topology on X. Prove that any map

$$X \xrightarrow{f} Y$$

is continuous.

E5.2. In the lecture notes

Task E5.2.1. Let X, Y, and Z be sets. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be maps. Prove that

$$\{x \in X \mid g\left(f(x)\right) \in U\} = \left\{x \in X \mid f(x) \in g^{-1}(U)\right\}.$$

This was appealed to in the proof of Proposition 5.3.1.

Task E5.2.2. Let $(X, \mathcal{O}_X$ and $(Y, \mathcal{O}_Y)$ be topological spaces.

(1) Let U_X be a subset of X which belongs to $\mathcal{O}_X$. Check that you understand why $U_X \times Y$ belongs to $\mathcal{O}_{X \times Y}$.

(2) Let U_Y be a subset of Y which belongs to $\mathcal{O}_Y$. Check that you understand why $X \times U_Y$ belongs to $\mathcal{O}_{X \times Y}$.

These observations were appealed to in the proof of Proposition 5.4.3.

Task E5.2.3. Do the same as in Task E2.2.2 for the proof of Proposition 5.2.2.

Task E5.2.4. Do the same as in Task E2.2.2 for the proof of Proposition 5.3.1.

Task E5.2.5. Do the same as in Task E2.2.2 for the proof of Proposition 5.4.3.

Task E5.2.6. Prove that the map

$$D^2 \times I \xrightarrow{f} D^2$$

of Example 5.1.2 is continuous. You may wish to proceed as follows.

(1) Express the map

$$\mathbb{R}^3 \xrightarrow{f_0} \mathbb{R}$$

given by $(x, y, t) \mapsto (1-t)x$ as a composition of four maps.

(I) The map

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

given by $(x, y, t) \mapsto (x, t)$.

(II) The twist map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

given by $(x, y) \mapsto (y, x)$.

(III) The map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

given by $(x, y) \mapsto (1 - x, y)$.

(IV) The map

$$\mathbb{R}^2 \xrightarrow{\times} \mathbb{R}$$

given by $(x, y) \mapsto xy$.

Appealing to Proposition 5.4.3, Task E5.3.17, Task E5.3.19, Task E5.3.14, Task E5.3.11, and Proposition 5.3.1, deduce that f_0 is continuous.

(2) In a similar way, prove that the map

$$\mathbb{R}^3 \xrightarrow{f_1} \mathbb{R}$$

given by $(x, y, t) \mapsto (1 - t)y$ is continuous.

(3) View $D^2 \times I$ as equipped with the subspace topology with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$. Appealing to Task E5.1.8, deduce from (1) that the map

$$D^2 \times I \xrightarrow{f_0} \mathbb{R}$$

given by $(x, y, t) \mapsto (1 - t)x$ is continuous, and deduce from (2) that the map

$$D^2 \times I \xrightarrow{f_1} \mathbb{R}$$

given by $(x, y, t) \mapsto (1 - t)y$ is continuous,

(4) Appealing to Task E5.3.17, deduce from (3) that the map

$$D^2 \times I \longrightarrow \mathbb{R}^2$$

given by $(x, y, t) \mapsto ((1-t)x, (1-t)y)$ is continuous.

(5) Appealing to Task E5.1.9, conclude from (4) that f is continuous.

Task E5.2.7. Let $k \in \mathbb{R}$. Construct a continuous map

$$I \longrightarrow S^1$$

which travels around the circle k times, as in Example 5.1.4. You may wish to proceed as follows.

(1) By Task E5.3.14, observe that the map

$$I \longrightarrow [0, k]$$

given by $t \mapsto kt$ is continuous.

(2) By Task E5.3.27 and Task E5.1.8, observe that the map

$$[0, k] \longrightarrow S^1$$

given by $t \mapsto \phi(t)$ is continuous, where

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is the map of Task E5.3.27.

(3) Appeal to Proposition 5.3.1.

Task E5.2.8. Use the map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

of Task E5.3.27 to construct the map

$$I \xrightarrow{f} S^1$$

of Example 5.1.8. Prove that f is not continuous.

Task E5.2.9. Use the map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

of Task E5.3.27 to construct the map

$$I \xrightarrow{f} D^2$$

of Example 5.1.10. Prove that f is not continuous.

Task E5.2.10. Let

$$\mathbb{R} \xrightarrow{f} D^2$$

be the map of Example 5.1.13. Prove that f is continuous. You may wish to proceed as follows.

(1) By Task E5.1.5, observe that the map

$$\left[-\infty, -\tfrac{1}{2}\right[\longrightarrow D^2$$

given by $x \mapsto (-\frac{1}{2}, 0)$ is continuous.

(2) By Task E5.1.3, Task E5.1.5, and Task E5.3.17, observe that the map

$$\left[-\tfrac{1}{2}, 0\right] \longrightarrow D^2$$

given by $x \mapsto (x, 0)$ is continuous.

(3) By Task E5.1.3, Task E5.1.5, and Task E5.3.17, observe that the map

$$\left[0, \tfrac{1}{2}\right] \longrightarrow D^2$$

given by $x \mapsto (0, x)$ is continuous.

(4) By Task E5.1.5, observe that the map

$$\left]\tfrac{1}{2}, \infty\right] \longrightarrow D^2$$

given by $x \mapsto (0, \frac{1}{2})$ is continuous.

(5) Appeal to (2) of Task E5.3.23.

E5.3. For a deeper understanding

Assumption E5.3.1. Throughout this section, let $\mathbb{R}$ be equipped with the standard topology $\mathcal{O}_{\mathbb{R}}$.

Remark E5.3.2. The proofs needed for Tasks E5.3.5 – E5.3.7 and Task E5.3.9 all follow the pattern of the proof of the following proposition, which is given to help you along.

Proposition E5.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Then the map

$$X \xrightarrow{|f|} \mathbb{R}$$

given by $x \mapsto |f(x)|$ is continuous.

Proof. By Corollary E4.2.6, to prove that f is continuous, it suffices to prove that $|f|^{-1}(]a,b[)$ belongs to $\mathcal{O}_X$, for every open interval $]a,b[$. We have that

$$\begin{aligned}
|f|^{-1}(]a,b[) &= \left\{x' \in X \mid \left|f(x')\right| \in]a,b[\right\} \\
&= \left\{x' \in X \mid f(x') \in]a,b[\right\} \cup \left\{x' \in X \mid -f(x') \in]a,b[\right\} \\
&= \left\{x' \in X \mid f(x') \in]a,b[\right\} \cup \left\{x' \in X \mid f(x') \in]-b,-a[\right\} \\
&= f^{-1}(]a,b[) \cup f^{-1}(]-b,-a[).
\end{aligned}$$

Both $]a,b[$ and $]-b,-a[$ belong to $\mathcal{O}_{\mathbb{R}}$. Since f is continuous, we deduce that both $f^{-1}(]a,b[)$ and $f^{-1}(]-b,-a[)$ belong to $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a topology on X, this implies that

$$f^{-1}(]a,b[) \cup f^{-1}(]-b,-a[)$$

belongs to $\mathcal{O}_X$. Hence $|f|^{-1}(]a,b[)$ belongs to $\mathcal{O}_X$. $\square$

Remark E5.3.4. In a nutshell, the proof of Proposition E5.3.3 proceeds by expressing $\left|f^{-1}\right|(]a,b[)$ as a union of inverse images under f of subsets of $\mathbb{R}$ which belong to $\mathcal{O}_{\mathbb{R}}$. It is this idea that is also at the heart of the proofs needed for Tasks E5.3.5 – E5.3.7 and Task E5.3.9.

Task E5.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Prove that, for any $k \in \mathbb{R}$, the map

$$X \xrightarrow{kf} \mathbb{R}$$

given by $x \mapsto k \cdot f(x)$ is continuous. You may wish to proceed by considering separately the cases $k = 0$, $k > 0$, and $k < 0$.

Task E5.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R} \end{tikzcd}$$

be continuous maps. Prove that the map

$$X \xrightarrow{f+g} \mathbb{R}$$

given by $x \mapsto f(x) + g(x)$ is continuous. You may wish to proceed as follows.

(1) Observe that, by Task E4.2.12 and Task E4.2.11, to prove that $f+g$ is continuous, it suffices to prove that for any $y \in \mathbb{R}$, the sets

$$(f+g)^{-1}\left(]-\infty, y[\right)$$

and

$$(f+g)^{-1}\left(]y, \infty[\right)$$

belong to $\mathcal{O}_X$.

(2) Prove that $\{x \in X \mid f(x) + g(x) < y\}$ is equal to

$$\bigcup_{y' \in \mathbb{R}} \left(\left\{ x \in X \mid f(x) < y - y' \right\} \cap \left\{ x \in X \mid g(x) < y' \right\} \right),$$

and that $\{x \in X \mid f(x) + g(x) > y\}$ is equal to

$$\bigcup_{y' \in \mathbb{R}} \left(\left\{ x \in X \mid f(x) > y - y' \right\} \cap \left\{ x \in X \mid g(x) > y' \right\} \right).$$

Task E5.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map, with the property that $f(x) \geq 0$ for all $x \in X$. Prove that the map

$$X \xrightarrow{f^2} \mathbb{R}$$

given by $x \mapsto f(x) \cdot f(x)$ is continuous.

Task E5.3.8. Let

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R} \end{tikzcd}$$

be continuous maps. Prove that the map

$$X \xrightarrow{fg} \mathbb{R}$$

given by $x \mapsto f(x) \cdot g(x)$ is continuous. You may wish to proceed as follows.

(1) Observe that fg is

$$\tfrac{1}{4}\left(|f+g|^2 - |f-g|^2\right).$$

(2) Appeal to Proposition E5.3.3 and Tasks E5.3.5 – E5.3.7.

Task E5.3.9. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Suppose that $f(x) \neq 0$ for all $x \in X$. Prove that the map

$$X \xrightarrow{\frac{1}{f}} \mathbb{R}$$

given by $x \mapsto \frac{1}{f(x)}$ is continuous. You may wish to proceed as follows.

(1) Observe that, by Task E4.2.12 and Task E4.2.11, to prove that $\frac{1}{f}$ is continuous, it suffices to prove that for any $y \in \mathbb{R}$, the sets

$$\left(\frac{1}{f}\right)^{-1}\left(]-\infty, y[\right)$$

and

$$\left(\frac{1}{f}\right)^{-1}\left(]y, \infty[\right)$$

belong to $\mathcal{O}_X$.

(2) Prove that, for all $y \in \mathbb{R}$, the set

$$\left\{x \in X \mid \frac{1}{f(x)} < y\right\}$$

is equal to the union of

$$\{x \in X \mid f(x) > 0\} \cap \{x \in X \mid (yf)(x) > 1\}$$

and

$$\{x \in X \mid f(x) < 0\} \cap \{x \in X \mid (yf)(x) < 1\}.$$

(3) Prove that, for all $y \in \mathbb{R}$, the set

$$\left\{x \in X \mid \frac{1}{f(x)} > y\right\}$$

is equal to the union of

$$\{x \in X \mid f(x) > 0\} \cap \{x \in X \mid (yf)(x) < 1\}$$

and

$$\{x \in X \mid f(x) < 0\} \cap \{x \in X \mid (yf)(x) > 1\}.$$

(4) Appeal to Task E5.3.5.

Task E5.3.10. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R} \end{tikzcd}$$

be continuous maps. Suppose that $g(x) \neq 0$ for all $x \in X$. Prove that the map

$$X \xrightarrow{\frac{f}{g}} \mathbb{R}$$

given by $x \mapsto \frac{f(x)}{g(x)}$ is continuous. You may wish to appeal to two of the previous tasks.

Task E5.3.11. Let $\mathbb{R}^2$ be equipped with the topology $\mathcal{O}_{\mathbb{R}^2}$. Prove that the map

$$\mathbb{R}^2 \xrightarrow{\times} \mathbb{R}$$

given by $(x, y) \mapsto xy$ is continuous. You may wish to appeal to Proposition 5.4.3, and to Task E5.3.8.

Task E5.3.12. Let $\mathbb{R}^2$ be equipped with the topology $\mathcal{O}_{\mathbb{R}^2}$. Prove that the map

$$\mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$$

given by $(x, y) \mapsto x + y$ is continuous. You may wish to appeal to Proposition 5.4.3, and to Task E5.3.6.

Terminology E5.3.13. Let X and Y be subsets of $\mathbb{R}$. Let

$$X \xrightarrow{f} Y$$

be a map given by

$$x \mapsto k_n x^n + k_{n-1} x^{n-1} + \ldots + k_1 x_1 + k_0,$$

where $k_i \in \mathbb{R}$ for all $0 \leq i \leq n$. We refer to f as a *polynomial map*.

Task E5.3.14. Let X be a subset of $\mathbb{R}$, equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let Y also be a subset of $\mathbb{R}$, equipped with the subspace topology $\mathcal{O}_Y$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that every polynomial map

$$X \xrightarrow{f} Y$$

is continuous. You may wish to proceed as follows.

(1) Demonstrate that a polynomial map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

is continuous. For this, you may wish to proceed by induction, appealing to Task E5.1.3, Task E5.3.5, Task E5.3.8, and Task E5.3.6.

(2) Appeal to Task E5.1.8 and to Task E5.1.9.

Corollary E5.3.15. Let X be a subset of $\mathbb{R}$, equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let Y also be a subset of $\mathbb{R}$, equipped with the subspace topology $\mathcal{O}_Y$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let

$$X \xrightarrow{f} Y$$

be a map given by $x \mapsto \frac{g_0(x)}{g_1(x)}$, where $g_1(x) \neq 0$ for all $x \in X$. Suppose that g_0 and g_1 are polynomial maps. Then f is continuous.

Proof. We can view f as a map

$$X \xrightarrow{f'} \mathbb{R}.$$

It follows immediately from Task E5.3.14 and Task E5.3.10 that f' is continuous. By Task E5.1.9, we conclude that f is continuous. $\square$

Task E5.3.16. Let $(X, \mathcal{O}_X)$ be a topological space. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map such that $f(x) \geq 0$ for all $x \in X$. Prove that, for any $n \in \mathbb{N}$, the map

$$X \xrightarrow{\sqrt[n]{f}} \mathbb{R}$$

given by $x \mapsto \sqrt[n]{f(x)}$, where $\sqrt[n]{f(x)}$ denotes the positive n^{th} root of $f(x)$, is continuous.

Task E5.3.17. Let $(X_0, \mathcal{O}_{X_0})$, $(X_1, \mathcal{O}_{X_1})$, $(Y_0, \mathcal{O}_{Y_0})$, and $(Y_1, \mathcal{O}_{Y_1})$ be topological spaces. Let

$$X_0 \xrightarrow{f_0} Y_0$$

and

$$X_1 \xrightarrow{f_1} Y_1$$

be continuous maps. Prove that the map

$$X_0 \times X_1 \xrightarrow{f_0 \times f_1} Y_0 \times Y_1$$

given by $(x_0, x_1) \mapsto (f_0(x_0), f_1(x_1))$ is continuous.

Terminology E5.3.18. Let X and Y be sets. We refer to the map

$$X \times Y \xrightarrow{\tau} Y \times X$$

given by $(x, y) \mapsto (y, x)$ as the *twist map*.

Task E5.3.19. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that the twist map

$$X \times Y \xrightarrow{\tau} Y \times X$$

is continuous. You may wish to appeal to Task E5.3.17.

Task E5.3.20. Let $(X, \mathcal{O}_X)$, $(Y_0, \mathcal{O}_{Y_0})$, and $(Y_1, \mathcal{O}_{Y_1})$ be topological spaces. Let

$$X \xrightarrow{f_0} Y_0$$

and

$$X \xrightarrow{f_1} Y_1$$

be continuous maps. Prove that the map

$$X \xrightarrow{f_0 \times f_1} Y_0 \times Y_1$$

given by $x \mapsto (f_0(x), f_1(x))$ is continuous. You may wish to appeal to Task E5.1.12 and Task E5.3.17.

Task E5.3.21. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Let A be a subset of Y, and let A be equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$. Let

$$A \xrightarrow{i} Y$$

be the inclusion map. Prove that a map

$$X \xrightarrow{f} A$$

is continuous if and only if the map

$$X \xrightarrow{i \circ f} Y$$

is continuous.

Notation E5.3.22. Let X and Y be sets. Let $\{A_j\}_{j \in J}$ be a set of subsets of X such that $X = \bigcup_{j \in J} A_j$. Suppose that, for every $j \in J$, we have a map

$$A_j \xrightarrow{f_j} Y.$$

Moreover, suppose that, for all $j_0, j_1 \in J$, the restriction of f_{j_0} to $A_{j_0} \cap A_{j_1}$ is equal to the restriction of f_{j_1} to $A_{j_0} \cap A_{j_1}$. We then obtain a map

$$X \longrightarrow Y$$

given by $x \mapsto f_j(x)$ if $x \in A_j$.

Task E5.3.23. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\{A_j\}_{j \in J}$ be a set of subsets of X such that $X = \bigcup_{j \in J} A_j$. For every $j \in J$, let A_j be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Suppose that, for every $j \in J$, we have a continuous map

$$A_j \xrightarrow{f_j} Y.$$

Moreover, suppose that, for all $j_0, j_1 \in J$, the restriction of f_{j_0} to A_{j_0} is equal to the restriction of f_{j_1} to A_{j_1}. Let

$$X \xrightarrow{f} Y$$

denote the map of Notation E5.3.22 corresponding to the maps $\{f_j\}_{j \in J}$.

(1) Suppose that A_j belongs to $\mathcal{O}_X$ for every $j \in J$. Prove that f is continuous. You may wish to appeal to Task E2.3.4.

(2) Suppose that $\{A_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_X$, and that A_j is closed with respect to $\mathcal{O}_X$, for every $j \in J$. Prove that f is continuous. You may wish to appeal to Task E8.3.9.

(3) Suppose that J finite. Give an example to demonstrate that if we do not assume that A_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$, then f is not necessarily continuous.

(4) Suppose that A_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$. Given an example to demonstrate that, if we do not assume that $\{A_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_X$, then f is not necessarily continuous.

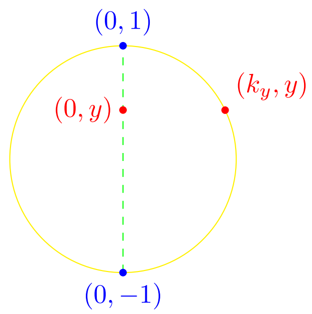
Remark E5.3.24. Taking into account Remark E8.3.6, we have that if J is finite, and A_j is closed with respect to $\mathcal{O}_X$, for every $j \in J$, then f is continuous.

Remark E5.3.25. The result of (1) and (2) of Task E5.3.23 is sometimes known as the *glueing lemma* or *pasting lemma*. Continuous maps constructed by means of (1) and (2) of Task E5.3.23 are sometimes said to be defined *piecewise*.

Notation E5.3.26. Given $y \in [-1, 1]$, let

$$k_y = \sqrt{1 - y^2}.$$

Here we take the positive square root. We have that $\|(k_y, y)\| = 1$.

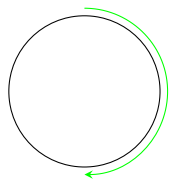


Let

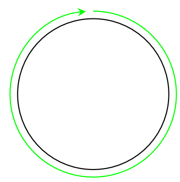
$$\mathbb{R} \xrightarrow{\phi} S^1$$

be the map defined as follows.

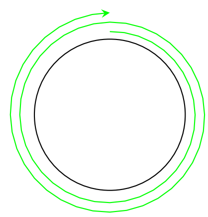
(1) Suppose that $x \in \left[0, \frac{1}{2}\right]$. Let $y = 1 - 4x$. We define $\phi(x)$ to be (k_y, y). We can picture ϕ on $\left[0, \frac{1}{2}\right]$ as follows.



(2) Suppose that $x \in \left[\frac{1}{2}, 1\right]$. Let $y = 4x - 3$. We define $\phi(x)$ to be $(-k_y, y)$. We can picture ϕ on $[0, 1]$ as follows.



(3) Suppose that $x \in [n, n+1]$, for some $n \in \mathbb{Z}$. We define $\phi(x)$ to be $\phi(x - n)$. We can picture ϕ on $[0, 2]$, for instance, as follows.



Task E5.3.27. Prove that the map

$$\mathbb{R} \xrightarrow{\phi} S^1$$

of Notation E5.3.26 is continuous. You may wish to proceed as follows.

(1) Let $\left[0, \frac{1}{2}\right]$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Observe that by Task E5.3.14 and Task E5.3.16, the map

$$\left[0, \tfrac{1}{2}\right] \xrightarrow{f} \mathbb{R}$$

given by $y \mapsto k_y$ is continuous.

(2) By Task E5.3.17, deduce from (1) that the map

$$\left[0, \tfrac{1}{2}\right] \xrightarrow{f \times id} \mathbb{R}^2$$

given by $y \mapsto (k_y, y)$ is continuous. By Task E5.1.9, deduce that the map

$$\left[0, \tfrac{1}{2}\right] \longrightarrow S^1$$

given by $x \mapsto \phi(x)$ is continuous.

(3) Let $\left[\frac{1}{2}, 1\right]$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. As in (1) and (2), demonstrate that the map

$$\left[\tfrac{1}{2}, 1\right] \longrightarrow S^1$$

given by $x \mapsto \phi(x)$ is continuous.

(4) Let the unit interval I be equipped with the topology $\mathcal{O}_I$. By (2) of Task E5.3.23, deduce from (2) and (3) that the map

$$I \longrightarrow S^1$$

given by $x \mapsto \phi(x)$ is continuous.

(5) Let $n \in \mathbb{Z}$. Let $[n, n+1]$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Task E5.3.14, observe that the map

$$[n, n+1] \xrightarrow{g} I$$

given by $x \mapsto x - n$ is continuous.

(6) Let $n \in \mathbb{Z}$. By Proposition 5.3.1, deduce from (4) and (5) that the map

$$[n, n+1] \longrightarrow S^1$$

given by $x \mapsto \phi(x - n)$ is continuous.

(7) By (2) of Task E5.3.23, deduce from (6) that

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is continuous.

Remark E5.3.28. The map ϕ allows us to construct paths around a circle without using, for instance, trigonometric maps. Sine and cosine do define continuous maps, but their construction, and the proof that they are continuous, is much more involved. We shall explore this in a later task.

Task E5.3.29. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that x belongs to X. Let $X \setminus f^{-1}(\{f(x)\})$ be equipped with the subspace topology $\mathcal{O}_{X \setminus f^{-1}(\{f(x)\})}$ with respect to $(X, \mathcal{O}_X)$. Let

$$X \xrightarrow{f} Y$$

be a map. Suppose that $f^{-1}(\{f(x)\})$ is closed in X with respect to $\mathcal{O}_X$. Let $Y \setminus \{f(x)\}$ be equipped with the subspace topology $\mathcal{O}_{Y \setminus \{f(x)\}}$ with respect to $(Y, \mathcal{O}_Y)$. Suppose that the map

$$X \setminus f^{-1}(\{f(x)\}) \xrightarrow{g} Y \setminus \{f(x)\}$$

given by $x' \mapsto f(x')$ is continuous. Prove that f is continuous. You may wish to proceed as follows.

(1) Let V be a subset of Y which is closed with respect to $\mathcal{O}_Y$. Suppose that $f(x)$ does not belong to V. Then V is a subset of $Y \setminus \{f(x)\}$. Thus $f^{-1}(V) = g^{-1}(V)$. Since g is continuous, deduce by Task **??** that $f^{-1}(V)$ is closed in X with respect to $\mathcal{O}_X$.

(2) Suppose that $f(x)$ belongs to V. Then

$$\begin{aligned} X \setminus f^{-1}(V) &= f^{-1}(Y \setminus V) \\ &= g^{-1}(Y \setminus V). \end{aligned}$$

Since V is closed in Y with respect to $\mathcal{O}_Y$, we have that $Y \setminus V$ belongs to $\mathcal{O}_Y$. Hence $Y \setminus V$ belongs to $\mathcal{O}_{Y \setminus \{f(x)\}}$. Since g is continuous, we thus have that $g^{-1}(Y \setminus V)$ belongs to $\mathcal{O}_{X \setminus f^{-1}(\{f(x)\})}$. Deduce that $X \setminus f^{-1}(V)$ belongs to $\mathcal{O}_{X \setminus f^{-1}(\{f(x)\})}$.

(3) Since $f^{-1}(\{f(x)\})$ is closed in X with respect to $\mathcal{O}_X$, we have that $X \setminus f^{-1}(\{f(x)\})$ belongs to $\mathcal{O}_X$. By Task E2.3.3 (1) and (2), deduce that $X \setminus f^{-1}(V)$ belongs to $\mathcal{O}_X$. Thus we have that $f^{-1}(V)$ is closed in X with respect to $\mathcal{O}_X$.

(3) By Task **??**, conclude from (1) and (2) that f is continuous.

6. Tuesday 21st January

6.1. Quotient topologies

Notation 6.1.1. Let X be a set, and let $\sim$ be an equivalence relation on X. We denote by $X/\sim$ the set

$$\{[x] \mid x \in X\}$$

of equivalence classes of X with respect to $\sim$.

Notation 6.1.2. We denote by

$$X \xrightarrow{\pi} X/\sim$$

the map given by $x \mapsto [x]$.

Terminology 6.1.3. We refer to π as the *quotient map* with respect to $\sim$.

Definition 6.1.4. Let $(X, \mathcal{O}_X)$ be a topological space. and let $\sim$ be an equivalence relation on X. Let $\mathcal{O}_{X/\sim}$ denote the set of subsets U of $X/\sim$ such that $\pi^{-1}(U)$ belongs to $\mathcal{O}_X$.

Proposition 6.1.5. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. Then $(X/\sim, \mathcal{O}_{X/\sim})$ is a topological space.

Proof. We verify that each of the conditions of Definition 1.1.1 holds.

(1) We have that $\pi^{-1}(\emptyset) = \emptyset$. Since $\mathcal{O}_X$ is a topology on X, we have that $\emptyset$ belongs to $\mathcal{O}_X$. Thus $\emptyset$ belongs to $\mathcal{O}_{X/\sim}$.

(2) We have that $\pi^{-1}(X/\sim) = X$. Since $\mathcal{O}_X$ is a topology on X, we have that X belongs to $\mathcal{O}_X$. Thus X belongs to $\mathcal{O}_X$.

(3) Let $\{U_i\}$ be a set of (possibly infinitely many) subsets of $X/\sim$ which belong to $\mathcal{O}_{X/\sim}$. By definition of $\mathcal{O}_{X/\sim}$, we have that $\pi^{-1}(U_i)$ belongs to $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a topology on X, we deduce that $\bigcup_{i \in I} \pi^{-1}(U_i)$ belongs to $\mathcal{O}_X$. We have that

$$\pi^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} \pi^{-1}(U_i).$$

Thus $\pi^{-1}\left(\bigcup_{i \in I} U_i\right)$ belongs to $\mathcal{O}_X$. We conclude that $\bigcup_{i \in I} U_i$ belongs to $\mathcal{O}_{X/\sim}$.

(4) Let U_0 and U_1 be subsets of $X/\sim$ which belong to $\mathcal{O}_{X/\sim}$. By definition of $\mathcal{O}_{X/\sim}$, we have that $\pi^{-1}(U_0)$ and $\pi^{-1}(U_1)$ belong to $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a topology on X, we deduce that $\pi^{-1}(U_0) \cap \pi^{-1}(U_1)$ belongs to $\mathcal{O}_X$. We have that

$$\pi^{-1}(U_0 \cap U_1) = \pi^{-1}(U_0) \cap \pi^{-1}(U_1).$$

Thus $\pi^{-1}(U_0 \cap U_1)$ belongs to $\mathcal{O}_X$. We conclude that $U_0 \cap U_1$ belongs to $\mathcal{O}_{X/\sim}$.

□

Remark 6.1.6. The proof of Proposition 6.1.5 does not appeal to anything specific to $X/\sim$ or to π. It relies only upon properties of π^{-1} which hold for any map.

Remark 6.1.7. Although we chose not to, it is possible to define the subspace and product topologies in a similar way. To investigate this is the topic of Task E6.2.1 and Task E6.2.2.

Terminology 6.1.8. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. We refer to $\mathcal{O}_{X/\sim}$ as the *quotient topology* upon $X/\sim$.

Remark 6.1.9. Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. Let $X/\sim$ be equipped with the quotient topology $\mathcal{O}_{X/\sim}$. Then

$$X \xrightarrow{\pi} X/\sim$$

is continuous. This is immediate from the definition of $\mathcal{O}_{X/\sim}$.

Remark 6.1.10. This introduces us to a more conceptual way to understand the definition of a subspace topology and of a product topology. The subspace topology ensures exactly that an inclusion map is continuous. The product topology ensures exactly that the projection maps are continuous. This is a consequence of Task E6.2.1 and Task E6.2.2.

6.2. Finite example of a quotient topology

Example 6.2.1. Let $X = \{a, b, c\}$ be a set with three elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}.$$

Let $\sim$ be the equivalence relation on X generated by $a \sim c$. Then

$$X/\sim = \{a', b'\},$$

where $a' = [a] = [c]$ and $b' = [b]$. The map

$$X \xrightarrow{\pi} X/\sim$$

is given by $a \mapsto a'$, $b \mapsto b'$, and $c \mapsto a'$. In order to determine which subsets of $X/\sim$ belong to $\mathcal{O}_{X/\sim}$, we have to calculate their inverse images under π. We know from Proposition 6.1.5 that and $\emptyset$ and $X/\sim$ belong to $\mathcal{O}_{X/\sim}$. Thus only the following calculations remain.

(1) We have that $\pi^{-1}(\{a'\}) = \{a, c\}$. Since $\{a, c\}$ belongs to $\mathcal{O}_X$, we deduce that $\{a'\}$ belongs to $\mathcal{O}_{X/\sim}$.

(2) We have that $\pi^{-1}(\{b'\}) = \{b\}$. Since $\{b\}$ does not belong to $\mathcal{O}_X$, we deduce that $\{b'\}$ does not belong to $\mathcal{O}_{X/\sim}$.

We conclude that

$$\mathcal{O}_{X/\sim} = \left\{\emptyset, \{a'\}, X\right\}.$$

Remark 6.2.2. Throughout the course, we shall make use the notion of an equivalence relation generated by a relation. A formal discussion can be found in A.4. However, you can harmlessly ignore it!

The relations that we shall consider express all that is important about our equivalence relations: which elements are to be identified with which, when we pass to $X/\sim$. For instance, in Example 6.2.1, the relation $a \sim c$ expresses that a is to be identified with c when we pass to $X/\sim$, and that no other identifications are to be made.

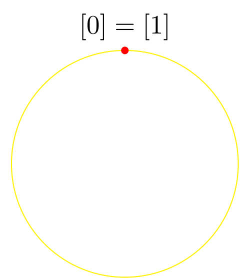
Formally, in order to construct $X/\sim$, we have to ensure that the conditions of Definition A.4.3 are satisfied. It is this that we achieve by passing to the equivalence relation generated by a relation. In full detail, the equivalence relation generated by $a \sim c$ is given by $a \sim a$, $b \sim b$, $c \sim c$, $a \sim c$, and $c \sim a$.

In all the examples which we shall consider, it is entirely straightforward to write down the equivalence relation generated by our relation. Since this would be tedious, and would not lend any insight into the corresponding quotient topology, we shall not do so.

6.3. The quotient topology obtained by glueing together the endpoints of I

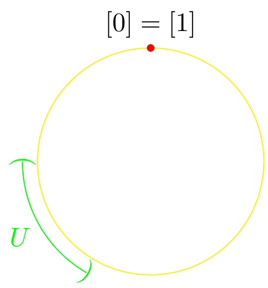
Example 6.3.1. Let $\sim$ be the equivalence relation on I generated by $0 \sim 1$.

Then $I/\sim$ is obtained by 'glueing 0 to 1'. We may picture it as follows.

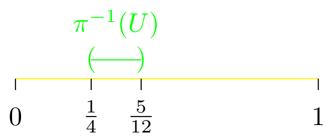


Remark 6.3.2. Let U be the subset of $I/{\sim}$ given by

$$\left\{[t] \mid \tfrac{1}{4} < t < \tfrac{5}{12}\right\}.$$



Then $\pi^{-1}(U)$ is the open interval $\left]\tfrac{1}{4}, \tfrac{5}{12}\right[$.

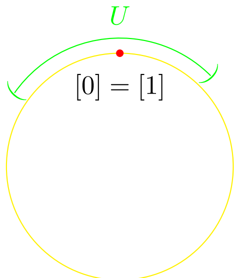


In particular, as in Example 2.3.3, we have that $\pi^{-1}(U)$ belongs to $\mathcal{O}_I$. Thus U belongs to $\mathcal{O}_{I/\sim}$.

Remark 6.3.3. Let U be the subset of $I/{\sim}$ given by

$$\left\{[t] \mid 0 \leq t < \tfrac{1}{8}\right\} \cup \left\{[t] \mid \tfrac{7}{8} < t \leq 1\right\}.$$

In particular, we have that $[0] = [1] \in U$.

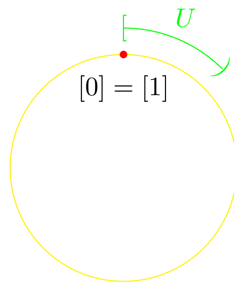


Then $\pi^{-1}(U)$ is $\left[0, \tfrac{1}{8}\right[\cup \left]\tfrac{7}{8}, 1\right]$.

As in Example 2.3.4, we have that $\left[0, \frac{1}{8}\right[$ belongs to $\mathcal{O}_I$. As in Example 2.3.5, we have that $\left]\frac{7}{8}, 1\right]$ belongs to $\mathcal{O}_I$. Thus $\pi^{-1}(U)$ belongs to $\mathcal{O}_I$. We conclude that U belongs to $\mathcal{O}_{I/\sim}$.

Remark 6.3.4. Let U be the subset of $I/\sim$ given by

$$\left\{[t] \mid \tfrac{7}{8} < t \leq 1\right\}.$$



Then $\pi^{-1}(U)$ is $\{0\} \cup \left]\frac{7}{8}, 1\right]$.

The subset $\{0\} \cup \left]\frac{7}{8}, 1\right]$ of I does not belong to $\mathcal{O}_I$. Thus U does not belong to $\mathcal{O}_{I/\sim}$.

Let $(X, \mathcal{O}_X)$ be a topological space, and let $\sim$ be an equivalence relation on X. Let U be a subset of X which belongs to $\mathcal{O}_X$. Then $\pi(U)$ does not necessarily belong to $\mathcal{O}_{X/\sim}$. The crucial point is that $\pi^{-1}(\pi(U))$ is not necessarily equal to U. Remark 6.3.4 demonstrates this, for we have the following.

(1) The subset U of $I/\sim$ considered in Remark 6.3.4 is $\pi\left(\left]\frac{7}{12}, 1\right]\right)$.

(2) As in Example 2.3.5, we have that $\left]\frac{7}{8}, 1\right]$ belongs to $\mathcal{O}_I$.

(3) We have that $\pi\left(\left]\frac{7}{12}, 1\right]\right)$ does not belong to $\mathcal{O}_{I/\sim}$. In particular

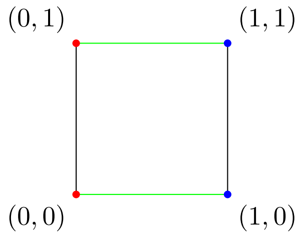
$$\pi^{-1}\left(\pi\left(\left]\tfrac{7}{12}, 1\right]\right)\right) = \{0\} \cup \left]\tfrac{7}{8}, 1\right],$$

which is not equal to $\left]\frac{7}{8}, 1\right]$.

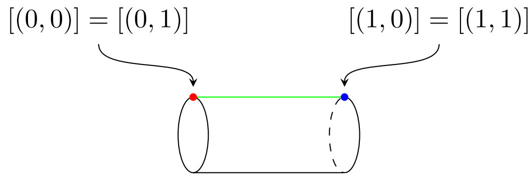
Remark 6.3.5. It is not a coincidence that we have depicted $I/\sim$ as a circle! In a sense which we shall define and investigate in the next lecture, $(I/\sim, \mathcal{O}_{I/\sim})$ is the 'same' topological space as $(S^1, \mathcal{O}_{S^1})$. To prove this is the topic of Task E7.3.10.

6.4. Further geometric examples of quotient topologies

Example 6.4.1. Let $\sim$ be the equivalence relation on I^2 generated by $(t,0) \sim (t,1)$, for all $t \in I$.

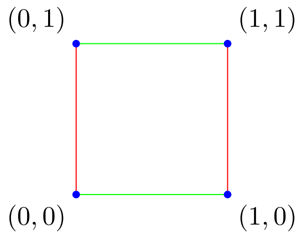


Then $I^2/\sim$ is obtained by 'glueing the upper horizontal edge of I^2 to the lower horizontal edge of I^2'. We may picture it as follows.

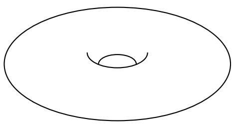


Remark 6.4.2. In the sense mentioned in Remark 6.3.5, $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is the 'same' topological space as the cylinder $(S^1 \times I, \mathcal{O}_{S^1 \times I})$.

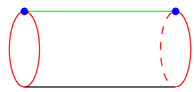
Example 6.4.3. Let $\sim$ be the equivalence relation on I^2 generated by $(s,0) \sim (s,1)$, for all $s \in I$, and by $(0,t) \sim (1,t)$ for all $t \in I$.



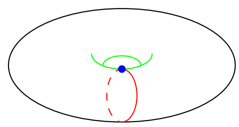
Then $I^2/\sim$ is obtained by 'glueing together the two horizontal edges of I^2', and moreover 'glueing together the two vertical edges of I^2'. We may picture $I^2/\sim$ as follows.



We can, for instance, first glue together the horizontal edges of I^2 as in Example 6.4.1, to obtain a cylinder.



We then glue the two circles at the end of the cylinder together.

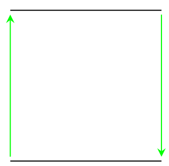


Remark 6.4.4. We can think of $I^2/{\sim}$ as a 'hollow doughnut'.

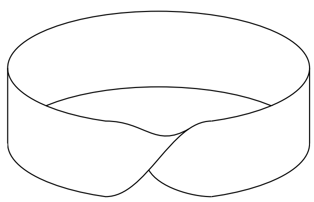
Terminology 6.4.5. We refer to $(I^2/{\sim}, \mathcal{O}_{I^2/\sim})$ as the *torus*.

Notation 6.4.6. We denote $(I^2/{\sim}, \mathcal{O}_{I^2/\sim})$ by $(T^2, \mathcal{O}_{T^2})$.

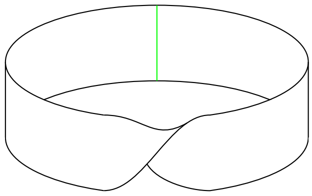
Example 6.4.7. Let $\sim$ be the equivalence relation on I^2 generated by $(0,t) \sim (1, 1-t)$, for all $t \in I$.



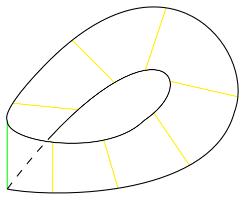
Then $I^2/{\sim}$ is obtained by 'glueing together the two horizontal edges of I^2 with a twist', so that the arrows in the figure above point in the same direction. We may picture $I^2/\sim$ as follows.



The glued vertical edges of I^2 can be thought of as a line in $I^2/{\sim}$, depicted below.



We can also picture $I^2/\!\sim$ as follows, from a different angle.

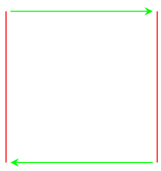


Terminology 6.4.8. We refer to $(I^2/\!\sim, \mathcal{O}_{I^2/\sim})$ as the *Möbius band*.

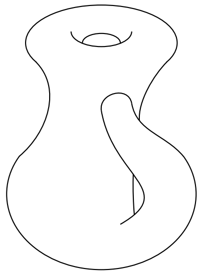
Notation 6.4.9. We denote $(I^2/\!\sim, \mathcal{O}_{I^2/\sim})$ by $(M^2, \mathcal{O}_{M^2})$.

Remark 6.4.10. If you find it difficult at first to visualise the glueing of M^2 from I^2, it is a very good idea to try it with a piece of ribbon or paper!

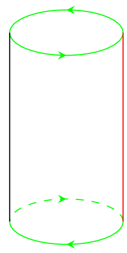
Example 6.4.11. Let $\sim$ be the equivalence relation on I^2 generated by $(s,0) \sim (1-s,1)$, for all $s \in I$, and by $(0,t) \sim (1,t)$, for all $t \in I$.



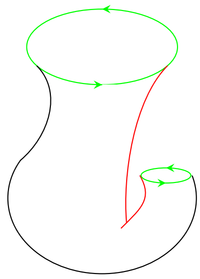
Then $I^2/\!\sim$ is obtained by 'glueing together the two vertical edges of I^2', and moreover 'glueing together the two horizontal edges of I^2 with a twist', so that the arrows point in the same direction. We cannot truly picture $I^2/\!\sim$ in $\mathbb{R}^3$. Nevertheless we can gain an intuitive feeling for it, through the following picture.



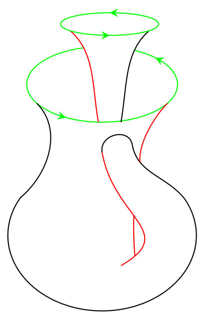
We can, for instance, first glue together the vertical edges, to obtain a cylinder.



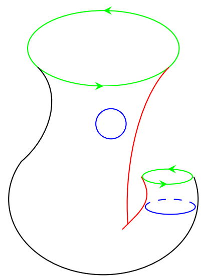
We can then bend this cylinder so that the arrows on the circles at its ends point in the same direction.



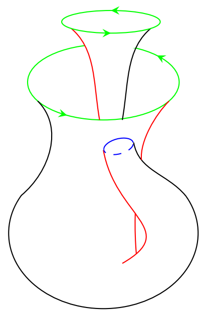
Next we can push the cylinder through itself.



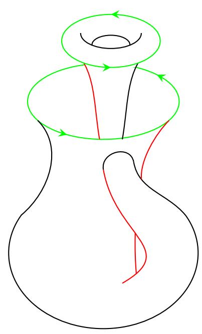
It is this step that is not possible in a true picture of $I^2/\sim$. It can be thought of glueing together two circles: a cross-section of the part of the cylinder which we have bent upwards, and a circle on the side of the cylinder which we have not bent upwards.



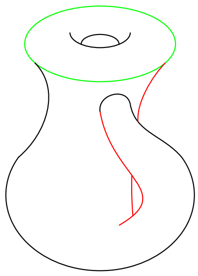
The equivalence relation $\sim$ does not prescribe that these two circles should be glued. We shall nevertheless proceed. The circle obtained after glueing the two circles together is pictured below.



Next we can fold back the end of the cylinder which we have pushed through. We obtain a 'mushroom with a hollow stalk'.



Finally we can glue the ends of the cylinder together, as prescribed by $\sim$.



Terminology 6.4.12. We refer to $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ as the *Klein bottle*.

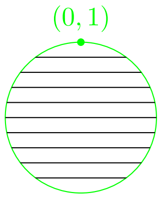
Notation 6.4.13. We denote $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ by $(K^2, \mathcal{O}_{K^2})$.

Remark 6.4.14. A rite of passage when learning about topology for the first time is to be confronted with the following limerick. I'm sure that I remember Colin Rourke enunciating it during the lecture in which I first met the Klein bottle, as an undergraduate at the University of Warwick!

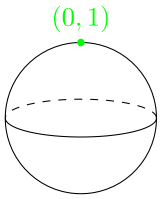
A mathematician named Klein
Thought the Möbius band was divine.
Said he: "If you glue
The edges of two,
You'll get a weird bottle like mine!"

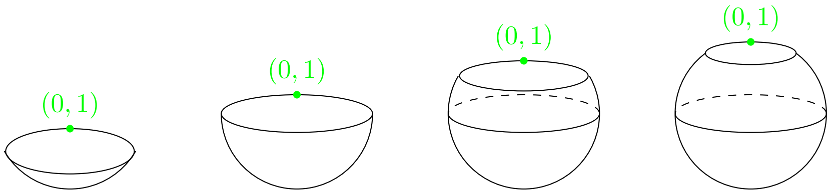
To investigate its meaning is the topic of Task **??**.

Example 6.4.15. Let $\sim$ be the equivalence relation on D^2 generated by $(x, y) \sim (0, 1)$ for all $(x, y) \in S^1$.



We obtain $D^2/\sim$ by 'contracting the boundary of D^2 to the point $(0, 1)$'. Imagine, for instance, that the boundary circle of D^2 is a loop of fishing line. Suppose that we have a reel at the point $(0, 1)$. Then $D^2/\sim$ is obtained by reeling in tight all of our fishing line. We obtain a 'hollow ball'.





Remark 6.4.16. We could have chosen any single point on S^1, instead of $(0, 1)$, in the definition of $\sim$.

Terminology 6.4.17. We refer to $(D^2/\sim, \mathcal{O}_{D^2/\sim})$ as the *2-sphere*.

Notation 6.4.18. We denote $(D^2/\sim, \mathcal{O}_{D^2/\sim})$ by $(S^2, \mathcal{O}_{S^2})$.

Remark 6.4.19. In the sense mentioned in Remark 6.3.5, $(S^2, \mathcal{O}_{S^2})$ is the 'same' topological space as the set

$$\{x \in \mathbb{R}^3 \mid \|x\| = 1\}$$

equipped with the subspace topology with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$.

E6. Exercises for Lecture 6

E6.1. Exam questions

Task E6.1.1. Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{e\}, \{a, e\}, \{c, d\}, \{c, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, X\}.$$

Let $\sim$ be the equivalence relation on X generated by $b \sim d$ and $c \sim e$. List the subsets of $X/\sim$ which belong to $\mathcal{O}_{X/\sim}$.

Task E6.1.2. Let $X = \{a, b\}$ be a set with two elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, X\}.$$

Let $Y = \{a', b', c', d', , e'\}$ be a set with five elements. Let $\mathcal{O}_Y$ be the topology on Y given by

$$\{\emptyset, \{a'\}, \{b', c'\}, \{a', b', c'\}, \{b', c', e'\}, \{a', b', c', e'\}, Y\}.$$

Let $\sim$ be the equivalence relation on Y generated by $b' \sim c'$ and $c' \sim e'$. Let $X \times X$ be equipped with the product topology $\mathcal{O}_{X \times X}$, and let $Y/\sim$ be equipped with the quotient topology $\mathcal{O}_{Y/\sim}$. Which of the following maps

$$X \times X \longrightarrow Y/\sim$$

are continuous?

(1) $(a, a) \mapsto [a']$, $(a, b) \mapsto [b']$, $(b, a) \mapsto [b']$, $(b, b) \mapsto [d']$

(2) $(a, a) \mapsto [b']$, $(a, b) \mapsto [b']$, $(b, a) \mapsto [d']$, $(b, b) \mapsto [d']$

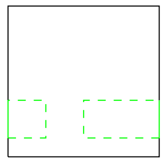
(3) $(a, a) \mapsto [b']$, $(a, b) \mapsto [b']$, $(b, a) \mapsto [a']$, $(b, b) \mapsto [d']$

(4) $(a, a) \mapsto [b']$, $(a, b) \mapsto [a']$, $(b, a) \mapsto [a']$, $(b, b) \mapsto [a']$

(5) $(a, a) \mapsto [a']$, $(a, b) \mapsto [d']$, $(b, a) \mapsto [a']$, $(b, b) \mapsto [d']$

Task E6.1.3. Let U be the subset of I^2 given by

$$\left([0, \tfrac{1}{4}[\times]\tfrac{1}{8}, \tfrac{3}{8}[\right) \cup \left(]\tfrac{1}{2}, 1] \times]\tfrac{1}{8}, \tfrac{3}{8}[\right).$$



For which of the following choices of $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ does $\pi(U)$ belong to $\mathcal{O}_{I^2/\sim}$?

(1) The torus.

(2) The Möbius band.

(3) The Klein bottle.

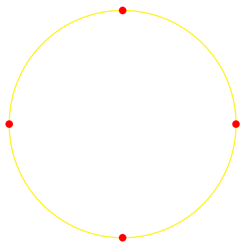
(4) The cylinder.

Task E6.1.4. Find a subset U of I^2 with the following properties.

(1) We have that $\pi(U)$ belongs to $\mathcal{O}_{I^2/\sim}$ both when $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is the Klein bottle, and when $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ is the Möbius band.

(2) It is not a subset of $]0,1[\times]0,1[$.

Task E6.1.5. Let $\sim$ be the equivalence relation on S^1 generated by $(1,0) \sim (0,1) \sim (-1,0) \sim (-1,-1)$.

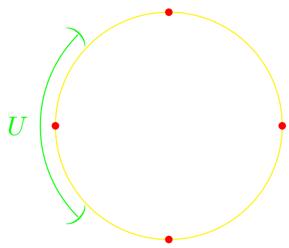


This task has two parts.

(1) Draw a picture of $S^1/\sim$. Indicate any important aspects.

(2) Let U be the 'open arc' given by

$$\left\{ (x,y) \in S^1 \mid -1 \leq x < -\tfrac{1}{2} \right\}.$$



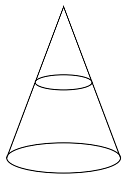
Does $\pi(U)$ belong to $\mathcal{O}_{S^1/\sim}$?

Task E6.1.6. Find an equivalence relation $\sim$ on D^2 with the following properties.

(1) We can picture $D^2/\sim$ as a 'hollow ball'.

(2) No three distinct elements of D^2 are identified by $\sim$.

Task E6.1.7. Find a subset X of $\mathbb{R}^2$, and an equivalence relation $\sim$ on X, such that $X/\sim$ can be pictured as a 'hollow cone'.



Let X be equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Give an example of a subset U of $X/\sim$ such that $\pi^{-1}(U)$ is the disjoint union of a pair of subsets U_0 and U_1 of X which belong to $\mathcal{O}_X$.

Task E6.1.8. Let $X = I^2 \cup ([3,4] \times [0,1])$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Define an equivalence relation $\sim$ on X such that $(X/\sim, \mathcal{O}_{X/\sim})$ can be thought of as two tori placed side by side.

E6.2. For a deeper understanding

Task E6.2.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Let

$$A \xrightarrow{i} X$$

denote the inclusion map. Let $\mathcal{O}'_A$ denote the set

$$\left\{ i^{-1}(U) \mid U \in \mathcal{O}_X \right\}.$$

Prove that $\mathcal{O}_A = \mathcal{O}'_A$.

Task E6.2.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\mathcal{O}_{X\times Y}$ denote the product topology on $X \times Y$ with respect to $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. Let

$$X \times Y \xrightarrow{p_1} X$$

and

$$X \times Y \xrightarrow{p_2} Y$$

denote the projection maps. Let $\mathcal{O}'_{X\times Y}$ denote the set

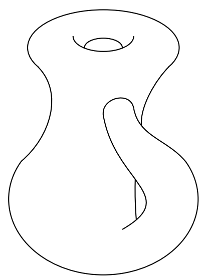
$$\left\{ p_1^{-1}(U) \mid U \in \mathcal{O}_X \right\} \cup \left\{ p_2^{-1}(U) \mid U \in \mathcal{O}_Y \right\}.$$

Prove that $\mathcal{O}'_{X\times Y}$ is a subbasis for $(X \times Y, \mathcal{O}_{X\times Y})$.

Remark E6.2.3. In other words, $\mathcal{O}_{X\times Y}$ is the smallest possible topology on $X \times Y$ for which p_1 and p_2 are continuous.

Task E6.2.4. In the notation of Task E6.2.2, find an example to prove that $\mathcal{O}'_{X\times Y}$ is not a basis for $(X \times Y, \mathcal{O}_{X\times Y})$.

Task E6.2.5. Find an equivalence relation $\sim$ on I^2 such that $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ can truly, unlike the Klein bottle, be pictured as follows.



Terminology E6.2.6. Let X and Y be sets. Let $\sim$ be an equivalence relation upon X. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Then f *respects* $\sim$ if, for all $x, x' \in X$ such that $x \sim x'$, we have that $f(x) = f(x')$.

Task E6.2.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\sim$ be an equivalence relation on X, and let $X/\sim$ be equipped with the quotient topology with respect to $(X, \mathcal{O}_X)$. Let

$$X \xrightarrow{f} Y$$

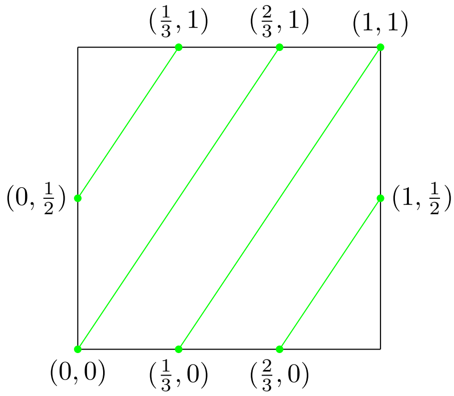
be a continuous map such that f respects $\sim$. Let

$$X/\sim \xrightarrow{g} Y$$

be the map given by $[x] \mapsto f(x)$, which is well defined since f respects $\sim$. Prove that g is continuous.

E6.3. Exploration — torus knots

Task E6.3.1. Let K denote the subset of I^2 pictured below.



In words: begin at $(0, 0)$, and follow a line of gradient $\frac{2}{3}$ until we hit a side of I^2; Jump over to the other side, and repeat this process. Eventually we end up at $(1, 1)$. Let

$$I^2 \xrightarrow{\pi} T^2$$

be the quotient map. Can you visualise or, even better, draw $\pi(K)$?

Remark E6.3.2. If you can draw $\pi(K)$, I would love to see it!

Remark E6.3.3. Later in the course, we shall investigate knots and links. As an apéritif, $\pi(K)$ is a gadget known as the *trefoil knot*, but wrapped around a torus!

Terminology E6.3.4. A knot which can be wrapped around a torus is known as a *torus knot*. Any pair of integers p and q whose greatest common divisor is 1 gives rise to a torus knot in a similar way, working with lines of gradient $\frac{p}{q}$ in place of $\frac{2}{3}$ above. For any pair of integers p and q, one obtains a link wrapped around a torus.

7. Monday 27th January

7.1. Homeomorphisms

Definition 7.1.1. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is *bijective* if there is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Remark 7.1.2. Here id_X and id_Y denote the respective identity maps, in the terminology of E5.1.2.

Notation 7.1.3. Let X and Y be sets, and let

$$X \xrightarrow{f} Y$$

be a bijective map. We often denote the corresponding map

$$Y \xrightarrow{g} X$$

by f^{-1}.

Remark 7.1.4. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is bijective in the sense of Definition 7.1.1 if and only if f is both injective and surjective. To prove this is Task E7.2.1.

Definition 7.1.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is a *homeomorphism* if the following hold.

(1) We have that f is continuous,

(2) There is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

Remark 7.1.6. An equivalent definition of a homeomorphism is the topic of Task E7.3.1.

Definition 7.1.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X, \mathcal{O}_X)$ is *homeomorphic* to $(Y, \mathcal{O}_Y)$ if there exists a homeomorphism

$$X \longrightarrow Y.$$

Remark 7.1.8. By Task E7.3.2, we have that $(X, \mathcal{O}_X)$ is homeomorphic to $(Y, \mathcal{O}_Y)$ if and only if there exists a homeomorphism

$$Y \longrightarrow X.$$

7.2. Examples of homeomorphisms between finite topological spaces

Example 7.2.1. Let $X = \{a, b, c\}$ be a set with three elements. Let

$$X \xrightarrow{f} X$$

be the bijective map given by $a \mapsto b$, $b \mapsto c$, and $c \mapsto a$. Let $\mathcal{O}_0$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b, c\}, X\}.$$

Let $\mathcal{O}_1$ be the topology on X given by

$$\{\emptyset, \{a, c\}, \{b\}, X\}.$$

Let us regard the copy of X in the source of f as equipped with the topology $\mathcal{O}_0$, and regard the copy of X in the target of f as equipped with the topology $\mathcal{O}_1$. We have the following.

$$\begin{aligned} f^{-1}(\emptyset) &= \emptyset \\ f^{-1}(\{a, c\}) &= \{b, c\} \\ f^{-1}(\{b\}) &= \{a\} \\ f^{-1}(X) &= X. \end{aligned}$$

Thus f is continuous. Let

$$X \xrightarrow{g} X$$

be the inverse to f, given by $a \mapsto c$, $b \mapsto a$, and $c \mapsto b$. We have the following.

$$\begin{aligned} g^{-1}(\emptyset) &= \emptyset \\ g^{-1}(\{a\}) &= \{b\} \\ g^{-1}(\{b,c\}) &= \{a,c\} \\ g^{-1}(X) &= X. \end{aligned}$$

Thus g is continuous. We conclude that f is a homeomorphism. In other words, we have that $(X, \mathcal{O}_0)$ and $(X, \mathcal{O}_1)$ are homeomorphic.

Example 7.2.2. Let X be as in Example 7.2.1. Let $\mathcal{O}_2$ be the topology on X given by

$$\{\emptyset, \{a,b\}, \{c\}, X\}.$$

Let f be as in Example 7.2.1. Let us again regard the copy of X in the source of f as equipped with the topology $\mathcal{O}_0$, but let us now regard the copy of X in the target of f as equipped with the topology $\mathcal{O}_2$. Then f is not continuous, since $f^{-1}(\{c\}) = \{b\}$, and $\{b\}$ does not belong to $\mathcal{O}_0$. Thus f is not a homeomorphism.

Remark 7.2.3. Let

$$X \xrightarrow{h} X$$

be the bijective map given by $a \mapsto c$, $b \mapsto b$, and $c \mapsto a$. We have the following.

$$\begin{aligned} h^{-1}(\emptyset) &= \emptyset \\ h^{-1}(\{a,b\}) &= \{b,c\} \\ h^{-1}(\{c\}) &= \{a\} \\ h^{-1}(X) &= X. \end{aligned}$$

Thus h is continuous. Moreover we have that h is its own inverse. We conclude that h is a homeomorphism. In other words, we have that $(X, \mathcal{O}_0)$ and $(X, \mathcal{O}_2)$ are homeomorphic.

Example 7.2.2 and Remark 7.2.3 demonstrate that a pair of topological spaces can be homeomorphic, even though a particular map that we consider might not be a homeomorphism. It is very important to remember this!

Example 7.2.4. Let X be as in Example 7.2.1. Let $\mathcal{O}_3$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\}.$$

Let f be as in Example 7.2.1. Let us regard the copy of X in the source of f as equipped with the topology $\mathcal{O}_3$, and regard the copy of X in the target of f as equipped with the topology $\mathcal{O}_1$. Since $\mathcal{O}_0$ is a subset of $\mathcal{O}_3$, the calculation of Example 7.2.1 demonstrates that f is continuous. The inverse of f is the map g of Example 7.2.1. We have that $g^{-1}(\{b\}) = \{c\}$, and $\{c\}$ does not belong to $\mathcal{O}_1$. Thus g is not continuous. We conclude that f is not a homeomorphism.

Remark 7.2.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Then $\mathcal{O}_X$ and $\mathcal{O}_Y$ must have the same cardinality. To prove this is Task E7.3.3. Thus $(X, \mathcal{O}_3)$ is not homeomorphic to $(X, \mathcal{O}_1)$.

Example 7.2.4 and Remark 7.2.5 demonstrate that there can be a continuous bijective map from one topological space to another, and yet these topological spaces might not be homeomorphic. It is very important to remember this! This phenomenon does not occur in group theory or linear algebra, for instance.

7.3. Geometric examples of homeomorphisms

Remark 7.3.1. Two geometric examples of topological spaces are, intuitively, homeomorphic if we can bend, stretch, compress, twist, or otherwise 'manipulate in a continuous manner', one to obtain the other. We can sharpen or smooth edges. We cannot cut or tear.

Remark 7.3.2. It may help you to think of geometric examples of topological spaces as made of dough, or of clay that has not yet been fired!

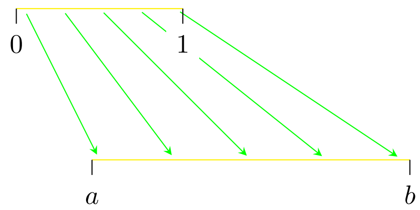
Example 7.3.3. Suppose that $a, b \in \mathbb{R}$, and that $a < b$. Let the open interval $]a, b[$ be equipped with the subspace topology $\mathcal{O}_{]a,b[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let the open interval $]0, 1[$ also be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Then $\left(]a, b[\,, \mathcal{O}_{]a,b[}\right)$ is homeomorphic to $\left(]0, 1[\,, \mathcal{O}_{]0,1[}\right)$. Intuitively we can stretch or shrink, and translate, $]0, 1[$ to obtain $]a, b[$. To be rigorous, the map

$$]0,1[\xrightarrow{f}]a,b[$$

given by $t \mapsto a(1-t)+bt$ is a homeomorphism.



For the following hold.

(1) By Task E5.3.14, we have that f is continuous.

(2) Let

$$]a,b[\xrightarrow{g}]0,1[$$

be the map given by $t \mapsto \frac{t-a}{b-a}$. By Task E5.3.14, we have that g is continuous. Moreover we have the following, for every $t \in]0,1[$.

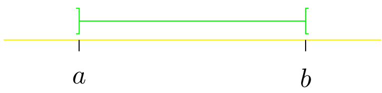
$$\begin{aligned} g\left(f(t)\right) &= g\left(a(1-t)+bt\right) \\ &= \frac{a(1-t)+bt-a}{b-a} \\ &= \frac{t(b-a)}{b-a} \\ &= t. \end{aligned}$$

Thus we have that $g \circ f = id_{]0,1[}$. We also have the following, for every $t \in]a,b[$.

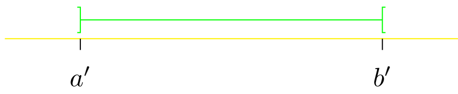
$$\begin{aligned} f\left(g(t)\right) &= f\left(\frac{t-a}{b-a}\right) \\ &= a\left(1-\frac{t-a}{b-a}\right) + b\left(\frac{t-a}{b-a}\right) \\ &= \frac{a\left(b-a\right)-a\left(t-a\right)+b\left(t-a\right)}{b-a} \\ &= \frac{t\left(b-a\right)}{b-a} \\ &= t. \end{aligned}$$

Thus we have that $f \circ g = id_{]a,b[}$.

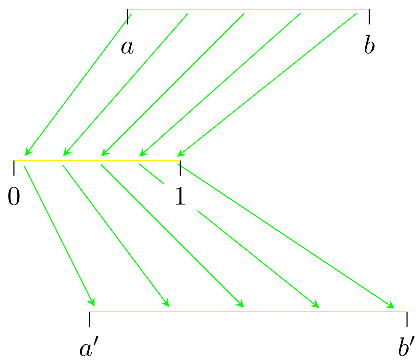
Example 7.3.4. Suppose that $a, b \in \mathbb{R}$, and that $a < b$. Suppose also that $a', b' \in \mathbb{R}$, and that $a' < b'$. Let $]a, b[$ be equipped with the subspace topology $\mathcal{O}_{]a,b[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Let $]a', b'[$ be equipped with the subspace topology $\mathcal{O}_{]a',b'[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



By Example 7.3.3 and Remark E7.1.11, we have that $\left(]a, b[\,, \mathcal{O}_{]a,b[}\right)$ is homeomorphic to $\left(]a', b'[\,, \mathcal{O}_{]a',b'[}\right)$. In other words, we use the homeomorphism of Example 7.3.3 to construct a homeomorphism from $(]a, b[\,, \mathcal{O}_{]a,b[})$ to $(]a', b'[\,, \mathcal{O}_{]a',b'[})$ in two steps.

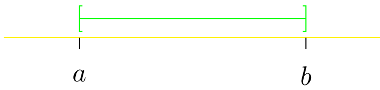


Remark 7.3.5. The technique of Example 7.3.4 and Example 7.3.4 is a good one to keep in mind when trying to prove that a pair of topological spaces are homeomorphic.

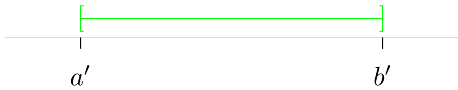
(1) Look for an intermediate special case, which in this case is where one of the topological spaces is $\left(]0, 1[\,, \mathcal{O}_{]0,1[}\right)$, for which we can explicitly write down a homeomorphism without too much difficulty.

(2) Apply a 'machine', which in this case is the fact that we can compose and invert homeomorphisms, to achieve our original goal.

Example 7.3.6. Suppose that $a, b \in \mathbb{R}$, and that $a < b$. Suppose also that $a', b' \in \mathbb{R}$, and that $a' < b'$. Let $[a, b]$ be equipped with the subspace topology $\mathcal{O}_{[a,b]}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

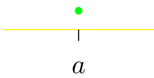


Let $[a', b']$ be equipped with the subspace topology $\mathcal{O}_{[a',b']}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

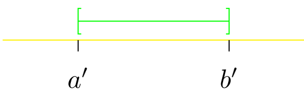


Then $\left([a,b], \mathcal{O}_{[a,b]}\right)$ is homeomorphic to $\left([a',b'], \mathcal{O}_{[a',b']}\right)$. Intuitively, we can stretch or shrink, and translate, $[a,b]$ to obtain $[a,b]$. To be rigorous, we can argue in exactly the same way as in Example 7.3.4 and Example 7.3.4, with the unit interval $(I, \mathcal{O}_I)$ as the intermediate special case.

Remark 7.3.7. The assumption that $a < b$ and $a' < b'$ is crucial in Example 7.3.6. Let $a \in \mathbb{R}$. Let $\{a\} = [a, a]$ be equipped with the subspace topology $\mathcal{O}_{\{a\}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Suppose that $a', b' \in \mathbb{R}$, and that $a' < b'$.



We have the following.

(1) A homeomorphism is in particular a bijection, as observed in Task E7.3.1.

(2) There is no bijective map

$$\{a\} \longrightarrow [a', b'].$$

To check that you understand this is Task E7.2.2.

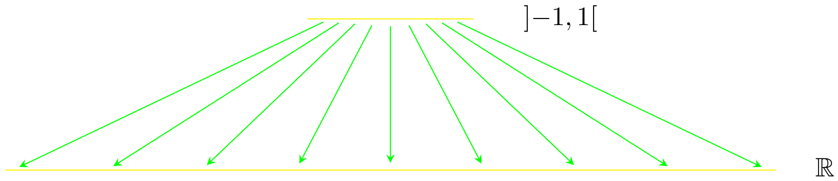
We conclude that $\left(\{a\}, \mathcal{O}_{\{a\}}\right)$ is not homeomorphic to $\left([a',b'], \mathcal{O}_{[a',b']}\right)$. Can you see where the argument of Example 7.3.4 breaks down? This is Task E7.2.3.

Remark 7.3.8. In a nutshell, we can shrink a closed interval to a closed interval which has as small a strictly positive length as we wish, but not to a point.

Example 7.3.9. Let the open interval $]-1,1[$ be equipped with the subspace topology $\mathcal{O}_{]-1,1[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Then $\left(]-1,1[\,, \mathcal{O}_{]-1,1[}\right)$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Intuitively, think a cylindrical piece of dough. The dough can be worked in such a way that the cylinder becomes a longer and longer piece of spaghetti. We can think of open intervals in topology in a similar way!

With dough, our piece of spaghetti would eventually snap, but the mathematical dough of which an open interval is made can be stretched as much as we like, to the end of time! If we 'wait long enough', our mathematical piece of spaghetti will be longer than the distance between any pair of real numbers! A way to visualise this is depicted below.



To be rigorous, the map

$$]-1,1[\xrightarrow{f} \mathbb{R}$$

given by $t \mapsto \frac{t}{1-|t|}$ is a homeomorphism. For the following hold.

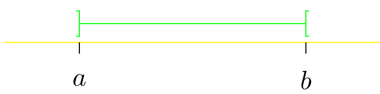
(1) We have that f is continuous. To prove this is the topic of Task E7.2.4.

(2) Let

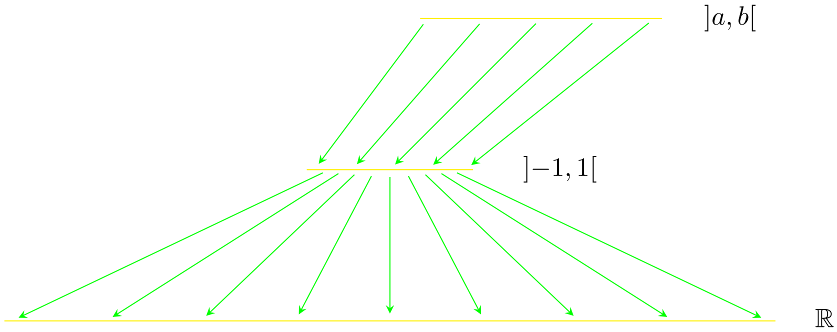
$$\mathbb{R} \xrightarrow{g}]-1,1[$$

be the map given by $t \mapsto \frac{t}{1+|t|}$. We have that g is continuous. To prove this is the topic of Task E7.2.5. Moreover we have that $g(f(t)) = t$ for all $t \in]-1,1[$, so that $g \circ f = id_{]-1,1[}$. In addition we have that $f(g(t)) = t$ for all $t \in \mathbb{R}$, so that $f \circ g = id_{\mathbb{R}}$. To prove the last two statements is the topic of Task E7.2.6.

Example 7.3.10. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{]a,b[}$ denote the subspace topology on $]a,b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

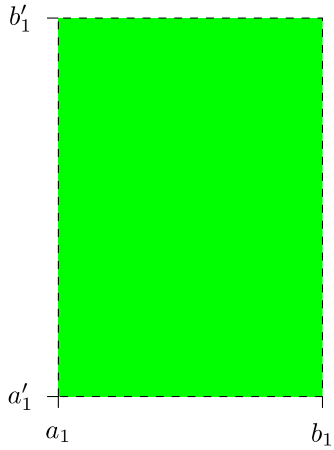


By Example 7.3.4, Example 7.3.9 and Remark E7.1.11, we have that $(]a,b[\,, \mathcal{O}_{]a,b[})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Following the technique described in Remark 7.3.5, we use the homeomorphisms of Example 7.3.4 and Example 7.3.9 to construct a homeomorphism from $(]a,b[\,, \mathcal{O}_{]a,b[})$ to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ in two steps.



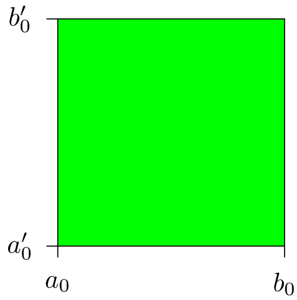
Example 7.3.11. Let $a_0, b_0, a_0', b_0' \in \mathbb{R}$ be such that $a_0 < b_0$ and $a_0' < b_0'$. Let X_0 be the 'open rectangle' given by $]a_0, b_0[\times]a_0', b_0'[$, equipped with the subspace topology $\mathcal{O}_{X_0}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $a_1, b_1, a_1', b_1' \in \mathbb{R}$ be such that $a_1 < b_1$ and $a_1' < b_1'$. Let X_1 be the 'open rectangle' given by $]a_1, b_1[\times]a_1', b_1'[$, equipped with the subspace topology $\mathcal{O}_{X_1}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

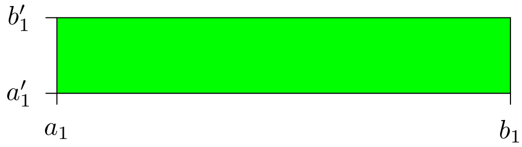


By Example 7.3.4, we have that $\left(\left]a_0, b_0\right[, \mathcal{O}_{]a_0,b_0[}\right)$ is homeomorphic to $\left(\left]a_1, b_1\right[, \mathcal{O}_{]a_1,b_1[}\right)$, and that $\left(\left]a_0', b_0'\right[, \mathcal{O}_{]a_0',b_0'[}\right)$ is homeomorphic to $\left(\left]a_1', b_1'\right[, \mathcal{O}_{]a_1',b_1'[}\right)$. By Task E7.1.14, we deduce that $(X_0, \mathcal{O}_{X_0})$ is homeomorphic to $(X_1, \mathcal{O}_{X_1})$.

Example 7.3.12. Let $a_0, b_0, a_0', b_0' \in \mathbb{R}$ be such that $a_0 < b_0$ and $a_0' < b_0'$. Let X_0 be the 'closed rectangle' given by $[a_0, b_0] \times [a_0', b_0']$, equipped with the subspace topology $\mathcal{O}_{X_0}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

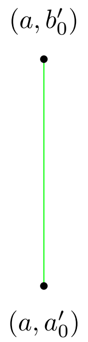


Let $a_1, b_1, a_1', b_1' \in \mathbb{R}$ be such that $a_1 < b_1$ and $a_1' < b_1'$. Let X_1 be the 'closed rectangle' given by $[a_1, b_1] \times [a_1', b_1']$, equipped with the subspace topology $\mathcal{O}_{X_1}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

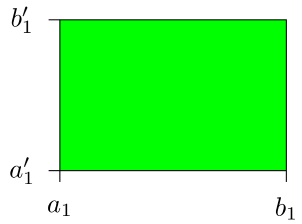


By Example 7.3.6, we have that $\left(\left[a_0, b_0\right], \mathcal{O}_{[a_0,b_0]}\right)$ is homeomorphic to $\left(\left[a_1, b_1\right], \mathcal{O}_{[a_1,b_1]}\right)$, and that $\left(\left[a_0', b_0'\right], \mathcal{O}_{[a_0',b_0']}\right)$ is homeomorphic to $\left(\left[a_1', b_1'\right], \mathcal{O}_{]a_1',b_1'[}\right)$. By Task E7.1.14, we deduce that $(X_0, \mathcal{O}_{X_0})$ is homeomorphic to $(X_1, \mathcal{O}_{X_1})$.

Remark 7.3.13. As in Remark 7.3.7, it is crucial in Example 7.3.12 that the inequalities are strict. For instance, let $a, a'_0, b'_0 \in \mathbb{R}$ be such that $a'_0 < b'_0$. Let X_0 be the line $\{a\} \times [a'_0, b'_0]$, equipped with the subspace topology $\mathcal{O}_{X_0}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

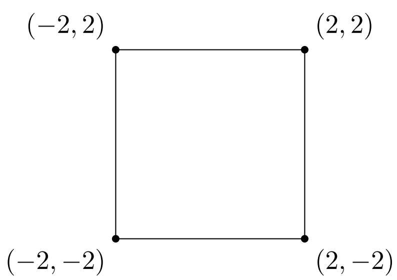


Let $a_1, b_1, a'_1, b'_1 \in \mathbb{R}$ be such that $a_1 < b_1$ and $a'_1 < b'_1$. Let X_1 be the 'closed rectangle' given by $[a_1, b_1] \times [a'_1, b'_1]$, equipped with the subspace topology $\mathcal{O}_{X_1}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Then $(X_0, \mathcal{O}_{X_0})$ is not homeomorphic to $(X_1, \mathcal{O}_{X_1})$. We cannot prove this yet, but we shall be able to soon, after we have studied *connectedness*.

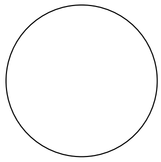
Example 7.3.14. Let X be the square depicted below, consisting of just the four lines, with no 'inside'.



In other words, X is given by

$$(\{-2, 2\} \times [-2, 2]) \cup ([-2, 2] \times \{-2, 2\}).$$

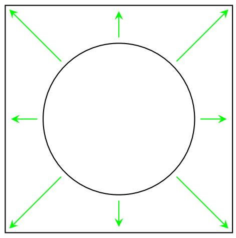
Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Then $(X, \mathcal{O}_X)$ is homeomorphic to the circle $(S^1, \mathcal{O}_{S^1})$.



A way to construct a homeomorphism

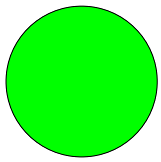
$$S^1 \xrightarrow{f} X$$

is to send each $x \in S^1$ to the unique $y \in X$ such that $y = kx$, where $k \in \mathbb{R}$ has the property that $k \geq 0$. To rigorously write down the details is the topic of Task E7.2.7.

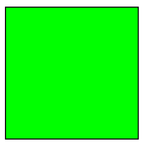


Remark 7.3.15. Think of a circular piece of string on a table. Even without stretching it, you could manipulate it so that it becomes a square!

Example 7.3.16. A similar argument to that of Example 7.3.14 demonstrates that the unit disc $(D^2, \mathcal{O}_{D^2})$



is homeomorphic to the unit square $(I^2, \mathcal{O}_{I^2})$.



To prove this is the topic of Task E7.2.9.

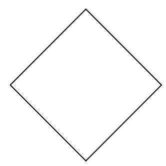
Example 7.3.17. Let Y denote the union of the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 0 \text{ and } |y| = 1 + x\right\}$$

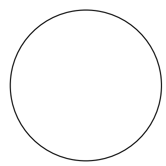
and the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } |y| = 1 - x\right\}.$$

Let $\mathcal{O}_Y$ denote the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



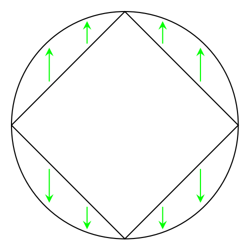
Then $(Y, \mathcal{O}_Y)$ is homeomorphic to the circle $(S^1, \mathcal{O}_{S^1})$.



A way to construct a homeomorphism

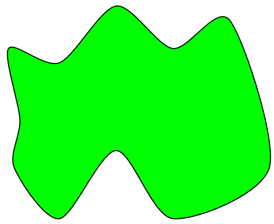
$$Y \xrightarrow{f} S^1$$

is to send each $(x, y_0) \in Y$ to the unique $(x, y_1) \in S^1$ such that $y_1 = ky_0$, where $k \in \mathbb{R}$ has the property that $k \geq 0$. To rigorously write down the details is the topic of Task **??**.



Remark 7.3.18. By Remark E7.1.11, we have that the topological space $(X, \mathcal{O}_X)$ of Example 7.3.14 is homeomorphic to the topological space $(Y, \mathcal{O}_Y)$ of Example 7.3.17, since both are homeomorphic to $(S^1, \mathcal{O}_{S^1})$. To prove this in a different way is the topic of Task 7.3.17.

Example 7.3.19. Let X be a 'blob' in $\mathbb{R}^2$, equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Then $(X, \mathcal{O}_X)$ is homeomorphic to the unit square $(I^2, \mathcal{O}_{I^2})$. If X were made of dough, it would be possible to knead it to obtain a square! To rigorously prove that $(X, \mathcal{O}_X)$ is homeomorphic to $(I^2, \mathcal{O}_{I^2})$ is the topic of Task **??**.

Remark 7.3.20. In Task **??**, we shall not explicitly describe a subset of $\mathbb{R}^2$ such as the 'blob' above. We shall work a little more abstractly, with subsets of $\mathbb{R}^2$ which can be 'cut into star shaped pieces'. Here 'star shaped' has a technical meaning, discussed before Task **??**.

E7. Exercises for Lecture 7

E7.1. Exam questions

Task E7.1.1. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

Let $Y = \{1, 2, 3, 4\}$ be a set with four elements. Let $\mathcal{O}_Z$ be the topology on Z given by

$$\{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{1, 3, 4\}, Z\}.$$

Let

$$X \xrightarrow{f} Y$$

be the map given by $a \mapsto 3$, $b \mapsto 1$, $c \mapsto 2$, and $d \mapsto 4$. Is f a homeomorphism?

Task E7.1.2. Let $X = \{a, b, c\}$ be a set with three elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b, c\}, X\}.$$

Let $Y = \{a', b'\}$ be a set with two elements. Let $\mathcal{O}_Y$ be the topology on Y given by

$$\left\{\emptyset, \{a'\}, Y\right\}.$$

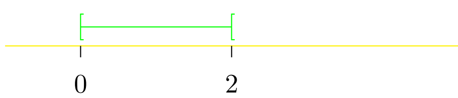
Let $Z = \{1, 2, \ldots, 6\}$ be a set with six elements. Let $\mathcal{O}_Z$ be the topology on Z given by

$$\{\emptyset, \{2\}, \{2, 5\}, \{1, 4\}, \{1, 3, 4, 6\}, \{1, 2, 4\}, \{1, 2, 4, 5\}, \{1, 2, 3, 4, 6\}, Z\}.$$

Let $X \times Y$ be equipped with the product topology $\mathcal{O}_{X \times Y}$. Find a homeomorphism

$$X \times Y \xrightarrow{f} Z.$$

Task E7.1.3. Let $[0, 2[$ be equipped with the subspace topology $\mathcal{O}_{[0,2[}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

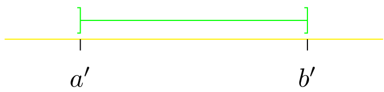


Let $]3,4]$ be equipped with the subspace topology $\mathcal{O}_{]3,4]}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $\left([0,2[\,,\mathcal{O}_{[0,2[}\right)$ is homeomorphic to $\left(]3,4]\,,\mathcal{O}_{]3,4]}\right)$.

Task E7.1.4. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Suppose that $a', b' \in \mathbb{R}$, and that $a' < b'$. Let $\mathcal{O}_{[a,b[}$ be the subspace topology on $[a,b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $\mathcal{O}_{[a',b'[}$ be the subspace topology on $]a',b']$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Generalise your argument from Task E7.1.3 to prove that $\left([a,b[\,,\mathcal{O}_{[a,b[}\right)$ is homeomorphic to $\left(]a',b']\,,\mathcal{O}_{]a',b]}\right)$.

Task E7.1.5. Suppose that a belongs to $\mathbb{R}$. Let $\mathcal{O}_{]a,\infty[}$ denote the subspace topology on $]a,\infty[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $(]a,\infty[\,,\mathcal{O}_{]a,\infty[})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. You may wish to proceed as follows.

(1) Let

$$]0,1[\xrightarrow{f}]a,\infty[$$

be the map given by $x \mapsto a + \frac{x}{1-x}$. Demonstrate that f is a homeomorphism. You may wish to appeal to Task E5.3.15.

(2) By Task E7.3.2, deduce that there is a homeomorphism

$$]a, \infty[\longrightarrow]0, 1[.$$

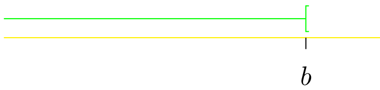
By Example 7.3.10, there is a homeomorphism

$$]0, 1[\longrightarrow \mathbb{R}.$$

By Task E7.1.10, conclude that there is a homeomorphism

$$]a, \infty[\longrightarrow \mathbb{R}.$$

Task E7.1.6. Suppose that b belongs to $\mathbb{R}$. Let $\mathcal{O}_{]-\infty,b[}$ denote the subspace topology on $]-\infty, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Prove that $(]-\infty, b[\,, \mathcal{O}_{]-\infty,b[})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Task E7.1.7. Let $k \in \mathbb{R}$, and let $c \in \mathbb{R}$. Let $L_{k,c}$ be the set given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid y = kx + c\right\} .$$

Let $\mathcal{O}_{L_{k,c}}$ denote the subspace topology on $L_{k,c}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Prove that $(L_{k,c}, \mathcal{O}_{L_{k,c}})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

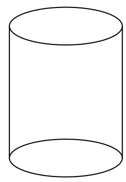
Task E7.1.8. Let $k \in \mathbb{R}$, and let $c \in \mathbb{R}$. Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $L^{[a,b]}_{k,c}$ be the set given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid y = kx + c \text{ and } a \leq x \leq b\right\} .$$

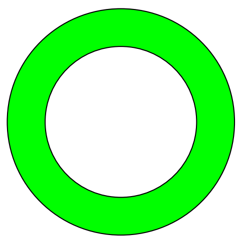
Let $\mathcal{O}_{L^{[a,b]}_{k,c}}$ denote the subspace topology on $L^{[a,b]}_{k,c}$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Prove that $\left(L^{[a,b]}_{k,c}, \mathcal{O}_{L^{[a,b]}_{k,c}}\right)$ is homeomorphic to $(I, \mathcal{O}_I)$. You may quote without proof anything from the lectures, and any of the other tasks.

Task E7.1.9. Find an intuitive argument to demonstrate that the cylinder $(S^1 \times I, \mathcal{O}_{S^1 \times I})$



is homeomorphic to an annulus $(A_k, \mathcal{O}_{A_k})$, where $0 < k < 1$.



Can you find a way to give a rigorous proof, along the lines of your intuitive argument?

Task E7.1.10. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ be topological spaces. Let

$$X \xrightarrow{f_0} Y$$

and

$$Y \xrightarrow{f_1} Z$$

be homeomorphisms. Prove that

$$X \xrightarrow{f_1 \circ f_0} Z$$

is a homeomorphism.

Remark E7.1.11. Together with Task E7.3.2, it follows that if any two of $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, and $(Z, \mathcal{O}_Z)$ are homeomorphic, then each is homeomorphic to the other two.

Remark E7.1.12. In other words, the relation on the set of topological spaces given by $(X_0, \mathcal{O}_{X_0}) \sim (X_1, \mathcal{O}_{X_1})$ if $(X_0, \mathcal{O}_{X_0})$ is homeomorphic to $(X_1, \mathcal{O}_{X_1})$ is an equivalence relation.

Remark E7.1.13. If it worries you, we do have to be careful about how we make sense of something as large as the set of topological spaces. This is a foundational matter which can be addressed in many different ways, and which we can harmlessly ignore!

Task E7.1.14. Let $(X_0, \mathcal{O}_{X_0})$, $(X_1, \mathcal{O}_{X_1})$, $(Y_0, \mathcal{O}_{Y_0})$, and $(Y_1, \mathcal{O}_{Y_1})$ be topological spaces. Let

$$X_0 \xrightarrow{f_0} Y_0$$

and

$$X_1 \xrightarrow{f_1} Y_1$$

be homeomorphisms. Prove that the map

$$X_0 \times X_1 \xrightarrow{f_0 \times f_1} Y_0 \times Y_1$$

given by $(x_0, x_1) \mapsto (f_0(x_0), f_1(x_1))$ is a homeomorphism.

Definition E7.1.15. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *open* if, for every subset U of X which belongs to $\mathcal{O}_X$, we have that $f(U)$ belongs to $\mathcal{O}_Y$.

Task E7.1.16. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Prove that f is open.

Definition E7.1.17. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. A map

$$X \xrightarrow{f} Y$$

is *closed* if, for every subset V of X which is closed with respect to $\mathcal{O}_X$, we have that $f(V)$ is closed with respect to $\mathcal{O}_Y$.

Task E7.1.18. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Prove

$$X \xrightarrow{f} Y$$

is closed.

Task E7.1.19. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that

$$X \xrightarrow{f} Y$$

is a homeomorphism. Let A be a subset of X. Let A be equipped with the subspace topology $\mathcal{O}_A$ with respect to $(X, \mathcal{O}_X)$. Let $f(A)$ be equipped with the subspace topology $\mathcal{O}_{f(A)}$ with respect to $(Y, \mathcal{O}_Y)$. Prove that $(A, \mathcal{O}_A)$ is homeomorphic to $\big(f(A), \mathcal{O}_{f(A)}\big)$.

Task E7.1.20. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that

$$X \xrightarrow{f} Y$$

is a homeomorphism. Let A be a subset of X. Let $X \setminus A$ be equipped with the subspace topology $\mathcal{O}_{X \setminus A}$ with respect to $(X, \mathcal{O}_X)$. Let $Y \setminus f(A)$ be equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$. Deduce from Task E7.1.19 that $\big(X \setminus A, \mathcal{O}_{X \setminus A}\big)$ is homeomorphic to $\big(Y \setminus f(A), \mathcal{O}_{Y \setminus f(A)}\big)$.

E7.2. In the lecture notes

Task E7.2.1. Let X and Y be sets. Prove that a map

$$X \xrightarrow{f} Y$$

is bijective in the sense of Definition 7.1.1 if and only if it is both injective and surjective.

Task E7.2.2. Let $X = \{x\}$ be a set with one element. Let Y be a set with at least two elements. Why can there not be a bijective map between X and Y? This was appealed to in Task 7.3.7.

Task E7.2.3. In the notation of Example 7.3.6, where does the analogue of the argument of Example 7.3.6 for closed intervals break down if we assume that $a = b$?

Task E7.2.4. Prove that the map

$$]-1,1[\xrightarrow{f} \mathbb{R}$$

given by $t \mapsto \frac{t}{1-|t|}$ is continuous. You may wish to proceed as follows.

(1) Prove that the map

$$]-1,1[\xrightarrow{g_1} \mathbb{R}$$

given by $t \mapsto 1 - |t|$ is continuous. For this, you may wish to express g_1 as a composition of maps, allowing you to deduce continuity from Task E5.3.3and from Task E5.3.14.

(2) Taking g_0 to be the inclusion map

$$]-1,1[\longrightarrow \mathbb{R}$$

and g_1 to be the map of (1), deduce the continuity of f from Proposition 5.2.2, (1), and Task E5.3.10.

Task E7.2.5. Prove that the map

$$\mathbb{R} \xrightarrow{g}]-1,1[$$

given by $t \mapsto \frac{t}{1+|t|}$ is continuous. You may wish to proceed in a similar way as in Task E7.2.4.

Task E7.2.6. Let

$$]-1,1[\xrightarrow{f} \mathbb{R}$$

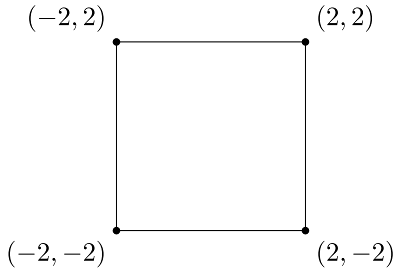
be the map of Task E7.2.4. Let

$$\mathbb{R} \xrightarrow{g}]-1,1[$$

be the map of Task E7.2.5. Prove that for all $t \in]-1,1[$ we have that $g(f(t)) = t$, and that for all $t \in \mathbb{R}$ we have that $f(g(t)) = t$.

Task E7.2.7. Let X be the square of Example 7.3.14, given by

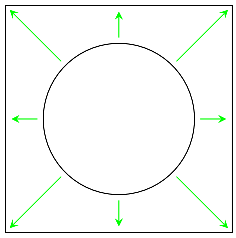
$$(\{-2,2\} \times [-2,2]) \cup ([-2,2] \times \{-2,2\}).$$



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Construct a homeomorphism

$$S^1 \xrightarrow{f} X$$

in the manner indicated in Example 7.3.14.



You may wish to proceed as follows.

(1) Let A_{east} be the subset of S^1 given by

$$\left\{(x,y) \in S^1 \mid x > 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq y \leq \tfrac{1}{\sqrt{2}}.\right\}.$$

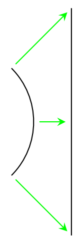
Let A_{east} be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Let B_{east} be the subset of X given by

$$\left\{(2,y) \in \mathbb{R}^2 \mid -2 \leq y \leq 2\right\}.$$

Let B_{east} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Prove that the map

$$A_{\mathsf{east}} \xrightarrow{f_{\mathsf{east}}} B_{\mathsf{east}}$$

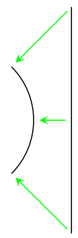
given by $(x, y) \mapsto \left(2, \frac{2y}{x}\right)$ is continuous. Quote any tasks which you appeal to.



(2) Prove that the map

$$B_{\mathsf{east}} \xrightarrow{g_{\mathsf{east}}} A_{\mathsf{east}}$$

given by $(x, y) \mapsto \frac{1}{\|(x,y)\|}(x, y)$ is continuous. In particular, quote any tasks which you appeal to.



(3) Verify that $g_{\mathsf{east}} \circ f_{\mathsf{east}} = id_{A_{\mathsf{east}}}$, and that $f_{\mathsf{east}} \circ g_{\mathsf{east}} = id_{B_{\mathsf{east}}}$. Conclude that f_{east} is a homeomorphism.

(4) Let A_{west} be the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid x < 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq y \leq \tfrac{1}{\sqrt{2}}\right\}.$$

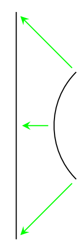
Let A_{west} be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Let B_{west} be the subset of X given by

$$\left\{(-2, y) \in \mathbb{R}^2 \mid -2 \leq y \leq 2\right\}.$$

Let B_{west} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. As in (1) – (3), prove that the map

$$A_{\mathsf{west}} \xrightarrow{f_{\mathsf{west}}} B_{\mathsf{west}}$$

given by $(x, y) \mapsto \left(-2, \frac{2y}{x}\right)$ is a homeomorphism. Quote any tasks which you appeal to.



(5) Let A_{north} be the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid y > 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq x \leq \tfrac{1}{\sqrt{2}}\right\}.$$

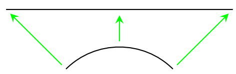
Let A_{north} be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Let B_{north} be the subset of X given by

$$\left\{(x, 2) \in \mathbb{R}^2 \mid -2 \leq x \leq 2\right\}.$$

Let B_{north} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Along the lines of (1) – (3), prove that the map

$$A_{\mathsf{north}} \xrightarrow{f_{\mathsf{north}}} B_{\mathsf{north}}$$

given by $(x, y) \mapsto \left(\frac{2x}{y}, 2\right)$ is a homeomorphism. Quote any tasks which you appeal to.



(6) Let A_{south} be the subset of S^1 given by

$$\left\{(x, y) \in S^1 \mid y < 0 \text{ and } -\tfrac{1}{\sqrt{2}} \le x \le \tfrac{1}{\sqrt{2}}\right\}.$$

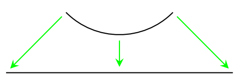
Let A_{north} be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Let B_{north} be the subset of X given by

$$\left\{(x, 2) \in \mathbb{R}^2 \mid -2 \le x \le 2\right\}.$$

Let B_{south} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Along the lines of (1) – (3), prove that the map

$$A_{\mathsf{south}} \xrightarrow{f_{\mathsf{south}}} B_{\mathsf{south}}$$

given by $(x, y) \mapsto \left(\frac{2x}{y}, -2\right)$ is a homeomorphism. Quote any tasks which you appeal to.

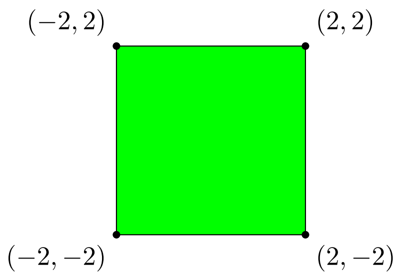


(7) Appeal to Task E7.3.6 three times to build a homeomorphism

$$S^1 \longrightarrow X$$

from the homeomorphisms f_{east}, f_{south}, f_{west}, and f_{north}.

Task E7.2.8. Let X be the subset $[-2,2] \times [-2,2]$ of $\mathbb{R}^2$, equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



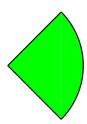
Construct a homeomorphism

$$D^2 \xrightarrow{f} X.$$

You may wish to proceed by adapting your argument from Task E7.2.7, in the following manner.

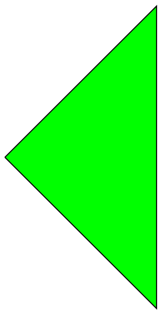
(1) Let A_{east} be the subset of D^2 given by

$$\left\{ (x,y) \in D^2 \mid x > 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq y \leq \tfrac{1}{\sqrt{2}} \right\}.$$



Let A_{east} be equipped with the subspace topology with respect to $(D^2, \mathcal{O}_{D^2})$. Let B_{east} be the subset of X given by

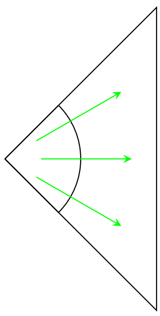
$$\left\{ (x,y) \in X \mid x > 0 \text{ and } -\tfrac{1}{\sqrt{2}} \leq y \leq \tfrac{1}{\sqrt{2}} \right\}.$$



Let B_{east} be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Prove that the map

$$A_{\mathsf{east}} \xrightarrow{f_{\mathsf{east}}} B_{\mathsf{east}}$$

given by $(x, y) \mapsto \left(\frac{2}{\|(x,y)\|}, \frac{2y}{x}\right)$ is a homeomorphism. Quote any tasks which you appeal to.

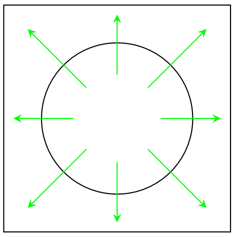


(2) Modify (4) – (6) of Task E7.2.7 in a similar way.

(3) Let $D^2 \setminus \{0\}$ be equipped with the subspace topology $\mathcal{O}_{D^2 \setminus \{0\}}$ with respect to $(D^2, \mathcal{O}_{D^2})$. Let $X \setminus \{0\}$ be equipped with the subspace topology $\mathcal{O}_{X \setminus \{0\}}$ with respect to $(X, \mathcal{O}_X)$. Appeal to Task E7.3.6 three times to build a homeomorphism

$$D^2 \setminus \{0\} \xrightarrow{f} X \setminus \{0\}$$

from the homeomorphisms f_{east}, f_{south}, f_{west}, and f_{north} of (1) and (2).



(4) By Task E7.3.9, deduce that the homeomorphism f of (3) gives rise to a homeomorphism

$$D^2 \longrightarrow X.$$

Task E7.2.9. Prove that $(D^2, \mathcal{O}_{D^2})$ is homeomorphic to $(I^2, \mathcal{O}_{I^2})$. You may wish to appeal to Task E7.2.8, Example 7.3.12, and Task E7.1.10.

E7.3. For a deeper understanding

Task E7.3.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Prove that a map

$$X \xrightarrow{f} Y$$

is a homeomorphism if and only if f is bijective, continuous, and open.

Task E7.3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. By definition of a homeomorphism, there is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. Prove that g is a homeomorphism.

Task E7.3.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Prove that there is a bijection between $\mathcal{O}_X$ and $\mathcal{O}_Y$.

Remark E7.3.4. In particular, if X and Y are finite sets such that $\mathcal{O}_X$ has a different cardinality to $\mathcal{O}_Y$, then $(X, \mathcal{O}_X)$ cannot be homeomorphic to $(Y, \mathcal{O}_Y)$.

Task E7.3.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let $\{A_j\}_{j \in J}$ be a set of subsets of X such that $X = \bigcup_{j \in J} A_j$. For every $j \in J$, let A_j be equipped with the subspace topology with respect to $(X, \mathcal{O}_X)$. Suppose that the following hold.

(1) For all j_0 and j_1 which belong to J, the restriction of f_{j_0} to $A_{j_0} \cap A_{j_1}$ is equal to the restriction of f_{j_1} to $A_{j_0} \cap A_{j_1}$.

(2) We have that A_j belongs to $\mathcal{O}_X$ for every j which belongs to J.

(3) For every j which belongs to J, we have a continuous map

$$A_j \xrightarrow{f_j} Y$$

such that the map

$$A_j \xrightarrow{f'_j} f(A_j)$$

given by $x \mapsto f_j(x)$ is a homeomorphism, where $f(A_j)$ is equipped with the subspace topology with respect to $(Y, \mathcal{O}_Y)$.

(4) Let

$$X \xrightarrow{f} Y$$

denote the map of Notation E5.3.22 corresponding to the maps $\{f_j\}_{j \in J}$. Suppose that f is bijective.

Prove that f is a homeomorphism. You may wish to proceed as follows.

(1) By (1) of Task E5.3.23, observe that f is continuous.

(2) Suppose that j belongs to J. Since f'_j is a homeomorphism, there is a continuous map

$$f(A_j) \xrightarrow{g'_j} A_j$$

such that $g'_j \circ f'_j = id_{A_j}$ and $f'_j \circ g'_j = id_{f(A_j)}$. Let

$$f(A_j) \xrightarrow{g_j} X$$

be the map given by $y \mapsto g'_j(y)$. By Task E5.1.10, observe that g_j is continuous.

(3) Suppose that j_0 and j_1 belong to J. We have that

$$f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1}) = f(A_{j_0}) \cap f(A_{j_1}).$$

Since f is bijective, we have that

$$f(A_{j_0}) \cap f(A_{j_1}) = f(A_{j_0} \cap A_{j_1}).$$

By definition of f, we have that

$$f(A_{j_0} \cap A_{j_1}) = f_{j_0}(A_{j_0} \cap A_{j_1})$$

and that

$$f(A_{j_0} \cap A_{j_1}) = f_{j_1}(A_{j_0} \cap A_{j_1}).$$

Thus we have that

$$f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1}) = f_{j_0}(A_{j_0} \cap A_{j_1})$$

and that

$$f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1}) = f_{j_1}(A_{j_0} \cap A_{j_1}).$$

Deduce that the restriction of g_{j_0} to $f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1})$ is equal to the restriction of g_{j_0} to $f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1})$. Let

$$Y \xrightarrow{g} X$$

be the map of Notation E5.3.22 corresponding to the maps $\{g_j\}_{j \in J}$.

(4) By Task E7.1.16, observe that $f_j(A_j)$ belongs to $\mathcal{O}_Y$.

(5) By (1) of Task E5.3.23, deduce from (2) and (4) that g is continuous.

(6) Observe that $g \circ f = id_X$, and that $f \circ g = id_Y$.

Task E7.3.6. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, $\{A_j\}_{j \in J}$, $\{f_j\}_{j \in J}$, and f be as in Task E7.3.5, except that instead of assuming that A_j belongs to $\mathcal{O}_X$ for every $j \in J$, suppose that $\{A_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_X$, and that A_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$. Suppose that $\{f(A_j)\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_Y$. Prove that f is a homeomorphism. You may wish to proceed as follows.

(1) By (2) of Task E5.3.23, observe that f is continuous.

(2) Define

$$f(A_j) \xrightarrow{g_j} Y$$

as in (2) of Task E7.3.5. By Task E5.1.10, observe that g_j is continuous.

(3) As in (3) of Task E7.3.5, demonstrate that the restriction of g_{j_0} to $f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1})$ is equal to the restriction of g_{j_0} to $f_{j_0}(A_{j_0}) \cap f_{j_1}(A_{j_1})$. Let

$$Y \xrightarrow{g} X$$

be the map of Notation E5.3.22 corresponding to the maps $\{g_j\}_{j \in J}$.

(4) By Task E7.1.18, observe that $f_j(A_j)$ is closed in Y with respect to $\mathcal{O}_Y$.

(5) By (2) of Task E5.3.23, deduce from (2), (4), and our assumption that $\{f(A_j)\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_Y$, that g is continuous.

(6) As in (6) of Task E7.3.5, observe that $g \circ f = id_X$, and that $f \circ g = id_Y$.

Task E7.3.7. Let $\mathcal{O}_{[0,1[}$ be the subspace topology on $[0,1[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

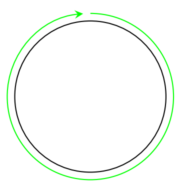
Let

$$[0,1[\xrightarrow{f} S^1$$

be the map given by $t \mapsto \phi(t)$, where

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is the map of Task E5.3.27.



Prove that f is a continuous bijection. Find a set $\{A_j\}_{j \in J}$ of subsets of $[0,1[$ with the following properties.

(1) We have that $\{A_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_{[0,1[}$.

(2) For every j which belongs to J, we have that A_j is closed in $[0,1[$ with respect to $\mathcal{O}_{[0,1[}$.

(3) Suppose that j belongs to J. Let A_j be equipped with the subspace topology with respect to $\big([0,1[\,, \mathcal{O}_{[0,1[}\big)$. Let $f(A_j)$ be equipped with the subspace topology with respect to $\big(S^1, \mathcal{O}_{S^1}\big)$. Then the map

$$A_j \xrightarrow{f_j} f(A_j)$$

given by $t \mapsto f(t)$ is a homeomorphism.

(4) We have that $\{f(A_j)\}_{j \in J}$ is not locally finite.

Remark E7.3.8. In Task E11.3.2, you are asked to prove that f is not homeomorphism.

Task E7.3.9. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that x belongs to X, and that $\{x\}$ is closed in X with respect to $\mathcal{O}_X$. Let $X \setminus \{x\}$ be equipped with the subspace topology $\mathcal{O}_{X \setminus \{x\}}$ with respect to $(X, \mathcal{O}_X)$. Let

$$X \xrightarrow{f} Y$$

be a bijective map. Let $Y \setminus f(x)$ be equipped with the subspace topology $\mathcal{O}_{Y \setminus \{f(x)\}}$ with respect to $(Y, \mathcal{O}_Y)$. Suppose that the map

$$X \setminus \{x\} \xrightarrow{g} Y \setminus \{f(x)\}$$

given by $x' \mapsto f(x')$ is a homeomorphism. Prove that f is a homeomorphism. You may wish to appeal to Task E5.3.29.

Task E7.3.10. As in Example 6.3.1, let $\sim$ be the equivalence relation on I generated by $0 \sim 1$.

Prove that $(I/{\sim}, \mathcal{O}_{I/\sim})$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

You may wish to proceed as follows.

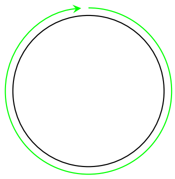
(1) Let

$$I \xrightarrow{\phi'} S^1$$

be the map given by $t \mapsto \phi(t)$, where

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is the map of Notation E5.3.26. By Task E5.3.27 and Task E5.2.3, observe that ϕ' is continuous.



(2) Observe that $\phi'(0) = \phi'(1)$. By Task E6.2.7, deduce that the map

$$I/\sim \xrightarrow{f} S^1$$

given by $[t] \mapsto \phi'(t)$ is continuous.

(3) Let A_0 be the set given by

$$\left\{(x, y) \in S^1 \mid x \geq 0\right\}.$$

Let A_0 be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Appealing to Task E2.3.1, Proposition 5.4.3, Task E5.3.14, and Proposition 5.3.1, observe that the map

$$A_0 \longrightarrow I$$

given by $(x, y) \mapsto -\frac{y}{4} + \frac{1}{4}$ is continuous.

(4) Let

$$I \xrightarrow{\pi} I/\sim$$

denote the quotient map. By Remark 6.1.9 and Proposition 5.3.1, deduce from (3) that the map

$$A_0 \xrightarrow{g_0} I/{\sim}$$

given by $(x, y) \mapsto [-\frac{y}{4} + \frac{1}{4}]$ is continuous.

(5) Let A_1 be the set given by

$$\left\{(x, y) \in S^1 \mid x \leq 0\right\}.$$

Let A_1 be equipped with the subspace topology with respect to $(S^1, \mathcal{O}_{S^1})$. Appealing to Task E2.3.1, Proposition 5.4.3 and Task E5.3.14, observe that the map

$$A_1 \longrightarrow I$$

given by $(x, y) \mapsto \frac{y}{4} + \frac{3}{4}$ is continuous.

(6) By Remark 6.1.9 and Proposition 5.3.1, deduce from (5) that the map

$$A_1 \xrightarrow{g_1} I/{\sim}$$

given by $(x, y) \mapsto [\frac{y}{4} + \frac{3}{4}]$ is continuous.

(7) Let

$$S^1 \xrightarrow{g} I/{\sim}$$

denote the map given by

$$(x, y) \mapsto \begin{cases} g_0(x, y) & \text{if } (x, y) \text{ belongs to } A_0, \\ g_1(x, y) & \text{if } (x, y) \text{ belongs to } A_1. \end{cases}$$

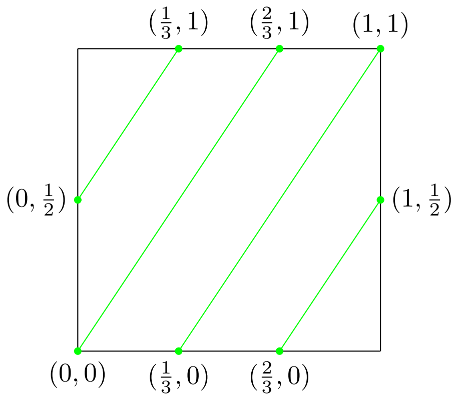
By (2) of Task E5.3.23, deduce from (4) and (6) that g is continuous.

(8) Observe that $g \circ f = id_{I/\sim}$, and that $f \circ g = id_{S^1}$.

(9) Conclude by (2), (7), and (8) that f is a homeomorphism.

E7.4. Exploration — torus knots

Task E7.4.1. Let K be the subset of T^2 of Task E6.3.1.

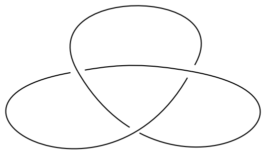


Let $\pi(K)$ be equipped with the subspace topology $\mathcal{O}_{\pi(K)}$ with respect to $(T^2, \mathcal{O}_{T^2})$. Prove that $\big(\pi(K), \mathcal{O}_{\pi(K)}\big)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

8. Tuesday 28th January

8.1. Further geometric examples of homeomorphisms

Example 8.1.1. Let K be a subset of $\mathbb{R}^3$ such as the following.

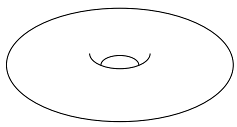


Let $\mathcal{O}_K$ denote the subspace topology on K with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$. Then $(K, \mathcal{O}_K)$ is an example of a *knot*. We have that $(K, \mathcal{O}_K)$ is homeomorphic to $(S^1, \mathcal{O}^1_S)$.

Remark 8.1.2. The crucial point is that both K and a circle can be obtained from a piece of string by glueing together the ends together. We may bend, twist, and stretch the string as much as we wish before we glue the ends together.

Remark 8.1.3. We shall explore knot theory later in the course.

Example 8.1.4. We have that $(T^2, \mathcal{O}_{T^2})$ is homeomorphic to $(S^1 \times S^1, \mathcal{O}_{S^1 \times S^1})$.

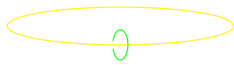


To prove this is the topic of Task E8.2.1.

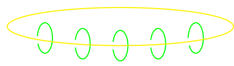
Remark 8.1.5. We can think of the left copy of S^1 in $S^1 \times S^1$ as the circle depicted below.

Suppose that x belongs to S^1.

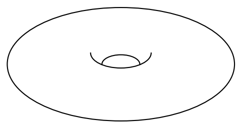
We can think of $\{x\} \times S^1$ as a circle around x.



In this way, we can think $S^1 \times S^1 = \bigcup_{x \in S^1} \{x\} \times S^1$ as a 'circle of circles'.



A 'circle of circles' is intuitively exactly a torus.



8.2. Neighbourhoods

Definition 8.2.1. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. A *neighbourhood* of x in X with respect to $\mathcal{O}_X$ is a subset U of X such that x belongs to U, and such that U belongs to $\mathcal{O}_X$.

In other references, you may see a neighbourhood U of x defined simply to be a subset of X to which x belongs, without the requirement that U belongs to $\mathcal{O}_X$.

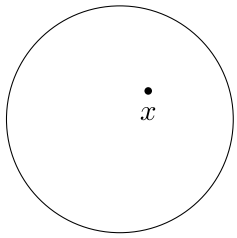
Example 8.2.2. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}.$$

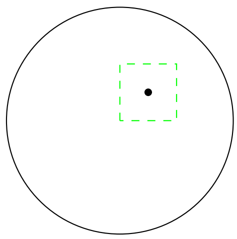
Here is a list of the neighbourhoods in X with respect to $\mathcal{O}_X$ of the elements of X.

Element	Neighbourhoods
a	$\{a\}$, $\{a,b\}$, $\{a,d\}$, $\{a,b,d\}$, $\{a,c,d\}$, X
b	$\{b\}$, $\{a,b\}$, $\{b,d\}$, $\{a,b,d\}$, $\{b,c,d\}$, X
c	$\{c,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, X
d	$\{d\}$, $\{a,d\}$, $\{b,d\}$, $\{c,d\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, X

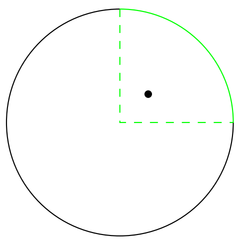
Example 8.2.3. Suppose that x belongs to D^2. For instance, we can take x to be $\left(\frac{1}{4}, \frac{1}{4}\right)$.



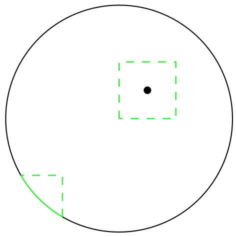
A typical example of a neighbourhood of x in D^2 with respect to $\mathcal{O}_{D^2}$ is a subset U of D^2 which is an 'open rectangle', and to which x belongs. When x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, we can, for instance, take U to be $\left]0, \frac{1}{2}\right[\times \left]0, \frac{1}{2}\right[$.



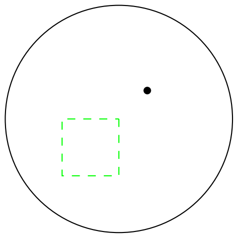
We could also take the intersection U with D^2 of any open rectangle in $\mathbb{R}^2$ to which x belongs. By definition of $\mathcal{O}_{D^2}$, we have that U belongs to $\mathcal{O}_{D^2}$. For instance, when x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, we can take U to be the intersection with D^2 of $]0, 1[\times]0, 1[$.



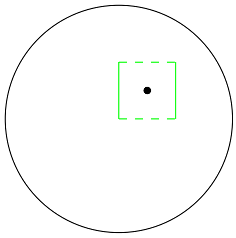
A disjoint union $U_0 \cup U_1$ of a pair of subsets of D^2 which both belong to $\mathcal{O}_{D^2}$, with the property that x belongs to either U_0 or U_1, is also a neighbourhood of x in D^2 with respect to $\mathcal{O}_{D^2}$. For $U_0 \cup U_1$ belongs to $\mathcal{O}_{D^2}$, and x belongs to $U_0 \cup U_1$. When x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, we can for instance take U_0 to be $\left]0, \frac{1}{2}\right[\times \left]0, \frac{1}{2}\right[$, and take U_1 to be the intersection with D^2 of $\left]-1, -\frac{1}{2}\right[\times \left]-1, -\frac{1}{2}\right[$.



A subset of D^2 to which x does not belong is not a neighbourhood of x in D^2 with respect to $\mathcal{O}_{D^2}$, even if it belongs to $\mathcal{O}_{D^2}$. When x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, the subset $\left]-\frac{1}{2}, 0\right[\times \left]-\frac{1}{2}, 0\right[$ is not a neighbourhood of x, for instance.



A subset of D^2 to which x belongs, but which does not belong to $\mathcal{O}_{D^2}$, is not a neighbourhood of x in D^2 with respect to $\mathcal{O}_{D^2}$. When x is $\left(\frac{1}{4}, \frac{1}{4}\right)$, the subset $\left]0, \frac{1}{2}\right[\times \left[0, \frac{1}{2}\right]$ is not a neighbourhood of x, for instance.



8.3. Limit points

Definition 8.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Suppose that x belongs to X. Then x is a *limit point* of A in X with respect to $\mathcal{O}_X$ if, for every neighbourhood U of x in X with respect to $\mathcal{O}_X$, there is an $a \in U$ such that a belongs to A.

Remark 8.3.2. In other words, x is a limit point of A in X with respect to $\mathcal{O}_X$ if and only if for every neighbourhood U of x in X with respect to $\mathcal{O}_X$, we have that $A \cap U \neq \emptyset$.

Remark 8.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Suppose that a belongs to A. Then a is a limit point of A in X with respect to $\mathcal{O}_X$, since every neighbourhood of a in X with respect to $\mathcal{O}_X$ contains a.

8.4. Examples of limit points

Example 8.4.1. Let $X = \{a, b\}$ be a set with two elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{b\}, X\}.$$

Let $A = \{b\}$. By Remark 8.3.3, we have that b is a limit point of A in X with respect to $\mathcal{O}_X$. Moreover, a is a limit point of A in X with respect to $\mathcal{O}_X$. For the only neighbourhood of a in X with respect to $\mathcal{O}_X$ is X, and we have that b belongs to X.

Example 8.4.2. Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, e\}, \{c, d\}, \{a, b, e\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}.$$

Let $A = \{d\}$. By Remark 8.3.3, we have that d is a limit point of A in X with respect to $\mathcal{O}_X$. To decide whether the other elements of X are limit points, we look at their neighbourhoods.

Element	Neighbourhoods
a	$\{a\}$, $\{a, b\}$, $\{a, b, e\}$, $\{a, c, d\}$, $\{a, b, c, d\}$, X
b	$\{b\}$, $\{a, b\}$, $\{b, e\}$, $\{a, b, e\}$, $\{b, c, d\}$, $\{a, b, c, d\}$, $\{b, c, d, e\}$, X
c	$\{c, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c, d\}$, $\{b, c, d, e\}$, X
e	$\{b, e\}$, $\{a, b, e\}$, $\{b, c, d, e\}$, X

For each element, we check whether d belongs to all of its neighbourhoods.

Element	Limit Point	Neighbourhoods to which d does not belong
a	✗	$\{a\}$, $\{a, b\}$, $\{a, b, e\}$
b	✗	$\{b\}$, $\{a, b\}$, $\{b, e\}$, $\{a, b, e\}$
c	✓	
e	✗	$\{b, e\}$, $\{a, b, e\}$

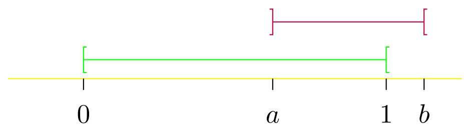
To establish that a, b, and e are not limit points, it suffices to observe that any *one* of the neighbourhoods listed in the table above does not contain d.

Example 8.4.3. Let $(X, \mathcal{O}_X)$ be as in Example 8.4.2. Let $A = \{b, d\}$. For each of the elements a, c, and e, we check whether every neighbourhood contains either b or d. The neighbourhoods are listed in a table in Example 8.4.2.

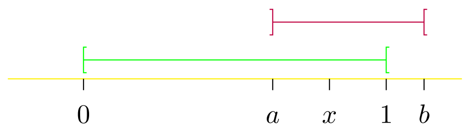
Element	Limit Point	Neighbourhoods U such that $A \cap U = \emptyset$
a	✗	$\{a\}$
c	✓	
e	✓	

Example 8.4.4. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $A = [0, 1[$.

Let U be a neighbourhood of 1 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ such that $a < 1 < b$ and which is a subset of U.

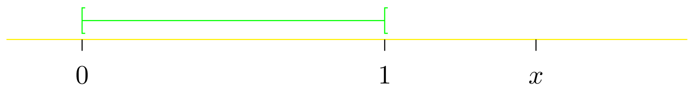


There is an $x \in \mathbb{R}$ such that $a < x < 1$, and $0 < x$. In particular, x belongs to $[0, 1[$.

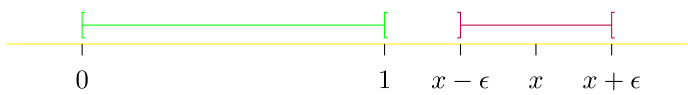


Since $]a, 1[$ is a subset of $]a, b[$, and since $]a, b[$ is a subset of U, we also have that x belongs to U. This proves that if U is a neighbourhood of 1 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, then $[0, 1[\cap U$ is not empty. Thus 1 is a limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Suppose now that $x \in \mathbb{R}$ has the property that $x > 1$.

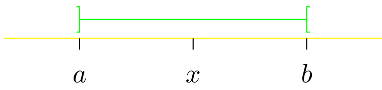


Let $\epsilon \in \mathbb{R}$ be such that $0 < \epsilon \leq x - 1$. Then $]x - \epsilon, x + \epsilon[$ is a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, but $[0, 1[\cap]x - \epsilon, x + \epsilon[$ is empty.

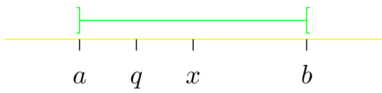


Thus x is not a limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. In a similar way, one can demonstrate that if $x \in \mathbb{R}$ has the property that $x < 0$, then x is not a limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. This is the topic of Task E8.2.2.

Example 8.4.5. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $A = \mathbb{Q}$, the set of rational numbers. Suppose that x belongs to $\mathbb{R}$. Let U be a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ such that $a < x < b$ which is a subset of U.



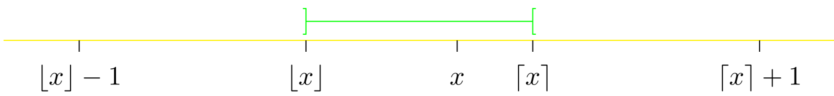
There is a $q \in \mathbb{Q}$ such that $a < q < x$. This is a consequence of the completeness of $\mathbb{R}$.



Since $]a, b[\cap \mathbb{Q}$ is a subset of $U \cap \mathbb{Q}$, we deduce that q belongs to U. We have proven that, for every neighbourhood U of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, $U \cap \mathbb{Q}$ is not empty. Thus x is a limit point of $\mathbb{Q}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Notation 8.4.6. Suppose that x belongs to $\mathbb{R}$. We denote by $\lfloor x \rfloor$ the largest integer z such that $z \leq x$. We denote by $\lceil x \rceil$ the smallest integer z such that $z \geq x$.

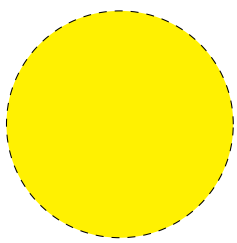
Example 8.4.7. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $A = \mathbb{Z}$, the set of integers. Suppose that x belongs to $\mathbb{R}$, and that x is not an integer. Then $]\lfloor x \rfloor, \lceil x \rceil[$ is a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.



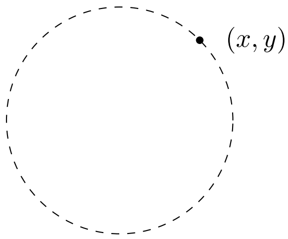
Moreover $\mathbb{Z} \cap]\lfloor x \rfloor, \lceil x \rceil[$ is empty. Thus x is not a limit point of $\mathbb{Z}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Example 8.4.8. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let A be the subset of $\mathbb{R}^2$ given by

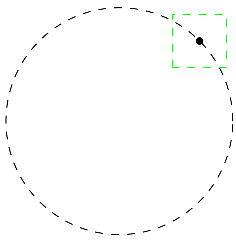
$$\left\{ (x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1 \right\}.$$



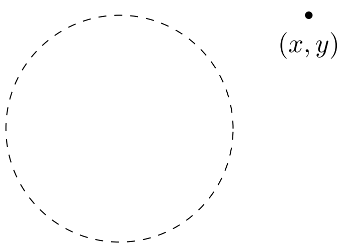
Suppose that $(x, y) \in \mathbb{R}^2$ belongs to S^1.



Then (x, y) is a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. Every neighbourhood of (x, y) in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ contains an 'open rectangle' U to which (x, y) belongs. We have that $A \cap U$ is not empty.



To fill in the details of this argument is the topic of Task E8.2.3. Suppose now that $(x, y) \in \mathbb{R}^2$ does not belong to D^2.



Then (x, y) is not a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. For let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon < \|(x, y)\| - 1.$$

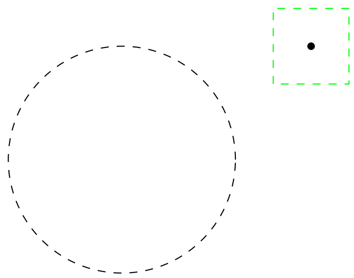
Let U_x be the open interval given by

$$\left]x - \tfrac{\epsilon\sqrt{2}}{\epsilon}, x + \tfrac{\epsilon\sqrt{2}}{\epsilon}\right[.$$

Let U_y be the open interval given by

$$\left]y - \tfrac{\epsilon\sqrt{2}}{\epsilon}, y + \tfrac{\epsilon\sqrt{2}}{\epsilon}\right[.$$

Then $U_x \times U_y$ is a neighbourhood of (x, y) in $\mathbb{R}^2$ whose intersection with A is empty.



To check this is the topic of Task E8.2.4.

8.5. Closure

Definition 8.5.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. The *closure* of A in X with respect to $\mathcal{O}_X$ is the set of limit points of A in X.

Notation 8.5.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. We shall denote the closure of A in X with respect to $\mathcal{O}_X$ by $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$.

Remark 8.5.3. The notation $\overline{A}$ is also frequently used to denote closure.

Remark 8.5.4. By Remark 8.3.3, we have that A is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$.

Definition 8.5.5. Let $(X, \mathcal{O}_X)$ be a topological space. A subset A of X is *dense* in X with respect to $\mathcal{O}_X$ if the closure of A in X with respect to $\mathcal{O}_X$ is X.

8.6. Examples of closure

Example 8.6.1. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.1. We found in Example 8.4.1 that the limit points of A in X with respect to $\mathcal{O}_X$ are a and b. Hence $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is X. Thus A is dense in X with respect to $\mathcal{O}_X$.

Example 8.6.2. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.2. We found in Example 8.4.2 that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is $\{c, d\}$. Thus A is not dense in X with respect to $\mathcal{O}_X$.

Example 8.6.3. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.3. We found in Example 8.4.3 that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is $\{b, c, d, e\}$. Thus A is not dense in X with respect to $\mathcal{O}_X$.

Example 8.6.4. We found in Example 8.4.4 that 1 is the only limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ which does not belong to $[0, 1[$. Thus $\mathsf{cl}_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}([0, 1[)$ is $[0, 1]$. In particular, $[0, 1[$ is not dense in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Example 8.6.5. We found in Example 8.4.5 that every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. In other words, $\mathsf{cl}_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}(\mathbb{Q})$ is $\mathbb{R}$. Thus $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Example 8.6.6. We found in Example 8.4.7 that if $x \in \mathbb{R}$ is not an integer, then x is not a limit point of $\mathbb{Z}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. In other words, $\mathsf{cl}_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}(\mathbb{Z})$ is $\mathbb{Z}$. In particular, $\mathbb{Z}$ is not dense in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

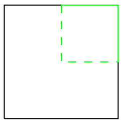
Example 8.6.7. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let A be as in Example 8.4.8. We found in Example 8.4.8 that if $(x, y) \in \mathbb{R}^2$ does not belong to A, then (x, y) is a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ if and only if (x, y) belongs to S^1. We conclude that $\mathsf{cl}_{(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})}(A)$ is D^2. In particular, A is not dense in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

E8. Exercises for Lecture 8

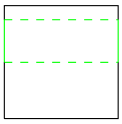
E8.1. Exam questions

Task E8.1.1. For which of the following subsets A of I^2 is $\pi(A)$ a neighbourhood of $\left[\left(\frac{3}{4}, \frac{3}{4}\right)\right]$ in K^2 with respect to $\mathcal{O}_{K^2}$? Take the equivalence relation on K^2 to be that of Example 6.4.11.

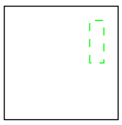
(1) $\left]\frac{1}{2}, 1\right] \times \left]\frac{1}{2}, 1\right]$



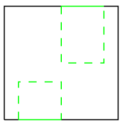
(2) $[0,1] \times \left]\frac{1}{2}, \frac{7}{8}\right[$



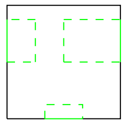
(3) $\left]\frac{3}{4}, \frac{7}{8}\right[\times \left]\frac{1}{2}, \frac{7}{8}\right[$



(4) $\left(\left]\frac{1}{2}, \frac{7}{8}\right[\times \left]\frac{1}{2}, 1\right]\right) \cup \left(\left]\frac{1}{8}, \frac{1}{2}\right[\times \left[0, \frac{1}{3}\right[\right)$



(5) $\left(\left]\frac{1}{2},1\right] \times \left]\frac{1}{2},\frac{7}{8}\right[\right) \cup \left(\left[0,\frac{1}{4}\right[\times \left]\frac{1}{2},\frac{7}{8}\right[\right) \cup \left(\left]\frac{1}{3},\frac{2}{3}\right[\times \left[0,\frac{1}{8}\right[\right)$

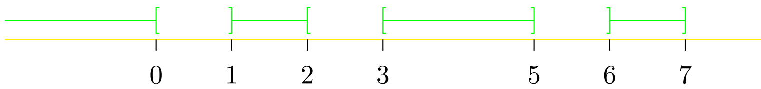


Task E8.1.2. Let $X = \{a,b,c,d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\}.$$

What is the closure of $\{b\}$ in X with respect to $\mathcal{O}_X$? Find a subset A of X with two elements, neither of which is b, with the property that A is dense in X with respect to $\mathcal{O}_X$.

Task E8.1.3. Let $A = \left]-\infty,0\right[\cup \left]1,2\right[\cup [3,5] \cup \left]6,7\right]$.



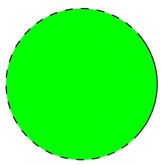
What is the closure of A in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$?

Task E8.1.4. Let A be the union of the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid -1 < x < \frac{3}{4} \text{ and } \|(x,y)\| < 1\right\}$$

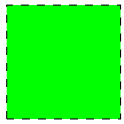
and the set

$$\left\{(x,y) \in \mathbb{R}^2 \mid \frac{3}{4} \leq x < 1 \text{ and } \|(x,y)\| \leq 1\right\}.$$

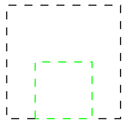


What is the closure of A in D^2 with respect to $\mathcal{O}_{D^2}$?

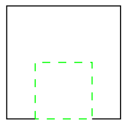
Task E8.1.5. Let $X = \left]0,1\right[\times \left]0,1\right[$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



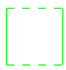
Let $A = \left]\frac{1}{4}, \frac{3}{4}\right[\times \left]0, \frac{1}{2}\right[$.



What is the closure of A in $(X, \mathcal{O}_X)$? What is the closure of A in $(I^2, \mathcal{O}_{I^2})$?



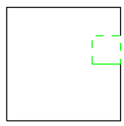
Find a subset Y of $\mathbb{R}^2$ such that the closure of A in Y with respect to $\mathcal{O}_Y$ is $\left[\frac{1}{4}, \frac{3}{4}\right] \times \left]0, \frac{1}{2}\right[$, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Task E8.1.6. Let $A = \left]\frac{3}{4}, 1\right[\times \left[\frac{1}{2}, \frac{3}{4}\right[$. Let

$$I^2 \xrightarrow{\pi} T^2$$

denote the quotient map.



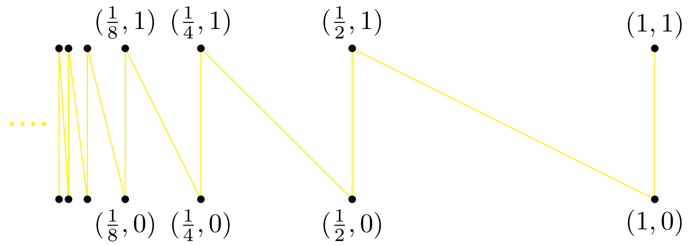
What is the closure of $\pi(A)$ in $(T^2, \mathcal{O}_{T^2})$?

Task E8.1.7. Let A be the subset of $\mathbb{R}^2$ given by the union of the sets

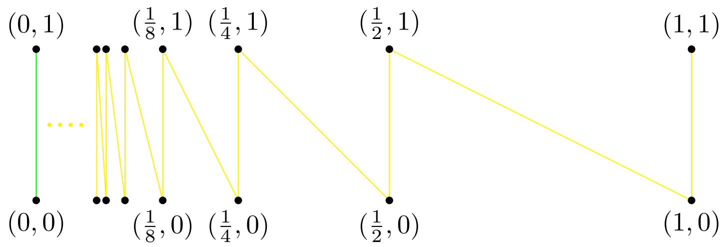
$$\bigcup_{n \in \mathbb{N}} \left\{ \left(\frac{1}{2^{n-1}}, y \right) \mid y \in [0, 1] \right\}$$

and

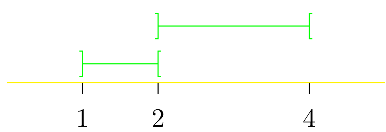
$$\bigcup_{n\in\mathbb{N}} \left\{ (x, -2^n x + 2) \mid x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \right\}.$$



Prove that the closure of X in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ is the union of X and the line $\{0\} \times [0,1]$.



Task E8.1.8. Let $X =]1,2[\cup]2,4[$. What is the closure of X in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$?



E8.2. In the lectures

Task E8.2.1. Prove that $(T^2, \mathcal{O}_{T^2})$ is homeomorphic to $(S^1 \times S^1, \mathcal{O}_{S^1 \times S^1})$, as discussed in Example 8.1.4. You may wish to proceed as follows.

(1) As in Example 6.3.1, work with S^1 throughout this task as the quotient of I by the equivalence relation generated by $0 \sim 1$. In particular, think of $\mathcal{O}_{S^1}$ as the quotient topology $\mathcal{O}_{I/\sim}$.

(2) Let

$$I \xrightarrow{\pi_{S^1}} S^1$$

denote the quotient map. Appealing to Remark 6.1.9 and Task **??**, observe that the map

$$I \times I \xrightarrow{\pi_{S^1} \times \pi_{S^1}} S^1 \times S^1$$

is continuous.

(3) Appealing to Task E6.2.7, deduce from (2) that the map

$$T^2 \xrightarrow{f} S^1 \times S^1$$

given by $[(s,t)] \mapsto ([s],[t])$ is continuous.

(4) Let $t \in I$. Appealing to Task E5.3.14, Task E5.1.5, and Task E5.3.17, observe that the map

$$I \xrightarrow{f_t^0} I^2$$

given by $s \mapsto (t,s)$ is continuous.

(5) Let

$$I^2 \xrightarrow{\pi_{T^2}} T^2$$

denote the quotient map. Appealing to Task 5.3.1, deduce from (1) and Remark 6.1.9 that the map

$$I \xrightarrow{\pi_{T^2} \circ f_t^0} T^2$$

given by $s \mapsto [(s,t)]$ is continuous.

(6) Observe that $\pi_{T^2}\left(f_t^0(0)\right) = \pi_{T^2}\left(f_t^0(1)\right)$. By Task E6.2.7, deduce that the map

$$S^1 \xrightarrow{g_t^0} T^2$$

given by $[s] \mapsto [(t,s)]$ is continuous.

(7) As in (4) – (6), use the map

$$I \xrightarrow{f_t^1} I^2$$

given by $s \mapsto (s,t)$ to prove that the map

$$S^1 \xrightarrow{g_t^1} T^2$$

given by $[t] \mapsto [(s,t)]$ is continuous.

(8) Let

$$S^1 \times S^1 \xrightarrow{g} T^2$$

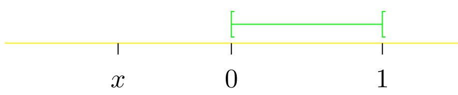
denote the map given by $([s],[t]) \mapsto [(s,t)]$. Observe that $g \circ f = id_{T^2}$, and that $f \circ g = id_{S^1 \times S^1}$.

(9) Let U be a subset of T^2 which belongs to $\mathcal{O}_{T^2}$. Suppose that $([x],[y])$ belongs to $g^{-1}(U)$. Let U_x denote the subset $\left(g_y^1\right)^{-1}(U)$ of S^1. By (6), we have that U_x belongs to $\mathcal{O}_{S^1}$. Let U_y denote the subset $\left(g_x^0\right)^{-1}(U)$ of S^1. By (5), we have that U_y belongs to $\mathcal{O}_{S^1}$. Observe that $([x],[y])$ belongs to $U_x \times U_y$, and that $U_x \times U_y$ is a subset of $g^{-1}(U)$.

(10) By definition of $\mathcal{O}_{S^1 \times S^1}$, deduce from (8) that $g^{-1}(U)$ belongs to $\mathcal{O}_{S^1 \times S^1}$. Conclude that g is continuous.

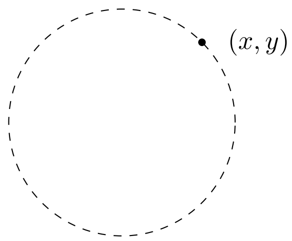
(11) Observe that (2), (8), and (10) together establish that f is a homeomorphism.

Task E8.2.2. Let $x \in \mathbb{R}$ be such that $x < 0$.



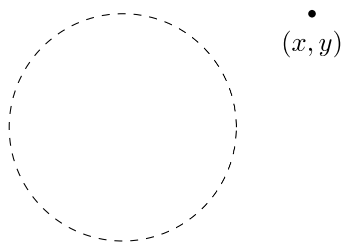
Prove that x is not a limit point of $[0,1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$).

Task E8.2.3. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.8. Suppose that $(x,y) \in \mathbb{R}^2$ belongs to S^1,



Prove that (x, y) is a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

Task E8.2.4. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.8. Suppose that $(x, y) \in \mathbb{R}^2$ does not belong to D^2.



Prove that (x, y) is not a limit point of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$, following the argument outlined in Example 8.4.8. You may find it helpful to look back at Example 3.2.3.

E8.3. For a deeper understanding

Task E8.3.1. Let $(X, \mathcal{O}_X)$. Let U be a subset of X. Prove that U belongs to $\mathcal{O}_X$ if and only if, for every x which belongs to X, there is a neighbourhood U_x of x in $(X, \mathcal{O}_X)$ such that U_x is a subset of U.

Remark E8.3.2. Task E8.3.1 gives a 'local characterisation' of subsets of X which belong to $\mathcal{O}_X$.

Task E8.3.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a map. Prove that f is continuous if and only for every $x \in X$, and every neighbourhood $U_{f(x)}$ of $f(x)$ in Y with respect to $\mathcal{O}_Y$, there is a neighbourhood U_x of x in X with respect to $\mathcal{O}_X$ such that $f(U_x)$ is a subset of $U_{f(x)}$. You may wish to proceed as follows.

(1) Suppose that f satisfies this condition. Let U be a subset of Y which belongs to $\mathcal{O}_Y$. Suppose that x belongs $f^{-1}(U)$. Observe that U is a neighbourhood of $f(x)$ in Y with respect to $\mathcal{O}_Y$.

(2) By assumption, there is thus a neighbourhood U_x of x in X with respect to $\mathcal{O}_X$ such that $f(U_x)$ is a subset of U. Deduce that U_x is a subset of $f^{-1}(U)$.

(3) By Task E8.3.1, deduce that $f^{-1}(U)$ belongs to $\mathcal{O}_X$. Conclude that f is continuous.

(4) Conversely, suppose that f is continuous. Suppose that x belongs to X, and that $U_{f(x)}$ is a neighbourhood of $f(x)$ in Y with respect to $\mathcal{O}_Y$. We have that $f\left(f^{-1}(U_{f(x)})\right)$ is a subset of $U_{f(x)}$. Since f is continuous, observe that $f^{-1}(U_{f(x)})$ is moreover a neighbourhood of x in X with respect to $\mathcal{O}_X$.

Remark E8.3.4. Task E8.3.3 gives a 'local characterisation' of continuous maps.

Definition E8.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. A set $\{A_j\}_{j \in J}$ of (possibly infinitely many) subsets of X is *locally finite* with respect to $\mathcal{O}_X$ if, for every $x \in X$, there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ with the property that the set of $j \in J$ such that $U \cap A_j$ is non-empty is finite.

Remark E8.3.6. If J is finite, then $\{A_j\}_{j \in J}$ is locally finite.

Task E8.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{V_j\}_{j \in J}$ be a set of subsets of X which is locally finite with respect to $\mathcal{O}_X$. Suppose that V_j is closed with respect to $\mathcal{O}_X$, for every $j \in J$. Let K be a (possibly infinite) subset of J. Prove that $\bigcup_{j \in K} V_j$ is closed with respect to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Let $x \in X \setminus \left(\bigcup_{j \in K} V_j\right)$. Observe that since $\{V_j\}_{j \in J}$ is locally finite with respect to $\mathcal{O}_X$, there is a neighbourhood U_x of x in $(X, \mathcal{O}_X)$ with the property that the set L of $j \in J$ such that $U_x \cap V_j$ is non-empty is finite.

(2) Let $U = U_x \cap \left(\bigcap_{j \in L} X \setminus V_j\right)$. Prove that U belongs to $\mathcal{O}_X$.

(3) Observe that $x \in U$.

(4) Prove that $U \cap \left(\bigcup_{j \in K} V_j\right)$ is empty, and thus that U is a subset of $X \setminus V$.

(5) By Task E8.3.1, deduce that $X \setminus \left(\bigcup_{j \in K} V_j\right)$ belongs to $\mathcal{O}_X$.

Task E8.3.8. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{V_j\}_{j \in J}$ be a locally finite set of subsets of X, with the property that $X = \bigcup_{j \in J} V_j$. For every $j \in J$, let $\mathcal{O}_{V_j}$ denote the subspace topology on V_j with respect to $(X, \mathcal{O}_X)$. Suppose that V_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$. Let V be a subset of X such that $V \cap V_j$ is closed with respect to $\mathcal{O}_{V_j}$ for every $j \in J$. Prove that V is closed with respect to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Appealing to Task E2.3.3 (3), observe that $V \cap V_j$ is closed with respect to $\mathcal{O}_X$.

(2) Prove that since $\{V_j\}_{j \in J}$ is locally finite, so is $\{V \cap V_j\}_{j \in J}$.

(3) By Task E8.3.7, deduce that $\bigcup_{j \in J} V \cap V_j$ is closed with respect to $\mathcal{O}_X$.

(4) Observe that $V = \bigcup_{j \in J} V \cap V_j$.

Task E8.3.9. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{V_j\}_{j \in J}$ be a locally finite set of subsets of X, with the property that $X = \bigcup_{j \in J} V_j$. For every $j \in J$, let $\mathcal{O}_{V_j}$ denote the subspace topology on V_j with respect to $(X, \mathcal{O}_X)$. Suppose that V_j is closed with respect to $\mathcal{O}_X$ for every $j \in J$. Let U be a subset of X such that $U \cap V_j$ belongs to $\mathcal{O}_{V_j}$ for every $j \in J$. Prove that U belongs to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Since $U \cap V_j$ belongs to $\mathcal{O}_{V_j}$, observe that $V_j \setminus (U \cap V_j)$ is closed with respect to $\mathcal{O}_{V_j}$, for every $j \in J$.

(2) Observe that $V_j \setminus (U \cap V_j) = V_j \cap (X \setminus U)$.

(3) By Task E8.3.8, deduce that $X \setminus U$ is closed with respect to $\mathcal{O}_X$.

Task E8.3.10. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\mathcal{O}'_X$ be a topology on X such that $\mathcal{O}'_X$ is a subset of $\mathcal{O}_X$. Let A be a subset of X. Suppose that x is a limit point of A in X with respect to $\mathcal{O}_X$. Prove that x is a limit point of A in X with respect to $\mathcal{O}'_X$.

Task E8.3.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of X, and let B be a subset of Y. Prove that $\mathsf{cl}_{(X \times Y, \mathcal{O}_{X \times Y})}(A \times B)$ is

$$\mathsf{cl}_{(X, \mathcal{O}_X)}(A) \times \mathsf{cl}_{(Y, \mathcal{O}_Y)}(B).$$

Task E8.3.12. Let $(X, \mathcal{O}_X)$ be a topological space. Let A and B be subsets of X such that A is a subset of B. Prove that $\mathsf{cl}_{(X, \mathcal{O}_X)}(A)$ is a subset of $\mathsf{cl}_{(X, \mathcal{O}_X)}(B)$.

Task E8.3.13. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Let B be a subset of A which belongs to $\mathcal{O}_X$. Prove that $\mathsf{cl}_{(A, \mathcal{O}_A)}(B)$ is $A \cap \mathsf{cl}_{(X, \mathcal{O}_X)}(B)$. You may wish to proceed as follows.

(1) Suppose that x belongs to $\mathsf{cl}_{(A, \mathcal{O}_A)}(B)$. In particular, we have that x belongs to A. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$. By definition of $\mathcal{O}_A$, observe that $A \cap U$ is a neighbourhood of x in A with respect to $\mathcal{O}_A$.

(2) Since x belongs to $\mathsf{cl}_{(A, \mathcal{O}_A)}(B)$, observe that $B \cap (A \cap U)$ is not empty.

(3) Since $B \cap (A \cap U)$ is $(B \cap A) \cap U$, and since B is a subset of A, deduce that $B \cap U$ is not empty.

(4) Deduce that x belongs to $\mathsf{cl}_{(X, \mathcal{O}_X)}(B)$. Conclude that $\mathsf{cl}_{(A, \mathcal{O}_A)}(B)$ is a subset of $A \cap \mathsf{cl}_{(X, \mathcal{O}_X)}(B)$.

(5) Conversely, suppose that x belongs to $A \cap \mathsf{cl}_{(X, \mathcal{O}_X)}(B)$. Suppose that U is a neighbourhood of x in A with respect to $\mathcal{O}_A$. By definition of $\mathcal{O}_A$, observe that there is a subset U' of X which belongs to $\mathcal{O}_X$ with the property that $U = A \cap U'$.

(6) Since x belongs to $\mathsf{cl}_{(X,\mathcal{O}_X)}(B)$, observe that $B \cap U'$ is not empty.

(7) Since B is a subset of A, we have that $B = B \cap A$. Deduce that $(B \cap A) \cap U' = B \cap (A \cap U') = B \cap U$ is not empty.

(8) Deduce that x belongs to $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$. Conclude that $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$ is a subset of $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$.

(9) By (4) and (8), deduce that $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$ is $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$.

9. Monday 3rd February

9.1. A local characterisation of closed sets

Proposition 9.1.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let V be a subset of X. Then V is closed with respect to $\mathcal{O}_X$ if and only if $\mathsf{cl}_{(X,\mathcal{O}_X)}(V)$ is V.

Proof. Suppose first that V is closed with respect to $\mathcal{O}_X$. Suppose that x does not belong to V. We make the following observations.

(1) By definition of $X \setminus V$, we have that x belongs to $X \setminus V$. Moreover, since V is closed with respect to $\mathcal{O}_X$, we have that $X \setminus V$ belongs to $\mathcal{O}_X$. In other words, $X \setminus V$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$.

(2) By definition of $X \setminus V$ once more, we have that $V \cap (X \setminus V)$ is empty.

Together (1) and (2) establish that x is not a limit point of V in X with respect to $\mathcal{O}_X$, for any x which does not belong to V. We conclude that $\mathsf{cl}_{(X,\mathcal{O}_X)}(V)$ is V.

Suppose now that $\mathsf{cl}_{(X,\mathcal{O}_X)}(V)$ is V. Suppose that $x \in X$ does not belong to V. By definition of $\mathsf{cl}_{(X,\mathcal{O}_X)}(V)$, we have that x is not a limit point of V in X with respect to $\mathcal{O}_X$. By definition of a limit point, we deduce that there is a neighbourhood U_x of x such that $V \cap U_x$ is empty. We make the following observations.

(1) We have that

$$X \setminus V = \bigcup_{x \in X \setminus V} \{x\}.$$

We also have that x belongs to U_x for every $x \in X \setminus V$, or, in other words, that $\{x\}$ is a subset of U_x for every $x \in X \setminus V$. Thus we have that $\bigcup_{x \in X \setminus V} \{x\}$ is a subset of $\bigcup_{x \in X \setminus V} U_x$. We deduce that $X \setminus V$ is a subset of $\bigcup_{x \in X \setminus V} U_x$.

(2) We have that

$$V \cap \left(\bigcup_{x \in X \setminus V} U_x \right) = \bigcup_{x \in X \setminus V} (V \cap U_x).$$

Since $V \cap U_x$ is empty for every $x \in X \setminus V$, we have that $\bigcup_{x \in X \setminus V} (V \cap U_x)$ is empty. We deduce that $V \cap \left(\bigcup_{x \in X \setminus V} U_x \right)$ is empty. In other words, $\bigcup_{x \in X \setminus V} U_x$ is a subset of $X \setminus V$.

(3) Since U_x belongs to $\mathcal{O}_X$, for every $x \in X \setminus V$, and since $\mathcal{O}_X$ is a topology on X, we have that $\bigcup_{x \in X \setminus V} U_x$ belongs to $\mathcal{O}_X$.

By (1) and (2) together, we have that $\bigcup_{x \in X \setminus V} U_x = X \setminus V$. By (3), we deduce that $X \setminus V$ belongs to $\mathcal{O}_X$. Thus V is closed with respect to $\mathcal{O}_X$. $\square$

Remark 9.1.2. We now, by Proposition 9.1.1, have two ways to understand closed sets. The first is 'global' in nature: that $X \setminus V$ belongs to $\mathcal{O}_X$. The second is 'local' in nature: that every limit point of V belongs to V.

For certain purposes in mathematics it can be appropriate to work 'locally', whilst for others it can be appropriate to work 'globally'. To know that 'local' and 'global' variants of a particular mathematical concept coincide allows us to move backwards and forwards between these points of view. This is often a very powerful technique.

9.2. Boundary

Definition 9.2.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. The *boundary* of A in X with respect to $\mathcal{O}_X$ is the set of $x \in X$ such that, for every neighbourhood U of x in X with respect to $\mathcal{O}_X$, there is an $a \in U$ which belongs to A, and there is a $y \in U$ which belongs to $X \setminus A$.

Notation 9.2.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. We shall denote the boundary of A in X with respect to $\mathcal{O}_X$ by $\partial_{(X,\mathcal{O}_X)}A$.

Remark 9.2.3. Suppose that $x \in X$ belongs to $\partial_{(X,\mathcal{O}_X)}A$. Then x is a limit point of A in X with respect to $\mathcal{O}_X$.

Remark 9.2.4. Let x be a limit point of A which does not belong to A. Then x belongs to $\partial_{(X,\mathcal{O}_X)}A$.

However, as we shall see in Example 9.3.1, it is not necessarily the case that if a belongs to A, then a belongs to $\partial_{(X,\mathcal{O}_X)}A$. In particular, not every limit point of A belongs to $\partial_{(X,\mathcal{O}_X)}A$.

9.3. Boundary in a finite example

Example 9.3.1. , Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,e\}, \{c,d\}, \{a,b,e\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}, \{b,c,d,e\}, X\}.$$

Let $A = \{b, d\}$. The neighbourhoods in X with respect to $\mathcal{O}_X$ of each of the elements of A are listed in a table in Example 8.4.2. To determine $\partial_{(X,\mathcal{O}_X)}A$, we check, for each element of A, whether each of its neighbourhoods both contain either a, c, or e, and contain either b or d. We determined the limit points of A in X with respect to $\mathcal{O}_X$ in Example 8.4.2, which saves us a little work.

Element	Belongs to $\partial_{(X,\mathcal{O}_X)}A$?	Reason
a	✗	Not a limit point.
b	✗	The neighbourhood $\{b\}$ does not contain any element of $X \setminus A$.
c	✓	Limit point which does not belong to A.
d	✓	Every neighbourhood of d contains both c and d. We have that d belongs to A, and that c belongs to $X \setminus A$.
e	✓	Limit point which does not belong to A.

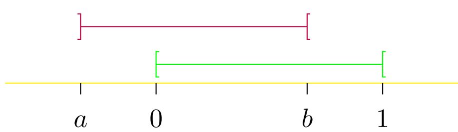
Thus $\partial_{(X,\mathcal{O}_X)}A = \{c, d, e\}$.

9.4. Geometric examples of boundary

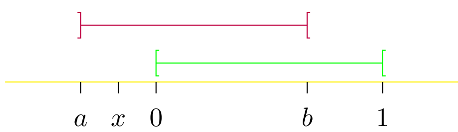
Example 9.4.1. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $A = [0, 1[$.

By Example 8.4.4, we have that 1 is a limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ which does not belong to $[0, 1[$. Thus 1 belongs to $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})} [0, 1[$. By Example **??**, we have that all other limit points of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ belong to $[0, 1[$. To determine $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})} [0, 1[$, it therefore remains to check which elements of $[0, 1[$ have the property that each of their neighbourhoods contains at least one element of $\mathbb{R} \setminus [0, 1[$.

Let U be a neighbourhood of 0 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ such that $a < 0 < b$, and which is a subset of U.

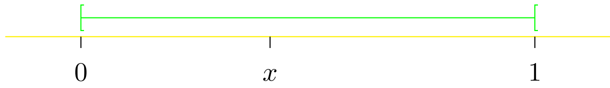


There is an $x \in \mathbb{R}$ such that $a < x < 0$. In particular, x belongs to $\mathbb{R} \setminus [0, 1[$.

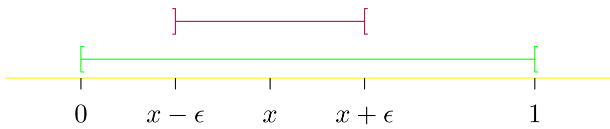


Since $]a, 0[$ is a subset of $]a, b[$, and since $]a, b[$ is a subset of U, we also have that x belongs to U. This proves that if U is a neighbourhood of 0 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, then $(\mathbb{R} \setminus [0, 1[) \cap U$ is not empty. Thus 0 belongs to $\partial_{(\mathbb{R}, \mathcal{O}_{\mathbb{R}})} [0, 1[$.

Suppose now that $0 < x < 1$.

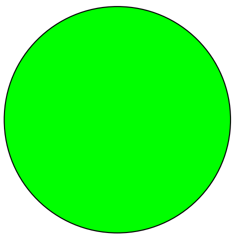


Let $0 < \epsilon \leq \min\{x, 1 - x\}$. Then $]x - \epsilon, x + \epsilon[$ is a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, and $(\mathbb{R} \setminus [0, 1[) \cap]x - \epsilon, x + \epsilon[$ is empty.

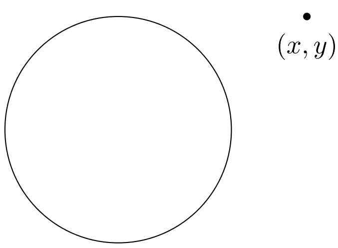


Thus x does not belong to $\partial_{(\mathbb{R}, \mathcal{O}_{\mathbb{R}})} [0, 1[$. We conclude that $\partial_{(\mathbb{R}, \mathcal{O}_{\mathbb{R}})} [0, 1[$ is $\{0, 1\}$.

Example 9.4.2. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $A = D^2$.



Suppose that (x, y) belongs to $\mathbb{R}^2 \setminus D^2$.



Let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon < \|(x, y)\| - 1.$$

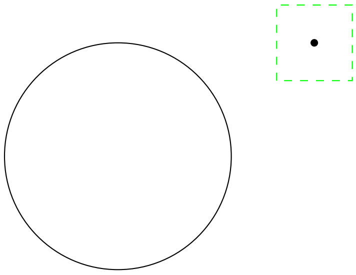
Let U_x be the open interval given by

$$\left] x - \tfrac{\epsilon\sqrt{2}}{\epsilon}, x + \tfrac{\epsilon\sqrt{2}}{\epsilon} \right[.$$

Let U_y be the open interval given by

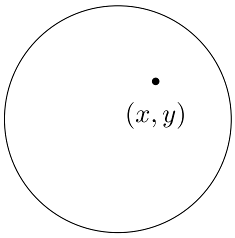
$$\left] y - \tfrac{\epsilon\sqrt{2}}{\epsilon}, y + \tfrac{\epsilon\sqrt{2}}{\epsilon} \right[.$$

Then $U_x \times U_y$ is a neighbourhood of (x, y) in $\mathbb{R}^2$ whose intersection with D^2 is empty.



This can be proven by the same argument as is needed to carry out Task E8.2.4. Thus (x, y) is not a limit point of D^2 in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. In particular, (x, y) does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$.

Suppose now that $(x, y) \in \mathbb{R}^2$ has the property that $\|(x, y)\| < 1$.



Let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon < 1 - \|(x, y)\|.$$

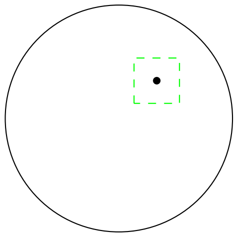
Let U_x be the open interval given by

$$\left] x - \tfrac{\epsilon\sqrt{2}}{\epsilon}, x + \tfrac{\epsilon\sqrt{2}}{\epsilon} \right[.$$

Let U_y be the open interval given by

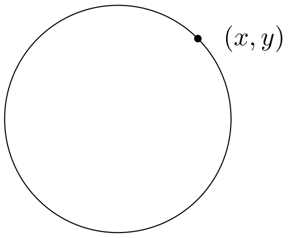
$$\left] y - \tfrac{\epsilon\sqrt{2}}{\epsilon}, y + \tfrac{\epsilon\sqrt{2}}{\epsilon} \right[.$$

Then $U_x \times U_y$ is a neighbourhood of (x, y) in $\mathbb{R}^2$ whose intersection with D^2 is empty. To check this is Task E9.2.2.

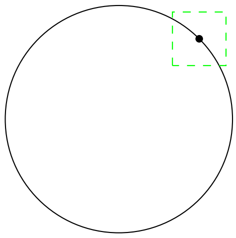


In other words, $\left(\mathbb{R}^2 \setminus D^2\right) \cap (U_x \times U_y)$ is empty. Thus (x, y) does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$.

Suppose now that $\|(x, y)\| = 1$. In other words, we have that (x, y) belongs to S^1.

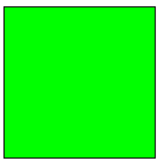


Every neighbourhood of (x, y) in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ contains an 'open rectangle' U to which (x, y) belongs. Both $D^2 \cap U$ and $\left(\mathbb{R}^2 \setminus D^2\right) \cap U$ are not empty.

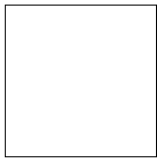


Thus (x, y) belongs to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$. To fill in the details of this argument is the topic of Task E9.2.4. We conclude that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$ is S^1.

Example 9.4.3. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $A = I^2$.

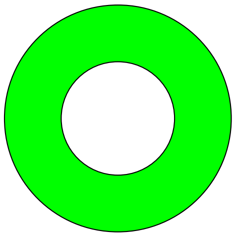


Then $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} I^2$ is the 'border around I^2'.



In other words, $\partial_{(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})} I^2$ is $(\{0,1\} \times I) \cup (I \times \{0,1\})$. To prove this is the topic of Task E9.2.3.

Example 9.4.4. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let A be an annulus A_k, for some $k \in \mathbb{R}$ with $0 < k < 1$, as in Notation 4.1.17.

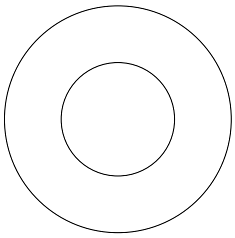


Then $\partial_{(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})} A_k$ is the union of the outer and the inner circle of A_k. In other words, the union of the set

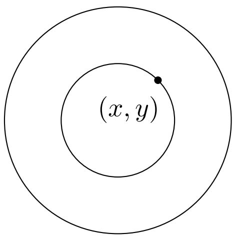
$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1\right\}$$

and the set

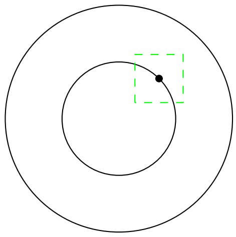
$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = k\right\}.$$



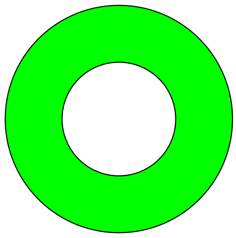
To prove this is the topic of Task E9.2.5. Suppose, for instance, that $(x, y) \in \mathbb{R}^2$ belongs to the inner circle of A_k.



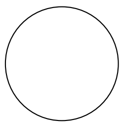
Every neighourhood of (x, y) contains an 'open rectangle' around (x, y) which overlaps both A_k and the open disc which we can think of as having been cut out from D^2 to obtain A_k.



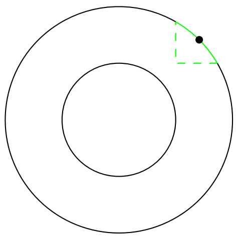
Example 9.4.5. Let $(X, \mathcal{O}_X)$ be $(D^2, \mathcal{O}_{D^2})$. Let A be an annulus A_k as in Example 9.4.4.



Then $\partial_{(D^2, \mathcal{O}_{D^2})} A_k$ is the inner circle of A_k.



To prove this is the topic of Task E9.2.6. In particular if $(x, y) \in S^1$ then, unlike in Example 9.4.4, (x, y) does not belong to $\partial_{(D^2, \mathcal{O}_{D^2})} A_k$. For there is a neighbourhood of (x, y) in D^2 with respect to $\mathcal{O}_{D^2}$ which does not overlap $D^2 \setminus A_k$, the open disc which we can think of as having cut out of D^2 to obtain A_k.



Example 9.4.4 and Example 9.4.5 demonstrate that given a set A, and a topological space $(X, \mathcal{O}_X)$ such that A is a subset of X, the boundary of A in X with respect to $\mathcal{O}_X$ depends upon $(X, \mathcal{O}_X)$. The next examples illustrate this further.

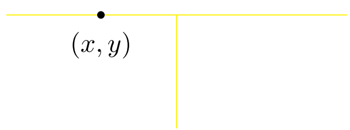
Example 9.4.6. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let T denote the subset of $\mathbb{R}^2$ given by the union of

$$\{(0, y) \mid 0 \leq y \leq 1\}$$

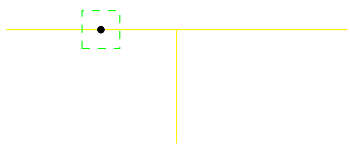
and

$$\{(x, 1) \mid -1 \leq x \leq 1\}.$$

Then $\partial_X \mathsf{T}$ is T. We have that T^2 is closed in $\mathbb{R}^2$. To prove this is the topic of Task E9.2.7. Thus every limit point of T^2 belongs to T. Suppose that (x, y) belongs to T.



Then the intersection with $\mathbb{R}^2 \setminus \mathsf{T}$ of every neighbourhood of (x, y) in $\mathbb{R}^2$ is not empty. To prove this is the topic of Task E9.2.8.



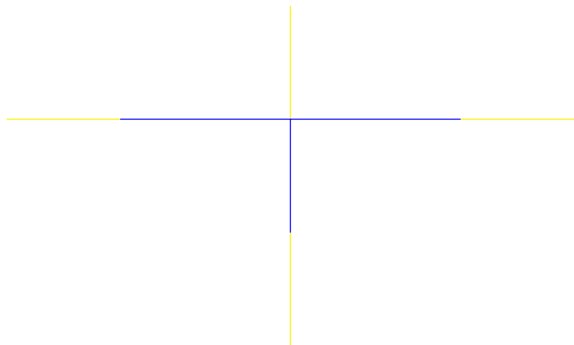
Example 9.4.7. Let X be the subset of $\mathbb{R}^2$ given by the union of

$$\{(0, y) \mid -1 \leq y \leq 2\}$$

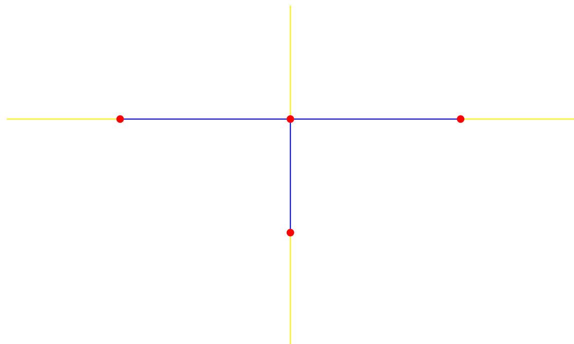
and

$$\{(x, 1) \mid -2 \leq x \leq 2\}.$$

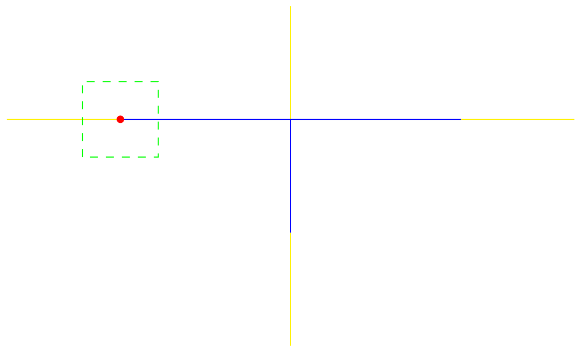
Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let T be as in Example 9.4.6.



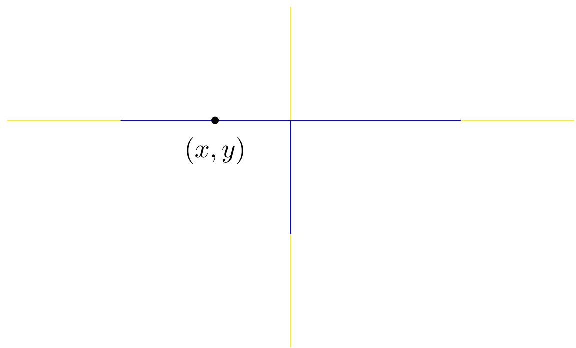
Then $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$ is $\{(-1,1),(0,1),(1,1),(0,0)\}$.



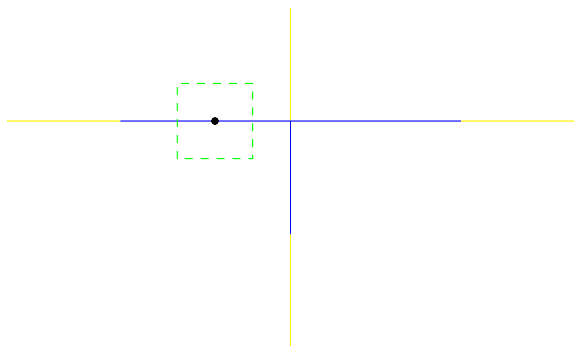
Every neighbourhood of each of these four points contains both a segment of $X \setminus \mathsf{T}$ and a segment of T. A typical neighbourhood of $(-1,1)$, for instance, is the intersection of an 'open rectangle' around $(-1,1)$ in $\mathbb{R}^2$ with T as depicted below.



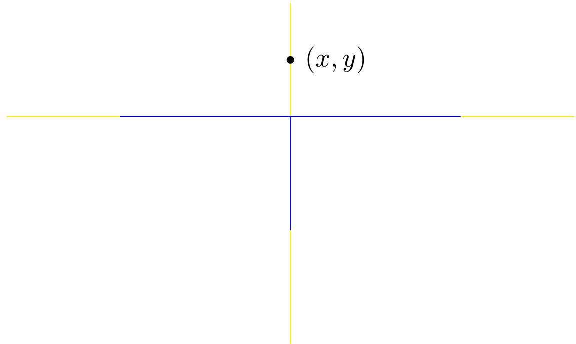
Let (x, y) be a point of T which is not one of these four.



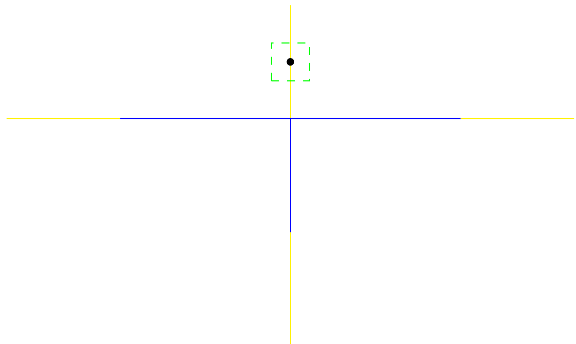
Then we can find a neighbourhood of (x, y) whose intersection with $X \setminus \mathsf{T}$ is empty. For instance, an intersection of a sufficiently small 'open rectangle' around (x, y) in $\mathbb{R}^2$ with X.



Suppose that $(x, y) \in X$ does not belong to T.



Then we can find a neighbourhood of (x, y) whose intersection with $X \setminus \mathsf{T}$ is empty. For instance, an intersection of a sufficiently small 'open rectangle' around (x, y) in $\mathbb{R}^2$ with X.



To fill in the details of this argument is the topic of Task E9.2.9.

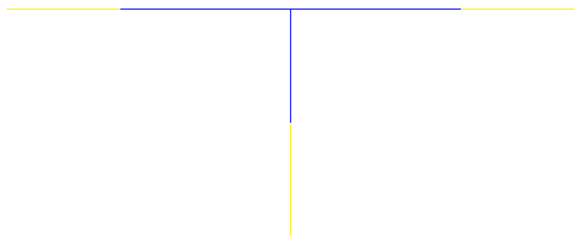
Example 9.4.8. Let X be the subset of $\mathbb{R}^2$ given by the union of

$$\{(0, y) \mid -2 \leq y \leq 1\}$$

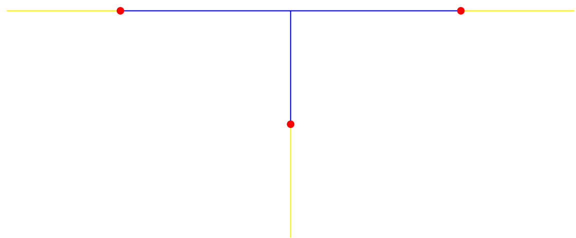
and

$$\{(x, 1) \mid -2 \leq x \leq 2\}.$$

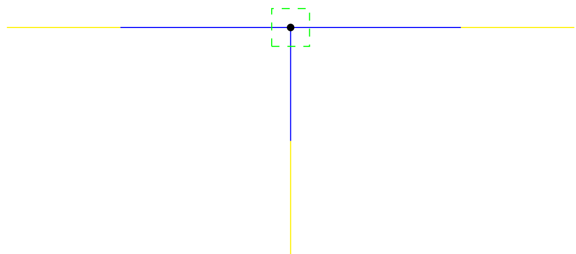
Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let T be as in Example 9.4.6.



Then $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$ is $\{(-1,1),(1,1),(0,0)\}$.



In particular $(0,1)$ does not belong to $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$, unlike in Example 9.4.7. We can find a neighbourhood of $(0,1)$ whose intersection with $X \setminus \mathsf{T}$ is empty, such as the intersection of a sufficiently small 'open rectangle' around $(0,1)$ in $\mathbb{R}^2$ with X.



To give the details of the calculation of $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$ is the topic of Task E9.2.10.

9.5. Connected topological spaces

Terminology 9.5.1. Let X be a set. Let X_0 and X_1 be subsets of X. The union $X_0 \cup X_1$ of X_0 and X_1 is *disjoint* if $X_0 \cap X_1$ is the empty set.

Notation 9.5.2. Let X be a set. Let X_0 and X_1 be subsets of X. If $X = X_0 \cup X_1$, and this union is disjoint, we write $X = X_0 \sqcup X_1$.

Definition 9.5.3. A topological space $(X, \mathcal{O}_X)$ is *connected* if there do not exist subsets X_0 and X_1 of X such that the following hold.

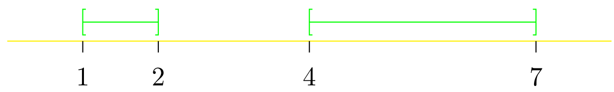
(1) Neither X_0 nor X_1 is empty, and both belong to $\mathcal{O}_X$.

(2 We have that $X = X_0 \sqcup X_1$.

9.6. An example of a topological space which is not connected

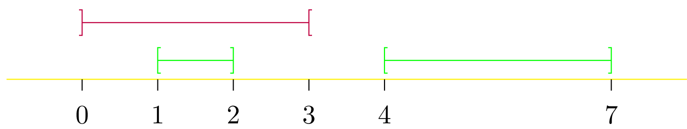
Remark 9.6.1. We shall have to work quite hard to prove that any of our geometric examples of topological spaces are connected. Instead, we shall begin with some examples of topological spaces which are not connected.

Example 9.6.2. Let $X = [1,2] \cup [4,7]$.



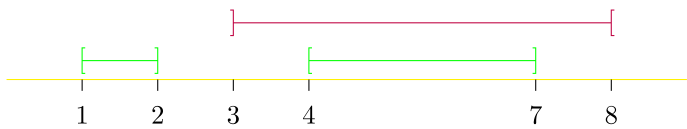
Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The following hold.

(1) By Example 1.6.3, we have that $]0,3[$ belongs to $\mathcal{O}_{\mathbb{R}}$. We have that $[1,2] = X \cap\,]0,3[$.



By definition of $\mathcal{O}_X$, we deduce that $[1,2]$ belongs to $\mathcal{O}_X$.

(2) By Example 1.6.3, we have that $]3,8[$ belongs to $\mathcal{O}_{\mathbb{R}}$. We have that $[4,7] = X \cap\,]3,8[$.



By definition of $\mathcal{O}_X$, we conclude that $[4,7]$ belongs to $\mathcal{O}_X$.

(3) We have that $X = [1,2] \sqcup [4,7]$, since $[1,2] \cap [4,7]$ is empty.

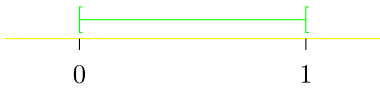
We conclude that $(X, \mathcal{O}_X)$ is not connected.

Remark 9.6.3. In (1), we could have chosen instead of $]0,3[$ any subset of $\mathbb{R}$ which belongs to $\mathcal{O}_{\mathbb{R}}$, which does not intersect $[4,7]$, and of which $[1,2]$ is a subset. In (2), we could have chosen instead of $]3,8[$ any subset of $\mathbb{R}$ which belongs to $\mathcal{O}_{\mathbb{R}}$, which does not intersect $[1,2]$, and of which $[4,7]$ is a subset.

E9. Exercises for Lecture 9

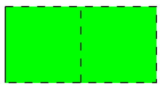
E9.1. Exam questions

Task E9.1.1. We saw in Example 9.4.1 that $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}\,[0,1[$ is $\{0,1\}$.



Prove that $\{0,1\}$ is also the boundary of each of $]0,1[$, $]0,1]$, and $[0,1]$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Task E9.1.2. Let A be the subset of $\mathbb{R}^2$ given by the union of $]0,1[\times]0,1[$ and $[-1,0[\times\,]0,1[$.



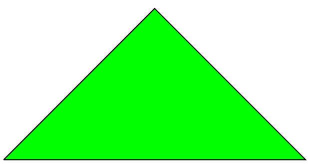
(1) What is the boundary of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$?

(2) What is the boundary of A in $\mathbb{R}\times\,]0,\infty[$, where $\mathbb{R}\times\,]0,\infty[$ is equipped with the subspace topology with respect to $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$?

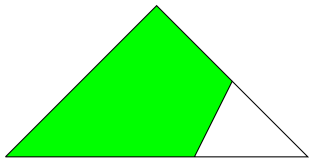
(3) Let X be the union of $]-\infty,0[\times\mathbb{R}$ and $]0,\infty[\times\mathbb{R}$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$. What is the boundary of A in X with respect to $\mathcal{O}_X$?

Task E9.1.3. Let $(X,\mathcal{O}_X)$ be as in Task E8.1.2. What is the boundary of $\{a,c\}$ in X with respect to $\mathcal{O}_X$? What is the boundary of $\{b,c\}$ in X with respect to $\mathcal{O}_X$? What is the boundary of $\{d\}$ in X with respect to $\mathcal{O}_X$?

Task E9.1.4. Let X be the subset of $\mathbb{R}^2$ which is a 'solid triangle'.

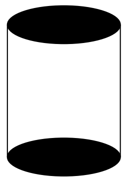


Let A be the subset of X depicted below. All of the lines, and the entire shaded area, belong to A.



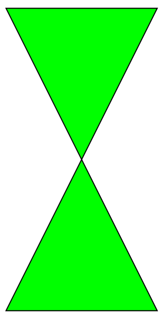
In other words, A is obtained from X by cutting out the inside of smaller 'solid triangle' inside it. What is $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} A$? What is $\partial_{(X, \mathcal{O}_X)} A$, where $\mathcal{O}_X$ is the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$?

Task E9.1.5. What is the boundary of $D^2 \times I$ in $\mathbb{R}^3$ with respect to $\mathcal{O}_{\mathbb{R}^3}$?

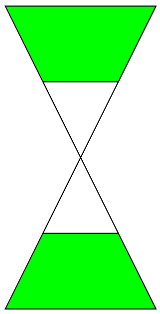


Give a proof by appealing to Example 9.4.2, Task E9.1.1, and Task E9.3.11. What is the boundary of $D^2 \times I$ in $\mathbb{R}^2 \times I$ with respect to $\mathcal{O}_{\mathbb{R}^2 \times I}$? What is the boundary of $D^2 \times I$ in $D^2 \times \mathbb{R}$ with respect to $\mathcal{O}_{D^2 \times \mathbb{R}}$?

Task E9.1.6 (Continuation exam, August 2013)**.** Let X be a subset of $\mathbb{R}^2$ as depicted below. In other words, we have two triangles which 'meet at their tips'.



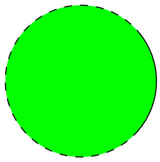
Let A be the subset of X obtained by removing the inside of a smaller copy of this shape, as depicted below. All of the lines, and the entirety of both shaded areas, belong to A.



What is $\partial_{(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})}A$? What is $\partial_{(X,\mathcal{O}_X)}A$?

Task E9.1.7. What is the boundary of $\mathbb{Q}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$?

Task E9.1.8. Let A be the subset of D^2 of Task E8.1.4.

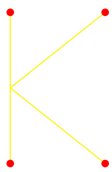


What is the boundary of A in D^2 with respect to $\mathcal{O}_{D^2}$?

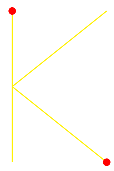
Task E9.1.9. View the letter K as a subset of $\mathbb{R}^2$.

For each of the following, find a subset X of $\mathbb{R}^2$ such that K is a subset of X, and such that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ is as described, where $\mathcal{O}_X$ denotes the subspace topology on X with respect to $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$.

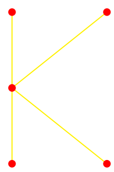
(1) We have that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ consists of the four points depicted below.



(2) We have that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ consists of the two points depicted below.



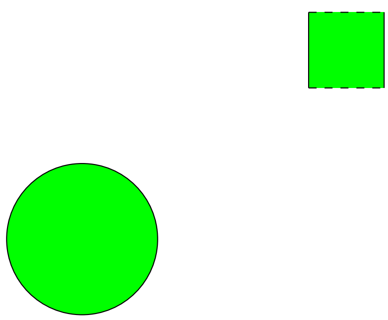
(3) We have that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ consists of the five points depicted below.



(4) We have that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ consists of the union of the two lines depicted below.

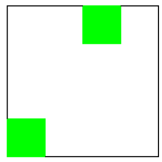
Task E9.1.10. Let $(X, \mathcal{O}_X)$ be a topological space. Explain why $\partial_{(X,\mathcal{O}_X)}X$ is the empty set.

Task E9.1.11. Let X be the union of D^2 and $[3,4] \times]2,3[$.



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that $(X, \mathcal{O}_X)$ is not connected.

Task E9.1.12. Let X be the subset of I^2 given by the union of $\left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{4}\right]$ and $\left[\frac{1}{2}, \frac{3}{4}\right] \times \left[\frac{3}{4}, 1\right]$.



Let

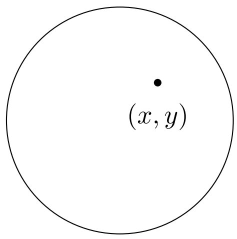
$$I^2 \xrightarrow{\pi} T^2$$

be the quotient map. Let $\mathcal{O}_{\pi(X)}$ denote the subspace topology on $\pi(X)$ with respect to $(T^2, \mathcal{O}_{T^2})$. Prove that $\left(\pi(X), \mathcal{O}_{\pi(X)}\right)$ is not connected.

E9.2. In the lecture notes

Task E9.2.1. Do the same as in Task E2.2.2 for the proof of Proposition 9.1.1.

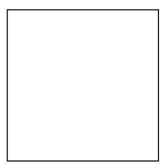
Task E9.2.2. Let $(x, y) \in \mathbb{R}^2$ be such that $\|(x, y)\| < 1$.



Prove that (x, y) does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$, following the argument outlined in Example 9.4.2. You may find it helpful to look back at Example 3.2.3.

Task E9.2.3. It was asserted in Example 9.4.3 that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} I^2$ is

$$(\{0,1\} \times I) \cup (I \times \{0,1\})\,.$$



Prove this first as follows, along the lines of Example 9.4.2.

(1) Demonstrate that if $(x, y) \in \mathbb{R}^2$ does not belong to I^2, then (x, y) is not a limit point of I^2 in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

(2) Demonstrate that if $0 < x < 1$ and $0 < y < 1$, then there is a neighbourhood U of (x, y) in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ such that $\left(\mathbb{R}^2 \setminus I^2\right) \cap U$ is empty.

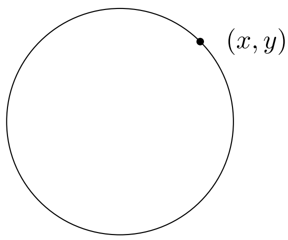
(3) Demonstrate that if (x, y) belongs to

$$(\{0, 1\} \times I) \cup (I \times \{0, 1\}),$$

then every neighbourhood U of (x, y) in $\mathbb{R}^2$ has the property that both $I^2 \cap U$ and $\left(\mathbb{R}^2 \setminus I^2\right) \cap U$ are not empty.

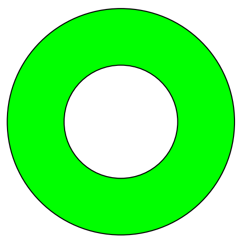
Give a second proof by appealing to Task E9.3.11. Give a third proof by appealing to Task E7.2.9 and Task E9.3.12.

Task E9.2.4. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.8. Suppose that $(x, y) \in \mathbb{R}^2$ belongs to S^1.



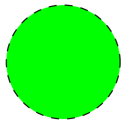
Prove that (x, y) belongs to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$.

Task E9.2.5. Let A be an annulus A_k, for some $k \in \mathbb{R}$ with $0 < k < 1$, as in Notation 4.1.17.



Prove that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} A_k$ is the union of the outer and the inner circle of the annulus, as claimed in Example 9.4.4. You may wish to proceed as follows.

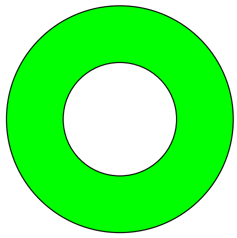
(1) Let B be the 'open disc' of radius k centred at $(0, 0)$.



Demonstrate that $\partial_{(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})}B$ is the circle of radius k centred at $(0,0)$.

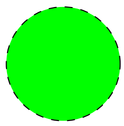
(2) Observing that A_k is $D^2 \setminus B$, appeal to Example 9.4.2 and Task E9.3.15.

Task E9.2.6. Let A be an annulus A_k, for some $k \in \mathbb{R}$ with $0 < k < 1$, as in Notation 4.1.17.



Prove that $\partial_{(D^2,\mathcal{O}_{D^2})}A_k$ is the inner circle of the annulus, as claimed in Example 9.4.5. You may wish to proceed as follows.

(1) Let B be the 'open disc' of radius k centred at $(0,0)$.



Appealing to (1) of Task E9.2.5and Task E9.3.13, observe that $\partial_{(D^2,\mathcal{O}_{D^2})}B$ is the circle of radius k centred at $(0,0)$.

(2) Appeal to Task E9.1.10 and Task E9.3.15.

Task E9.2.7. Let T be the subset of $\mathbb{R}^2$ of Example 9.4.6.

Prove that T is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. You may wish to proceed as follows.

(1) Observe that T is the union of $\{0\} \times [0,1]$ and $[0,1] \times \{1\}$.

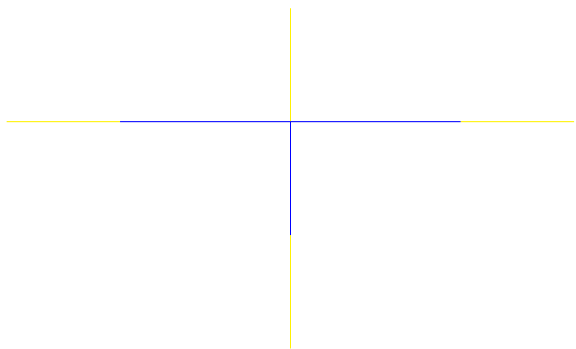
(2) Appealing to Task E3.3.1, observe that $\{0\} \times [0,1]$ and $[0,1] \times \{1\}$ are both closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

(3) Appealing to Task E9.3.5, conclude from (1) and (2) that T is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

Task E9.2.8. Let T be the subset of $\mathbb{R}^2$ of Example 9.4.6.

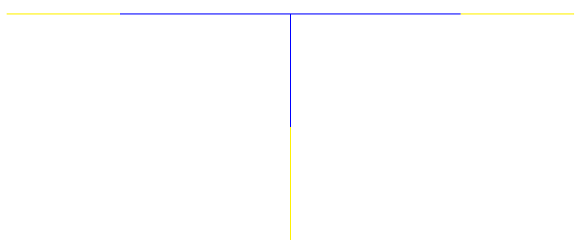
Prove that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} \mathsf{T}$ is T. You may wish to follow the argument outlined in Example 9.4.6.

Task E9.2.9. Let $(X, \mathcal{O}_X)$ and T be as in Example 9.4.7.



Prove that $\partial_{(X, \mathcal{O}_X)} \mathsf{T}$ is $\{(-1,1), (0,1), (1,1), (0,0)\}$. You may wish to follow the argument outlined in Example 9.4.7.

Task E9.2.10. Let $(X, \mathcal{O}_X)$ and T be as in Example 9.4.8.



Prove that $\partial_{(X, \mathcal{O}_X)} \mathsf{T}$ is $\{(-1,1), (1,1), (0,0)\}$.

E9.3. For a deeper understanding

Task E9.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let V be a subset of X which is closed with respect to $\mathcal{O}_X$. Let A be a subset of V. Prove that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is a subset of V. You may wish to appeal to Proposition 9.1.1.

Task E9.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Prove that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is equal to the intersection of all subsets V of X with the following properties.

(1) V is closed with respect to $\mathcal{O}_X$.

(2) A is a subset of V.

You may wish to appeal to Task E9.3.1.

Corollary E9.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Then $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is closed.

Proof. Follows immediately from Task E9.3.2 and the fact, observed as part of Remark E1.3.2, that an intersection of (possibly infinitely many) subsets of X which are closed with respect to $\mathcal{O}_X$ is closed with respect to $\mathcal{O}_X$. $\square$

Remark E9.3.4. In other words, $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is the smallest subset of X which contains A, and which is closed with respect to $\mathcal{O}_X$.

Task E9.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. Let A and B be subsets of X. Prove that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A \cup B)$ is $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cup \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$. You may wish to proceed as follows.

(1) By Corollary E9.3.3, we have $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ and $\mathsf{cl}_{(X,\mathcal{O}_X)}(B)$ are closed with respect to $\mathcal{O}_X$. By Remark E1.3.2, we thus have that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cup \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$ is closed with respect to $\mathcal{O}_X$. Deduce by Task E9.3.1 that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A \cup B)$ is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cup \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$.

(2) Observe that if $x \in X$ is a limit point of A or B in X with respect to $\mathcal{O}_X$, then x is a limit point of $A \cup B$ in X with respect to $\mathcal{O}_X$.

Task E9.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{A_i\}_{i \in I}$ be an infinite set of subsets of X. Give an example to demonstrate that $\mathsf{cl}_{(X,\mathcal{O}_X)}(\cup_{i \in I} A_i)$ is not necessarily $\cup_{i \in I} \mathsf{cl}_{(X,\mathcal{O}_X)}(A_i)$.

Task E9.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Let A and B be subsets of X. Prove that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A \cap B)$ is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$.

Task E9.3.8. Let $(X, \mathcal{O}_X)$ be a topological space. Let A and B be subsets of X. Give an example to demonstrate that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$ is not necessarily a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A \cap B)$. In particular, these sets are not necessarily equal.

Task E9.3.9. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a map. Prove that f is continuous if and only if for every subset A of X, we have that $f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}(A)\right)$ is a subset of $\mathsf{cl}_{(Y,\mathcal{O}_Y)}(f(A))$. You may wish to proceed as follows.

(1) Suppose that the condition holds. Let V be a subset of Y which is closed with respect to $\mathcal{O}_Y$. By one of the relations of Table A.2, observe that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)$ is a subset of

$$f^{-1}\left(f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)\right)\right).$$

(2) By hypothesis, we have that $f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)\right)$ is a subset of

$$\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right).$$

By one of the relations of Table A.2, deduce that

$$f^{-1}\left(f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)\right)\right)$$

is a subset of

$$f^{-1}\left(\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)\right).$$

(3) By (1) and (2), deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)$ is a subset of

$$f^{-1}\left(\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)\right).$$

(4) By one of the relations of Table A.2, observe that $f\left(f^{-1}(V)\right)$ is a subset of V. By Task E8.3.12, deduce that $\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)$ is a subset of $\mathsf{cl}_{(Y,\mathcal{O}_Y)}(V)$.

(5) Since V is closed in Y with respect to $\mathcal{O}_Y$, we have by Proposition 9.1.1 that $V = \mathsf{cl}_{(Y,\mathcal{O}_Y)}(V)$. By (4), deduce that $\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)$ is a subset of V.

(6) By (5) and one of the relations of Table A.2, deduce that $f^{-1}\left(\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)\right)$ is a subset of $f^{-1}(V)$.

(7) By (3) and (6), deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)$ is a subset of $f^{-1}(V)$.

(8) By Remark 8.5.4, we have that $f^{-1}(V)$ is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)$. By (7), deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right) = V$.

(9) By Proposition 9.1.1, deduce that $f^{-1}(V)$ is closed in X with respect to $\mathcal{O}_X$. By Task **??**, conclude that f is continuous.

(10) Conversely, suppose that f is continuous. Suppose that x is a limit point of A in X with respect to $\mathcal{O}_X$. Let $U_{f(x)}$ be a neighbourhood of $f(x)$ in Y with respect to $\mathcal{O}_Y$. Since f is continuous, observe that, by Task E8.3.3, there is a neighbourhood U_x of x in X with respect to $\mathcal{O}_X$ such that $f(U_x)$ is a subset of $U_{f(x)}$.

(11) Since x is a limit point of A in X with respect to $\mathcal{O}_X$, we have that $U_x \cap A$ is not empty. Thus $f(U_x \cap A)$ is not empty. Since $f(U_x \cap A)$ is a subset of $f(U_x) \cap f(A)$, deduce that $f(U_x) \cap f(A)$ is not empty.

(12) Since $f(U_x)$ is a subset of $U_{f(x)}$, deduce that $U_{f(x)} \cap f(A)$ is not empty.

(13) Conclude that $f(x)$ is a limit point of $f(A)$ in Y with respect to $\mathcal{O}_Y$. Thus $f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}(A)\right)$ is a subset of $\mathsf{cl}_{(Y,\mathcal{O}_Y)}(f(A))$.

Task E9.3.10. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Prove that $\partial_{(X,\mathcal{O}_X)}A$ is the intersection of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ and $\mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus A)$.

Task E9.3.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of X, and let B be a subset of Y. Prove that $\partial_{(X \times Y, \mathcal{O}_{X \times Y})} A \times B$ is the union of

$$\left(\partial_{(X,\mathcal{O}_X)}A\right) \times \mathsf{cl}_{(Y,\mathcal{O}_Y)}(B)$$

and

$$\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \times \left(\partial_{(Y,\mathcal{O}_Y)}B\right).$$

For proving that $\partial_{(X \times Y, \mathcal{O}_{X \times Y})} A \times B$ is a subset of this union, you may wish to make use of one of the set theoretic equalities listed in Remark A.1.1.

Task E9.3.12. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Let A be a subset of X. Prove that $\partial_{(Y,\mathcal{O}_Y)}f(A)$ is $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$. You may wish to proceed as follows.

(1) Suppose that y belongs to $\partial_{(Y,\mathcal{O}_Y)}f(A)$. Let U be a neighbourhood of $f^{-1}(y)$ in X with respect to $\mathcal{O}_X$. Observe that since f is a homeomophism, $f(U)$ is a neighbourhood of y in Y with respect to $\mathcal{O}_Y$.

(2) We have that

$$f^{-1}\left(f(A) \cap f(U)\right) = f^{-1}\left(f(A)\right) \cap f^{-1}\left(f(U)\right).$$

Since f is a bijection, we have that $f^{-1}(f(A)) = A$, and that $f^{-1}(f(U)) = U$. Deduce that

$$f^{-1}\left(f(A) \cap f(U)\right) = A \cap U.$$

(3) Observe that by (1) and the fact that y belongs to $\partial_{(Y,\mathcal{O}_Y)}f(A)$, we have that $f(A) \cap f(U)$ is not empty. Conclude by means of (2) that $A \cap U$ is not empty.

(4) We have that

$$\begin{aligned} f^{-1}\left((Y \setminus f(A)) \cap f(U)\right) &= f^{-1}\left(Y \setminus f(A)\right) \cap f^{-1}\left(f(U)\right) \\ &= \left(X \setminus f^{-1}\left(f(A)\right)\right) \cap f^{-1}\left(f(U)\right). \end{aligned}$$

In a similar manner as in (2) and (3), deduce that $(X \setminus A) \cap U$ is not empty.

(5) Observe that (2) and (3) demonstrate that $f^{-1}(y)$ belongs to $\partial_{(X,\mathcal{O}_X)}A$. Conclude that y belongs to $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$. Thus we have proven that $\partial_{(Y,\mathcal{O}_Y)}f(A)$ is a subset of $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$.

(6) Suppose now that x belongs to $\partial_{(X,\mathcal{O}_X)}A$. Let U be a neighbourhood of $f(x)$ in Y with respect to $\mathcal{O}_Y$. Observe that then $f^{-1}(U)$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$.

(7) Since x belongs to $\partial_{(X,\mathcal{O}_X)}A$, we have that $A \cap f^{-1}U$ is not empty. Thus

$$f\left(A \cap f^{-1}(U)\right)$$

is not empty. We have that $f\left(A \cap f^{-1}(U)\right)$ is a subset of

$$f(A) \cap f\left(f^{-1}(U)\right).$$

Since f is a surjection, we also have that $f\left(f^{-1}(U)\right) = U$. Deduce that $f(A) \cap U$ is not empty.

(8) Since x belongs to $\partial_{(X,\mathcal{O}_X)}A$, we have that $(X \setminus A) \cap f^{-1}U$ is not empty. Observe that since f is a bijection, we have that $f(X \setminus A) = Y \setminus f(A)$. In a similar manner as in (6), deduce that $(Y \setminus f(A)) \cap U$ is not empty.

(9) Observe that (7) and (8) demonstrate that $f(x)$ belongs to $\partial_{(Y,\mathcal{O}_Y)}f(A)$. Thus we have proven that $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$ is a subset of $\partial_{(Y,\mathcal{O}_Y)}f(A)$.

(10) Conclude from (5) and (9) that $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$ is $\partial_{(Y,\mathcal{O}_Y)}f(A)$.

Task E9.3.13. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Let B be a subset of A which belongs to $\mathcal{O}_X$. Prove that $\partial_{(A,\mathcal{O}_A)}B$ is $A \cap \partial_{(X,\mathcal{O}_X)}B$. You may wish to proceed as follows.

(1) By Task E9.3.10, we have that $\partial_{(A,\mathcal{O}_A)}B$ is the intersection of $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$ and $\mathsf{cl}_{(A,\mathcal{O}_A)}(A \setminus B)$.

(2) Observe that since B belongs to $\mathcal{O}_X$, and since B is a subset of A, we have that B belongs to $\mathcal{O}_A$. Thus $A \setminus B$ is closed in A with respect to $\mathcal{O}_A$. Deduce by Proposition 9.1.1 that $\mathsf{cl}_{(A,\mathcal{O}_A)}(A \setminus B)$ is $A \setminus B$.

(3) Since B belongs to $\mathcal{O}_X$, we have that $X \setminus B$ is closed in X with respect to $\mathcal{O}_X$. Deduce by Proposition 9.1.1 that $\mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus B)$ is $X \setminus B$. Thus $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus B)$ is $A \setminus B$.

(4) Observe that by (2) and (3), we have that $\mathsf{cl}_{(A,\mathcal{O}_A)}(A \setminus B)$ is $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus B)$.

(5) Observe that by Task E8.3.13, we have that $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$ is $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$.

(6) By (1), (4), and (5), conclude that $\partial_{(A,\mathcal{O}_A)}B$ is $A \cap \left(\mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus B) \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)\right)$.

(7) Conclude by Task E9.3.10 that $\partial_{(A,\mathcal{O}_A)}B$ is $A \cap \partial_{(X,\mathcal{O}_X)}B$.

Task E9.3.14. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let B be a subset of A. Prove that $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$ is a subset of the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$. You may wish to proceed as follows.

(1) Suppose that x belongs to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$. Suppose first that every neighbourhood U of x in X with respect to $\mathcal{O}_X$ has the property that $(X \setminus A) \cap U$ is not empty. Since x belongs $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$, we also have that $(A \setminus B) \cap U$ is not empty. In particular, $A \cap U$ is not empty. Deduce that x belongs to $\partial_{(X,\mathcal{O}_X)}A$.

(2) Suppose instead that there is a neighbourhood U of x in X with respect to $\mathcal{O}_X$ such that $(X \setminus A) \cap U$ is empty. We have that $X \setminus (A \setminus B)$ is the union of $X \setminus A$ and B. Since x belongs $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$, we have that $(X \setminus (A \setminus B)) \cap U$ is not empty. Deduce that $B \cap U$ is not empty.

(3) Let U' be any neighbourhood of x in X with respect to $\mathcal{O}_X$. Suppose that $B \cap U'$ is empty. We have that $U \cap U'$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$. Moreover, observe that $(X \setminus A) \cap (U \cap U')$ is empty, and that $B \cap (U \cap U')$ is empty. Conclude that $B \cap U'$ is not empty.

(4) Since x belongs to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$, we have that $(A \setminus B) \cap U'$ is not empty. In particular, we have that $(X \setminus B) \cap U'$ is not empty.

(5) Observe that, by (2) – (4), if there is a neighbourhood U of x in X with respect to $\mathcal{O}_X$ such that $(X \setminus A) \cap U$ is empty, then x belongs to $\partial_{(X,\mathcal{O}_X)}B$.

(6) Observe that, by (1) and (5), we have that $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$ is a subset of the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$.

Task E9.3.15. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X which is closed with respect to $\mathcal{O}_X$. Let B be a subset of X which belongs to $\mathcal{O}_X$. Prove that $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$ is the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$. You may wish to proceed as follows.

(1) Observe that, by Task E9.3.14, we have that $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$ is a subset of the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$.

(2) Since B is a subset of A, we have that $X \setminus A$ is a subset of $X \setminus B$. Deduce that $(X \setminus A) \cap B$ is empty.

(3) Suppose that $x \in X$ belongs to B. Then B is a neighbourhood of x in X with respect to $\mathcal{O}_X$. Deduce by (2) that x does not belong to $\partial_{(X,\mathcal{O}_X)}A$.

(4) Suppose that $x \in X$ belongs to $X \setminus A$. Since A is closed in X with respect to $\mathcal{O}_X$, we then have that $X \setminus A$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$. Deduce that x does not belong to $\partial_{(X,\mathcal{O}_X)}A$.

(5) Suppose that x belongs to $\partial_{(X,\mathcal{O}_X)}A$. By (3) and (4), we have that x belongs to $A \setminus B$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$. Observe that since x belongs to $A \setminus B$, we have that $(A \setminus B) \cap U$ is not empty.

(6) Since x belongs to $\partial_{(X,\mathcal{O}_X)}A$, we also have that $(X \setminus A) \cap U$ is not empty. We have that $X \setminus (A \setminus B)$ is the union of $X \setminus A$ and B. Deduce that $(X \setminus (A \setminus B)) \cap U$ is not empty.

(7) Conclude from (5) and (6) that if x belongs to $\partial_{(X,\mathcal{O}_X)}A$, then x belongs to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$.

(8) Arguing in a similar way, prove that if x belongs to $\partial_{(X,\mathcal{O}_X)}B$, then x belongs to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$.

(9) By (7) and (8), we have that the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$ is a subset of $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$. Conclude by (1) that the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$ is equal to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$.

E9.4. Exploration — limit points in a metric space

Task E9.4.1. Let (X, d) be a metric space. Let $\mathcal{O}_d$ be the topology on X corresponding to d of Task E3.4.9. Let A be a subset of X. Suppose that x belongs to X. Prove that x is a limit point of A in X with respect to $\mathcal{O}_d$ if and only if, for every $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, there is an a which belongs to A such that $d(x, a) < \epsilon$. You may wish to proceed as follows.

(1) Suppose that x is a limit point of A in X with respect to $\mathcal{O}_d$. By Task E4.3.2, we have that $B_\epsilon(x)$ is a neighourhood of x in X with respect to $\mathcal{O}_d$. Deduce that $A \cap B_\epsilon(x)$ is not empty, and thus that there is an a which belongs to A such that $d(x, a) < \epsilon$.

(2) Suppose instead that, for every $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, there is an a which belongs to A such that $d(x, a) < \epsilon$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_d$. By definition of $\mathcal{O}_d$, there is a $\zeta \in \mathbb{R}$ with $\epsilon > 0$ such that $B_\zeta(x)$ is a subset of U. By assumption, there is an a in A such that a belongs to $B_\zeta(x)$. Deduce that $A \cap U$ is not empty. Conclude that that x is a limit point of A in X with respect to $\mathcal{O}_d$.

Task E9.4.2. Let (X, d) be a metric space. Let A be a subset of X. Suppose that x belongs to X. Let X be equipped with the topology $\mathcal{O}_d$ corresponding to d of Task E3.4.9. Prove that if A is closed in X with respect to $\mathcal{O}_d$, then $d(x, A) > 0$ for every x which does not belong to A. You may wish to proceed as follows.

(1) Since A is closed in X with respect to $\mathcal{O}_d$, we have, by Proposition 9.1.1, that x is not a limit point of A in X with respect to $\mathcal{O}_X$. By Task E9.4.1, deduce that there is an $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ such that $d(x, a) \geq \epsilon$ for all a which belong to A.

(2) Deduce that $d(x, A) \geq \epsilon$, and thus that $d(x, A) > 0$.

10. Tuesday 4th February

10.1. Connectedness in finite examples

Example 10.1.1. Let $X = \{a, b\}$ be a set with two elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{b\}, X\}.$$

The only way to express X as a disjoint union of subsets which are not empty is:

$$X = \{a\} \sqcup \{b\}.$$

However, $\{a\}$ does not belong to $\mathcal{O}_X$. We conclude that $(X, \mathcal{O}_X)$ is connected.

Example 10.1.2. Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{c, d, e\}, \{a, b, c, d\}, \{a, c, d, e\}, X\}.$$

The following hold.

(1) We have that $X = \{a, b\} \sqcup \{c, d, e\}$.

(2) Both $\{a, b\}$ and $\{c, d, e\}$ belong to $\mathcal{O}_X$.

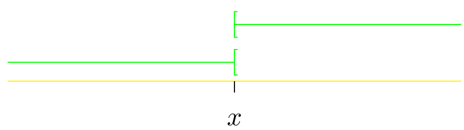
We conclude that $(X, \mathcal{O}_X)$ is not connected.

10.2. $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not connected

Example 10.2.1. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathcal{O}_{\mathbb{Q}}$ denote the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $x \in \mathbb{R}$ be irrational. For instance, we can take x to be $\sqrt{2}$. The following hold.

(1) Since x is irrational, we have that

$$\mathbb{Q} = (\mathbb{Q} \cap]-\infty, x[) \sqcup (\mathbb{Q} \cap]x, \infty[).$$



(2) By Example 1.6.3, we have that $]-\infty, x[$ belongs to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{Q}}$, we deduce that $\mathbb{Q} \cap]-\infty, x[$ belongs to $\mathcal{O}_{\mathbb{Q}}$.

(3) By Example 1.6.3, we have that $]x, \infty[$ belongs to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{Q}}$, we thus have that $\mathbb{Q} \cap]x, \infty[$ belongs to $\mathcal{O}_{\mathbb{Q}}$.

We conclude that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not connected.

10.3. A characterisation of connectedness

Proposition 10.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{0,1\}$ be equipped with the discrete topology. A topological space $(X, \mathcal{O}_X)$ is connected if and only if there does not exist a surjective, continuous map

$$X \longrightarrow \{0,1\}.$$

Proof. Suppose that there exists a surjective continuous map

$$X \xrightarrow{f} \{0,1\}.$$

The following hold.

(1) Both $\{0\}$ and $\{1\}$ belong to the discrete topology on $\{0,1\}$. Since f is continuous, we thus have that both $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ belong to $\mathcal{O}_X$.

(2) Since f is surjective, neither $f^{-1}(\{0\})$ nor $f^{-1}(\{1\})$ is empty.

(3) We have that

$$\begin{aligned} f^{-1}(\{0\}) \cup f^{-1}(\{1\}) &= f^{-1}(\{0,1\}) \\ &= X. \end{aligned}$$

(4) We have that

$$f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = \{x \in X \mid f(x) = 0 \text{ and } f(x) = 1\}.$$

Since f is a well-defined map, the set

$$\{x \in X \mid f(x) = 0 \text{ and } f(x) = 1\}$$

is empty. We deduce that

$$f^{-1}(\{0\}) \cap f^{-1}(\{1\})$$

is empty.

By (3) and (4), we have that

$$X = f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}).$$

We conclude, by (1) and (2), that $(X, \mathcal{O}_X)$ is not connected.

Conversely, suppose that $(X, \mathcal{O}_X)$ is not connected. Then there are subsets X_0 and X_1 of X with the following properties.

(1) Neither X_0 nor X_1 is empty, and both belong to $\mathcal{O}_X$.

(2) We have that $X = X_0 \sqcup X_1$.

Let

$$X \xrightarrow{f} \{0,1\}$$

be the map given by

$$\begin{cases} x \mapsto 0 & \text{if } x \in X_0, \\ x \mapsto 1 & \text{if } x \in X_1. \end{cases}$$

By (2), we have that f is well-defined. Since neither X_0 nor X_1 is empty, we have that f is surjective. Moreover we have that $f^{-1}(\{0\}) = X_0$, and that $f^{-1}(\{1\}) = X_1$. Since both X_0 and X_1 belong to $\mathcal{O}_X$, we deduce that f is continuous. □

Remark 10.3.2. For theoretical purposes, it is often very powerful to have a characterisation of a mathematical concept in terms of maps. We shall see that Proposition 10.3.1 is very useful for carrying out proofs involving connected topological spaces.

10.4. $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected

Proposition 10.4.1. The topological space $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

Remark 10.4.2. This is one of the most important facts in the course! It is a 'low-level' result, which relies fundamentally on the completeness of $\mathbb{R}$. Task E10.2.1 guides you through a proof.

To put it another way, Proposition 10.4.1 is the bridge between set theory and topology upon which connectedness rests. After we have proven it, we shall not need again to work in a 'low-level' way with $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ in matters concerning connectedness. We shall be able to argue entirely topologically.

Remark 10.4.3. Nevertheless Proposition 10.4.1 is intuitively clear. Something would be wrong with our notion of a connected topological space if it did not hold! It is for this very reason that Proposition 10.4.1 requires a 'low-level' proof. We have to think very carefully about how our intuitive understanding that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected can be captured rigorously within the framework in which we are working.

10.5. Continuous surjections with a connected source

Proposition 10.5.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is connected. Suppose that there exists a continuous, surjective map

$$X \xrightarrow{f} Y.$$

Then $(Y, \mathcal{O}_Y)$ is connected.

Proof. Let $\{0,1\}$ be equipped with the discrete topology. Suppose that

$$Y \xrightarrow{g} \{0,1\}$$

is a continuous, surjective map. Since f is continuous, we have by Proposition 5.3.1 that

$$X \xrightarrow{g \circ f} \{0,1\}$$

is continuous. Since f is surjective, we moreover have that $g \circ f$ is surjective. By Proposition 10.3.1, this contradicts our hypothesis that $(X, \mathcal{O}_X)$ is connected.

We deduce there does not exist a continuous, surjective map

$$Y \xrightarrow{g} \{0,1\}.$$

By Proposition 10.3.1, we conclude that $(Y, \mathcal{O}_Y)$ is connected. □

Corollary 10.5.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Suppose that $(X, \mathcal{O}_X)$ is connected. Then $(Y, \mathcal{O}_Y)$ is connected.

Proof. Since f is a homeomorphism, f is in particular a continuous bijection. By Task E7.2.1, a bijection in the sense of Definition 7.1.1 is in particular surjective. By Proposition 10.5.1, we deduce that $(Y, \mathcal{O}_Y)$ is connected. □

Corollary 10.5.3. Let $(X, \mathcal{O}_X)$ be a connected topological space. Let $\sim$ be an equivalence relation on X. Then $(X/\sim, \mathcal{O}_{X/\sim})$ is connected.

Proof. Let

$$X \xrightarrow{\pi} X/\sim$$

denote the quotient map with respect to $\sim$. By Remark 6.1.9, we have that π is continuous. Moreover π is surjective. By Proposition 10.5.1, we deduce that $(X/\sim, \mathcal{O}_{X/\sim})$ is connected. □

10.6. Geometric examples of connected topological spaces

Example 10.6.1. Let $]a, b[$ be an open interval. Let $\mathcal{O}_{]a,b[}$ denote the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Example 7.3.10, we have that $(]a, b[\,, \mathcal{O}_{]a,b[})$ is homeomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Corollary 10.5.2, we deduce that $(]a, b[\,, \mathcal{O}_{]a,b[})$ is connected.

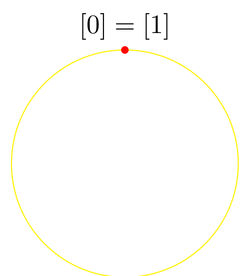
Example 10.6.2. Let $[a, b]$ be a closed interval, where $a < b$. Let $\mathcal{O}_{[a,b]}$ denote the subspace topology on $[a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

We have that $\mathsf{cl}_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}(]a, b[)$ is $[a, b]$. By Example 10.6.1 and Corollary E10.3.4, we deduce that $([a, b]\,, \mathcal{O}_{[a,b]})$ is connected.

Remark 10.6.3. We can go beyond Example 10.6.1 and Example 10.6.2. Let X be a subset of $\mathbb{R}$, and let $\mathcal{O}_X$ be equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $(X, \mathcal{O}_X)$ is connected if and only if X is an interval. To prove this is the topic of Task E10.3.5.

Example 10.6.4. As in Example 6.3.1, let $\sim$ be the equivalence relation on I generated by $0 \sim 1$.

By Example 10.6.2, we have that $(I, \mathcal{O}_I)$ is connected. By Corollary 10.5.3, we deduce that $(I/\sim, \mathcal{O}_{I/\sim})$ is connected.



By Task E7.3.10, there is a homeomorphism

$$I/\sim \longrightarrow S^1.$$

By Corollary 10.5.2, we deduce that $(S^1, \mathcal{O}_{S^1})$ is connected.

10.7. Products of connected topological spaces

Proposition 10.7.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be connected topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is connected.

Proof. Let $\{0, 1\}$ be equipped with the discrete topology. Let

$$X \times Y \xrightarrow{f} \{0, 1\}$$

be a continuous map. Our argument has two principal steps.

(1) Suppose that x belongs to X. By Task E5.1.5, we have that the map

$$Y \xrightarrow{c_x} X$$

given by $y \mapsto x_0$ for all y which belong to Y is continuous. By Task E5.1.3, we also have that the map

$$Y \xrightarrow{id} Y$$

is continuous. By Task E5.3.17, we deduce that the map

$$Y \xrightarrow{c_x \times id} X \times Y$$

given by $y \mapsto (x, y)$ for all y which belong to Y is continuous. By Proposition 5.3.1, we deduce that the map

$$Y \xrightarrow{f \circ (c_x \times id)} \{0, 1\}$$

given by $y \mapsto f(x, y)$ for all y which belong to Y is continuous. Since $(Y, \mathcal{O}_Y)$ is connected, we deduce, by Proposition 10.3.1, that $f \circ (c_x \times id)$ is not surjective. Since $\{0, 1\}$ has only two elements, we deduce that $f \circ (c_x \times id)$ is constant. In other words, we have that

$$f(x, y_0) = f(x, y_1)$$

for all y_0 and y_1 which belong to Y.

(2) Suppose that y belongs to Y. Let

$$X \xrightarrow{c_y} Y$$

denote the map given by $x \mapsto y$ for all x which belong to X. Arguing as in (1), we have that the map

$$X \xrightarrow{f \circ (id \times c_y)} \{0,1\}$$

given by $x \mapsto f(x,y)$ for all x which belong to X is continuous. To carry out this argument is the topic of Task E10.2.2. Since $(X, \mathcal{O}_X)$ is connected, we deduce, by Proposition 10.3.1, that $f \circ (id \times c_y)$ is not surjective. Since $\{0,1\}$ has only two elements, we deduce that $f \circ (id \times c_y)$ is constant. In other words, we have that

$$f(x_0, y) = f(x_1, y)$$

for all x_0 and x_1 which belong to X.

Suppose now that x_0 and x_1 belong to X, and that y_0 and y_1 belong to Y. By (1), taking x to be x_0, we have that

$$f(x_0, y_0) = f(x_0, y_1).$$

By (2), taking y to be y_1, we have that

$$f(x_0, y_1) = f(x_1, y_1).$$

We deduce that

$$f(x_0, y_0) = f(x_1, y_1).$$

Thus f is constant. In particular, f is not surjective. We have thus demonstrated that there does not exist a continuous surjection

$$X \times Y \longrightarrow \{0,1\}.$$

By Proposition 10.3.1, we conclude that $(X \times Y, \mathcal{O}_{X \times Y})$ is connected.

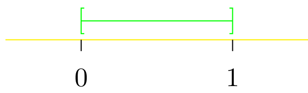
□

Remark 10.7.2. Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are topological spaces. The converse to Proposition 10.7.1 holds: if $(X \times Y, \mathcal{O}_{X \times Y})$ is connected, then both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected. To prove this is the topic of Task E10.3.8.

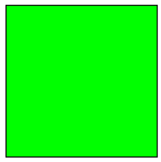
10.8. Further geometric examples of connected topological spaces

Example 10.8.1. By Proposition 10.4.1, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected. Applying Proposition 10.7.1 repeatedly, we deduce that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is connected, for any $n \in \mathbb{N}$.

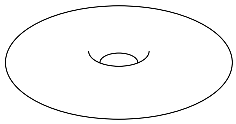
Example 10.8.2. By Example 10.6.2, we have that $(I, \mathcal{O}_I)$ is connected.



By Proposition 10.7.1, we deduce that $(I^2, \mathcal{O}_{I^2})$ is connected.



Example 10.8.3. By Example 10.8.2, we have that $(I^2, \mathcal{O}_{I^2})$ is connected. By Corollary 10.5.3, we deduce that $(T^2, \mathcal{O}_{T^2})$ is connected.

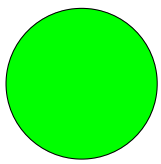


Remark 10.8.4. By a similar argument, $(M^2, \mathcal{O}_{M^2})$ and $(K^2, \mathcal{O}_{K^2})$ are connected. To check that you understand how we have built up to being able to prove this is the topic of Task E10.1.3.

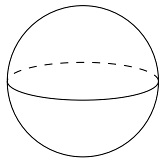
Example 10.8.5. By Example 10.8.2, we have that $(I^2, \mathcal{O}_{I^2})$ is connected. By Task E7.2.9, there is a homeomorphism

$$I^2 \longrightarrow D^2.$$

By Corollary 10.5.2, we deduce that $(D^2, \mathcal{O}_{D^2})$ is connected.



Example 10.8.6. By Example 10.8.5, we have that $(D^2, \mathcal{O}_{D^2})$ is connected. By Corollary 10.5.3, we deduce that $(S^2, \mathcal{O}_{S^2})$ is connected.



E10. Exercises for Lecture 10

E10.1. Exam questions

Task E10.1.1. Let $X = \{a, b, c, d\}$ be a set with four elements.

(1) Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

Is $(X, \mathcal{O}_X)$ connected?

(2) Let $\mathcal{O}_X$ be the topology on X given by

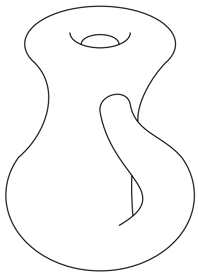
$$\{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}.$$

Is $(X, \mathcal{O}_X)$ connected?

(3) Find an equivalence relation $\sim$ on X with the property that $(X/\sim, \mathcal{O}_{X/\sim})$ is connected, where $\mathcal{O}_{X/\sim}$ is the quotient topology on $X/\sim$ with respect to the topology $\mathcal{O}_X$ on X of (2).

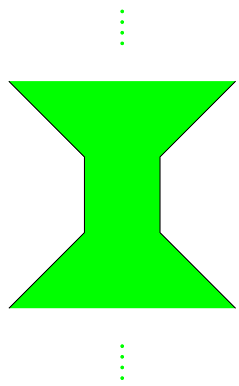
Task E10.1.2. Let $\mathbb{R} \setminus \mathbb{Q}$ be equipped with the subspace topology $\mathcal{O}_{\mathbb{R}\setminus\mathbb{Q}}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $(\mathbb{R} \setminus \mathbb{Q}, \mathcal{O}_{\mathbb{R}\setminus\mathbb{Q}})$ is not connected.

Task E10.1.3. Prove that $(K^2, \mathcal{O}_{K^2})$ is connected. You may appeal without proof to any results from the lecture, but may not assert without justification that any topological space except $(I, \mathcal{O}_I)$ is connected.

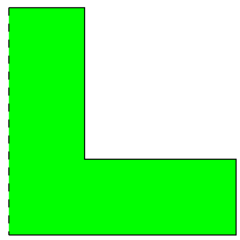


Task E10.1.4. Prove that the following topological spaces are connected. Where possible, give both a proof which makes use of Task E10.3.9, and a proof which does not. You may appeal to any results from the lectures or tasks. In addition, if you may assert the existence of homeomorphisms without proofs or explicit definitions.

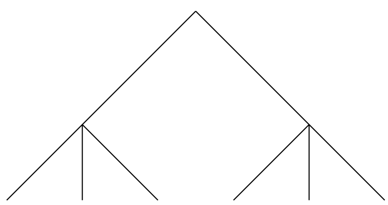
(1) The subset X of $\mathbb{R}^2$ depicted below, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



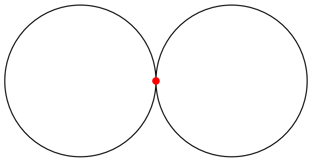
(2) The subset X of $\mathbb{R}^2$ depicted below, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



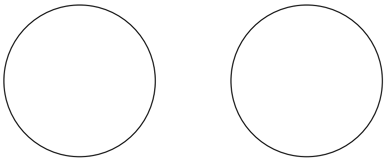
(3) The subset X of $\mathbb{R}^2$ depicted below, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



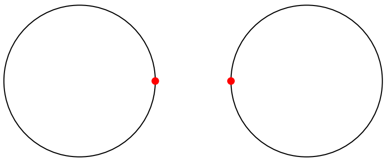
(4) The subset of $\mathbb{R}^2$ depicted below, consisting of two circles joined at a point, equipped with its subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



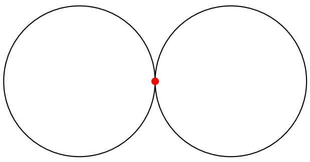
Task E10.1.5. Let X be a disjoint union of two circles of radius 1 in $\mathbb{R}^2$, centred at $(0,0)$ and $(3,0)$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



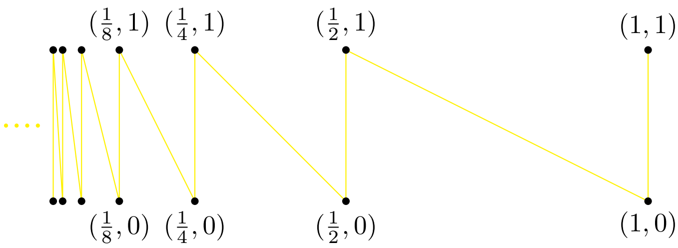
Let $\sim$ be the equivalence relation on X generated by $(1,0) \sim (2,0)$.



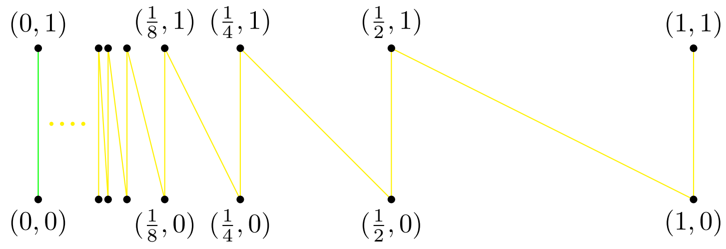
Without appealing to the fact that $(X/\sim, \mathcal{O}_{X/\sim})$ is homeomorphic to the topological space of Task E10.1.4 (5), prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is connected.



Task E10.1.6. Let A be the subset of $\mathbb{R}^2$ of Task E8.1.7.



Let X be the closure of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$, which, as you were asked to prove in Task E8.1.7, is the union of X and the line $\{0\} \times [0, 1]$.



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that $(X, \mathcal{O}_X)$ is connected. You may wish to proceed as follows.

(1) Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Prove that $(A, \mathcal{O}_A)$ is connected by appealing to Task E2.3.1, Task E7.1.8, Example 10.6.2, Corollary 10.5.2, and Task E10.3.9.

(2) Deduce that $(X, \mathcal{O}_X)$ is connected by Task E10.3.4.

Task E10.1.7. Let $\mathbb{R}$ be equipped with its standard topology $\mathcal{O}_{\mathbb{R}}$. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Can there be a continuous map

$$\mathbb{R} \longrightarrow \mathbb{Q}$$

which is a surjection?

E10.2. In the lecture notes

Task E10.2.1. Prove that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected, by filling in the details of the following argument. Let U be a subset of $\mathbb{R}$ which belongs to $\mathcal{O}_{\mathbb{R}}$. By Task E2.3.7, there is a set I and an open interval U_i for each $i \in I$ such that $U = \bigsqcup_{i \in I} U_i$. Suppose that U is neither $\emptyset$ nor $\mathbb{R}$. Then there is an $i \in I$ such that one of the following holds.

(1) We have that U_i is $]a, \infty[$, where $a \in \mathbb{R}$.

(2) We have that U_i is $]-\infty, b[$, where $b \in \mathbb{R}$,.

(3) We have that U_i is $]a, b[$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Treat each of the cases separately, as follows.

(1) Then a is a limit point of U in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, and a does not belong to U.

(2) Then b is a limit point of U in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, and b does not belong to U.

(3) Then both a and b are limit points of U in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, and neither a nor b belongs to U.

By Proposition 9.1.1, deduce in each case that U is not closed with respect to $\mathcal{O}_{\mathbb{R}}$. By Task E10.3.1, conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is connected.

Task E10.2.2. Carry out the argument needed for (2) of the proof of Proposition 10.7.1.

E10.3. For a deeper understanding

Task E10.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Prove that $(X, \mathcal{O}_X)$ is connected if and only if the only subsets of X which both belong to $\mathcal{O}_X$ and are closed with respect to $\mathcal{O}_X$ are $\emptyset$ and X. You may wish to proceed as follows.

(1) Suppose that $(X, \mathcal{O}_X)$ is connected. Let X_0 be a subset of X which belongs to $\mathcal{O}_X$. If X_0 is closed with respect to $\mathcal{O}_X$, we have that $X \setminus X_0$ belongs to $\mathcal{O}_X$. Moreover $X_0 \cap (X \setminus X_0)$ is empty. Since $(X, \mathcal{O}_X)$ is connected, conclude that X_0 is either $\emptyset$ or X.

(2) Suppose that X_0 is a subset of X which is neither $\emptyset$ nor X. Observe that $X \setminus X_0$ is then neither $\emptyset$ nor X. We have that $X = X_0 \sqcup (X \setminus X_0)$. If both X_0 and $X \setminus X_0$ belong to $\mathcal{O}_X$, deduce that $(X, \mathcal{O}_X)$ is not connected. Conclude that if X_0 both belongs to $\mathcal{O}_X$ and is closed with respect to $\mathcal{O}_X$, then $(X, \mathcal{O}_X)$ is not connected.

Task E10.3.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is connected. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Let $\mathcal{O}_{f(X)}$ denote the subspace topology on $f(X)$ with respect to $(Y, \mathcal{O}_Y)$. Prove that $\big(f(X), \mathcal{O}_{f(X)}\big)$ is connected. You may wish to proceed as follows.

(1) Let

$$X \xrightarrow{g} f(X)$$

be the map given by $x \mapsto f(x)$. By Task E5.1.9, observe that g is continuous.

(2) Moreover we have that g is surjective. By Proposition 10.5.1, conclude that $\big(f(X), \mathcal{O}_{f(X)}\big)$ is connected.

Task E10.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Suppose that $(A, \mathcal{O}_A)$ is connected. Let B be a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ with the property that A is a subset of B. Let $\mathcal{O}_B$ denote the subspace topology on B with respect to $(X, \mathcal{O}_X)$. Prove that $(B, \mathcal{O}_B)$ is connected. You may wish to proceed as follows.

(1) Let $\{0,1\}$ be equipped with the discrete topology $\mathcal{O}_{\mathsf{discrete}}$. Suppose that

$$B \xrightarrow{f} \{0,1\}$$

is continuous. Let

$$A \xrightarrow{i} B$$

denote the inclusion map. By Proposition 5.2.2, we have that i is continuous. By Proposition 5.3.1, deduce that

$$A \xrightarrow{f \circ i} \{0,1\}$$

is continuous.

(2) Since $(A, \mathcal{O}_A)$ is connected, deduce by Proposition 10.3.1 that $f \circ i$ is not surjective. Since $\{0,1\}$ has only two elements, deduce that $f \circ i$ is constant.

(3) By Task E8.3.13, we have that $\mathsf{cl}_{(B,\mathcal{O}_B)}(A)$ is $B \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(A)$. Since B is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ by assumption, deduce that $\mathsf{cl}_{(B,\mathcal{O}_B)}(A)$ is B.

(4) By Task E9.3.9, we have that $f\left(\mathsf{cl}_{(B,\mathcal{O}_B)}(A)\right)$ is a subset of

$$\mathsf{cl}_{(\{0,1\},\mathcal{O}_{\mathsf{discrete}})}(f(A)).$$

By (3), deduce that $f(B)$ is a subset of

$$\mathsf{cl}_{(\{0,1\},\mathcal{O}_{\mathsf{discrete}})}(f(A)).$$

(5) Demonstrate that $\mathsf{cl}_{(\{0,1\},\mathcal{O}_{\mathsf{discrete}})}(f(A))$ is $f(A)$. By (4), deduce that $f(B)$ is a subset of $f(A)$.

(6) By (2) and (5), we have that f is constant. In particular, we have that f is not surjective. By Proposition 10.3.1, conclude that $(B, \mathcal{O}_B)$ is connected.

Corollary E10.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Suppose that $(A, \mathcal{O}_A)$ is connected. Let $\mathcal{O}_{\mathsf{cl}_{(X,\mathcal{O}_X)}(A)}$ denote the subspace topology on $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Then $\left(\mathsf{cl}_{(X,\mathcal{O}_X)}(A), \mathcal{O}_{\mathsf{cl}_{(X,\mathcal{O}_X)}(A)}\right)$ is connected.

Proof. Follows immediately from Task E10.3.3, taking B to be $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$. $\square$

Task E10.3.5. Let X be a subset of $\mathbb{R}$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $(X, \mathcal{O}_X)$ is connected if and only if X is an interval. You may wish to proceed as follows.

(1) Suppose that X is an interval. The difference possibilities for X are listed below, where a and b belong to $\mathbb{R}$, and $a < b$. In each case, fill in the details of the outlined proof that $(X, \mathcal{O}_X)$ is connected.

Interval X	Proof that $(X, \mathcal{O}_X)$ is connected
$\mathbb{R}$	Proposition 10.4.1
$]a, b[$	Example 10.6.1
$[a, b]$	Example 10.6.2
$[a, b[$	Example 10.6.1 and Task E10.3.3
$]a, b]$	Example 10.6.1 and Task E10.3.3
$\emptyset$	By inspection.
$[a, a]$	By inspection.
$]a, \infty[$	Task E7.1.5, Corollary 10.5.2, and Proposition 10.4.1.
$]-\infty, b[$	Task E7.1.6, Corollary 10.5.2, and Proposition 10.4.1.
$[a, \infty[$	Corollary E10.3.4 and the case that X is $]a, \infty[$.
$]-\infty, b]$	Corollary E10.3.4 and the case that X is $]-\infty, b[$.

(2) Suppose that X is not an interval. By Task E1.3.3, there is an $x_0 \in X$, an $x_1 \in X$, and a $y \in \mathbb{R} \setminus X$, such that $x_0 < y < x_1$. Let X_0 be $X \cap]-\infty, y[$, and let X_1 be $X \cap]y, \infty[$. Observe that both X_0 and X_1 belong to $\mathcal{O}_X$, and that $X = X_0 \sqcup X_1$. Conclude that $(X, \mathcal{O}_X)$ is not connected.

Task E10.3.6. Let $(X, \mathcal{O}_X)$ be a connected topological space. Let $\mathbb{R}$ be equipped with the standard topology $\mathcal{O}_{\mathbb{R}}$. Let

$$X \xrightarrow{f} \mathbb{R}$$

be a continuous map. Suppose that x_0 and x_1 belong to X, and that $f(x_0) \leq f(x_1)$. Prove that, for every $x \in \mathbb{R}$ such that $f(x_0) \leq y \leq f(x_1)$, there is an $x_2 \in X$ such that $f(x_2) = y$. You may wish to proceed as follows.

(1) Let $\mathcal{O}_{f(X)}$ denote the subspace topology on $f(X)$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Since $(X, \mathcal{O}_X)$ is connected, deduce by Task E10.3.2 that $\big(f(X), \mathcal{O}_{f(X)}\big)$ is connected.

(2) By Task E10.3.5, deduce that $f(X)$ is an interval.

(3) Appeal to Task E1.3.3.

Remark E10.3.7. Taking $(X, \mathcal{O}_X)$ to be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, or to be an interval equipped with the subspace topology with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, the conclusion of Task E10.3.6 is exactly the *intermediate value theorem*. As you may recall from earlier courses, this is one of the handful of crucial facts upon which analysis rests.

Task E10.3.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X \times Y, \mathcal{O}_{X \times Y})$ is connected. Prove that both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are connected. You may wish to appeal to Proposition 5.4.3 and Proposition 10.5.1.

Task E10.3.9. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{A_j\}_{j \in J}$ be a set of subsets of X such that the following hold.

(1) For every $j \in J$, we have that $(A_j, \mathcal{O}_{A_j})$ is connected, where $\mathcal{O}_{A_j}$ denotes the subspace topology on A_j with respect to $(X, \mathcal{O}_X)$.

(2) We have that $\bigcup_{j \in J} A_j$ is X.

(3) We have that $\bigcap_{j \in J} A_j$ is not empty.

Prove that $(X, \mathcal{O}_X)$ is connected. You may wish to proceed as follows.

(1) Let $\{0, 1\}$ be equipped with the discrete topology. Let

$$X \xrightarrow{f} \{0, 1\}$$

be a continuous map. Suppose that j belongs to J. Let

$$A_j \xrightarrow{i_j} X$$

denote the inclusion map, given by $a \mapsto a$. By Proposition 5.2.2, we have that i_j is continuous. By Proposition 5.3.1, deduce that the map

$$A_j \xrightarrow{f \circ i_j} \{0, 1\}$$

given by $a \mapsto f(a)$ is continuous.

(2) Since $(A_j, \mathcal{O}_{A_j})$ is connected, deduce by Proposition 10.3.1 that $f \circ i_j$ is constant.

(3) Observe that the fact that $\bigcup_{j \in J} A_j$ is X, that $\bigcap_{j \in J} A_j$ is not empty, and that $f \circ i_j$ is constant for every $j \in J$, implies that f is constant.

(4) In particular, f is not surjective. Thus we have demonstrated that there does not exist a continuous surjection

$$X \longrightarrow \{0, 1\}.$$

By Proposition 10.3.1, conclude that $(X, \mathcal{O}_X)$ is connected.

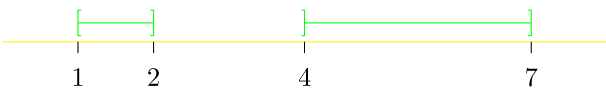
11. Monday 10th February

11.1. Using connectedness to prove that two topological spaces are not homeomorphic

Remark 11.1.1. To prove that a given topological space $(X, \mathcal{O}_X)$ is not homeomorphic to a particular topological space $(Y, \mathcal{O}_Y)$ is typically hard. In geometric examples, when X and Y are infinite, there are many infinitely many maps from X to Y. Thus we cannot simply list them all, and then check whether or not there is a homeomorphism amongst them.

We must proceed in a more sophisticated way. The theory of connectedness which we have developed furnishes us with our first powerful tool for proving that two topological spaces are not homeomorphic.

Example 11.1.2. Let $X = [1, 2] \cup]4, 7]$.



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Arguing as in Example 9.6.2, we have that $(X, \mathcal{O}_X)$ is not connected.

Let $\mathcal{O}_{]1,5[}$ be the subspace topology on $[1, 5]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

By Task E10.3.5, we have that $\left([1, 5], \mathcal{O}_{]1,5]}\right)$ is connected. Suppose that

$$[1, 5] \xrightarrow{f} X$$

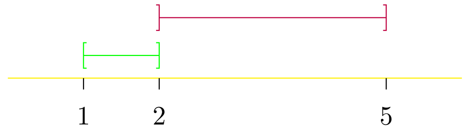
is a homeomorphism. By Corollary 10.5.2, we then have that $(X, \mathcal{O}_X)$ is connected.

Thus we have a contradiction. We conclude that there does not exist a homeomorphism

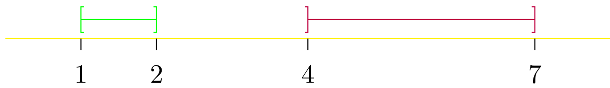
$$[1, 5] \xrightarrow{f} X.$$

In other words, we have that $(X, \mathcal{O}_X)$ is not homeomorphic to $\left([1, 5], \mathcal{O}_{]1,5]}\right)$.

Remark 11.1.3. We can 'snap off' the half open interval $]2,5]$ from $[1,5]$.



We can then 'move' this half open interval to $]4,7]$.



This defines a bijection

$$[1,5] \xrightarrow{f} X.$$

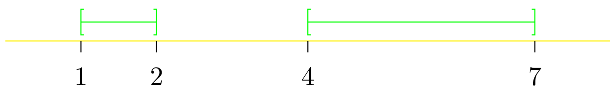
However, this bijection is not continuous. To 'snap off' is not allowed in topology! The details of this are the topic of Task E11.2.1.

It is very important to appreciate that to distinguish between $(X, \mathcal{O}_X)$ and

$$\left([1,5]\,, \mathcal{O}_{]1,5]}\right),$$

we must give a topological argument. From the point of view of set theory, $[1,5]$ and X are 'the same'.

Remark 11.1.4. Let $X = [1,2] \cup [4,7]$.

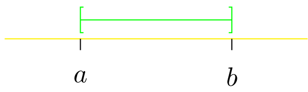


Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Exactly the same kind of argument as in Example 11.1.2 proves that $(X, \mathcal{O}_X)$ is not homeomorphic to $\left([1,5]\,, \mathcal{O}_{]1,5]}\right)$.

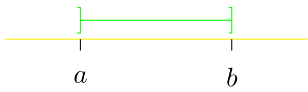
There is a bijection between $[1,5]$ and X, though it is harder to find than the bijection of Remark 11.1.3. This is the topic of Task E11.4.2. Once more, we see that it is necessary to give a topological argument to distinguish between $(X, \mathcal{O}_X)$ and $\left([1,5]\,, \mathcal{O}_{]1,5]}\right)$.

11.2. Using connectedness to distinguish between topological spaces by removing points

Example 11.2.1. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{[a,b]}$ be the subspace topology on $[a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



We have that $[a, b] \setminus \{a\}$ is $]a, b]$.



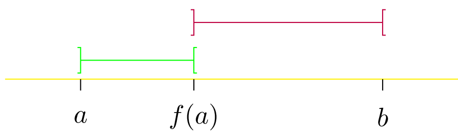
Let $\mathcal{O}_{]a,b]}$ be the subspace topology on $]a, b]$ with respect to $\left([a, b], \mathcal{O}_{[a,b]}\right)$. By Task E2.3.1 and Task E10.3.5, we have that $\left(]a, b], \mathcal{O}_{]a,b]}\right)$ is connected.

Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Suppose that

$$[a, b] \xrightarrow{f}]a, b[$$

is a homeomorphism. Let $\mathcal{O}_{]a,b[\setminus\{f(a)\}}$ be the subspace topology on $]a, b[\setminus \{f(a)\}$ with respect to $\left(]a, b[, \mathcal{O}_{]a,b[}\right)$. We have that $]a, b[\setminus \{f(a)\}$ is the union of $]a, f(a)[$ and $]f(a), b[$. This union is disjoint.



Moreover, both $]a, f(a)[$ and $]f(a), b[$ belong to $\mathcal{O}_{]a,b[\setminus\{f(a)\}}$. Thus

$$\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$$

is not connected. To generalise this argument is the topic of Task E11.2.5.

By Task E7.1.20, since f is a homeomorphism, the map

$$]a, b] \longrightarrow]a, b[\setminus \{f(a)\}$$

given by $x \mapsto f(x)$ is a homeomorphism. Since $(]a, b], \mathcal{O}_{]a,b]})$ is connected, we deduce, by Corollary 10.5.2, that

$$\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$$

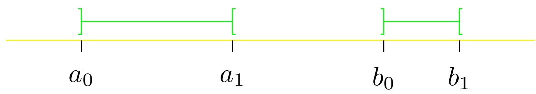
is connected.

Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$[a, b] \xrightarrow{f}]a, b[\,.$$

In other words, $\left([a, b], \mathcal{O}_{[a,b]}\right)$ is not homeomorphic to $\left(]a, b[, \mathcal{O}_{]a,b[}\right)$.

Remark 11.2.2. Suppose that $a_0 < a_1 < b_0 < b_1$ belong to $\mathbb{R}$. Let X be the union of $]a_0, a_1[$ and $]b_0, b_1[$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Let

$$\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$$

be as in Example 11.2.1. By Task E11.2.3, we have that

$$\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$$

is homeomorphic to $(X, \mathcal{O}_X)$. Thus we can picture $\left(]a, b[\setminus \{f(a)\}, \mathcal{O}_{]a,b[\setminus\{f(a)\}}\right)$ as follows.

Example 11.2.3. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{[a,b]}$ be the subspace topology on $[a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

We have that $[a, b] \setminus \{a, b\}$ is $]a, b[$.

Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a,b[$ with respect to $\left([a,b[\,, \mathcal{O}_{[a,b[}\right)$. By Task E2.3.1 and Task E10.3.5, we have that $\left(]a,b[\,, \mathcal{O}_{]a,b[}\right)$ is connected. Let $\mathcal{O}_{[a,b[}$ be the subspace topology on $[a,b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let

$$[a,b] \xrightarrow{f} [a,b[$$

be a homeomorphism. Let $\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}$ be the subspace topology on $[a,b[\setminus\{f(a),f(b)\}$ with respect to $\left([a,b[\,, \mathcal{O}_{[a,b[}\right)$. One of the following two possibilities must hold.

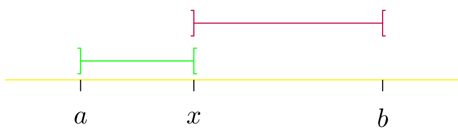
(I) One of $f(a)$ or $f(b)$ is a.

(II) Neither $f(a)$ nor $f(b)$ is a.

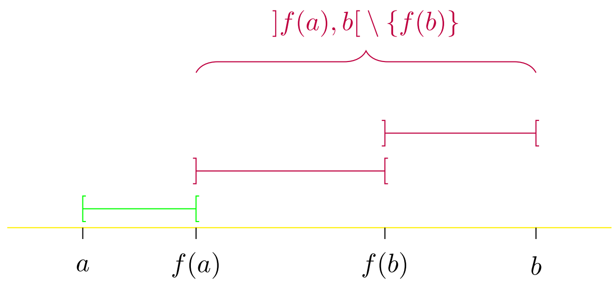
Suppose that (I) holds. Since f is bijective, one of $f(a)$ or $f(b)$ is not a. Let us denote whichever of $f(a)$ or $f(b)$ is not a by x. Then $[a,b[\setminus \{f(a),f(b)\}$ is $]a,b[\setminus \{x\}$. As in Example 11.2.1, we deduce that

$$\left([a,b[\setminus \{f(a),f(b)\}\,, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is not connected.



Suppose now that (II) holds. Since f is bijective, either $f(a) < f(b)$ or $f(a) > f(b)$. Suppose that $f(a) < f(b)$. We have that $[a,b[\setminus \{f(a),f(b)\}$ is the union of $[a,f(a)[$ and $]f(a),b[\setminus \{f(b)\}$, and this union is disjoint.



Moreover, both $[a, f(a)[$ and $]f(a), b[\setminus \{f(b)\}$ belong to $\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}$. Thus

$$\left([a, b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is not connected. A similar argument establishes that

$$\left([a, b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is not connected if $f(a) > f(b)$. This is the topic of Task E11.2.2.

By Task E7.1.20, since f is a homeomorphism, the map

$$]a, b[\longrightarrow [a, b[\setminus \{f(a), f(b)\}$$

given by $x \mapsto f(x)$ is a homeomorphism. Since $(]a, b[, \mathcal{O}_{]a,b[})$ is connected, we deduce, by Corollary 10.5.2, that

$$\left([a, b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

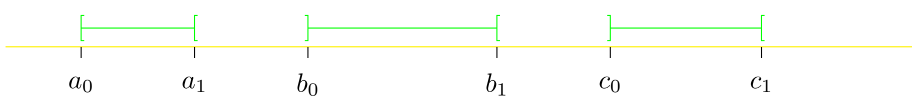
is connected.

Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$[a, b] \xrightarrow{f} [a, b[\,.$$

In other words, $\left([a, b], \mathcal{O}_{[a,b]}\right)$ is not homeomorphic to $\left([a, b[, \mathcal{O}_{[a,b[}\right)$.

Remark 11.2.4. Suppose that $a_0 < a_1 < b_0 < b_1 < c_0 < c_1$ belong to $\mathbb{R}$. Let X be the union of $]a_0, a_1[$, $]b_0, b_1[$, and $]c_0, c_1[$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Let

$$\left([a, b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

be as in case (II) of Example 11.2.3. By Task E11.2.4, we have that

$$\left([a,b[\setminus\{f(a),f(b)\},\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

is homeomorphic to $(X,\mathcal{O}_X)$. Thus we can picture

$$\left([a,b[\setminus\{f(a),f(b)\},\mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\right)$$

as follows.

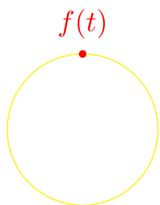
Example 11.2.5. Let $(I,\mathcal{O}_I)$ be the unit interval. Suppose that $0<t<1$.

Let $\mathcal{O}_{I\setminus\{t\}}$ be the subspace topology on $I\setminus\{t\}$ with respect to $(I,\mathcal{O}_I)$. Then $\left(I\setminus\{t\},\mathcal{O}_{I\setminus\{t\}}\right)$ is not connected.

Suppose that

$$I \xrightarrow{f} S^1$$

is a homeomorphism.



Let $\mathcal{O}_{S^1\setminus\{f(t)\}}$ be the subspace topology on $S^1\setminus\{f(t)\}$ with respect to $(S^1,\mathcal{O}_{S^1})$. We have that $\left(S^1\setminus\{f(t)\},\mathcal{O}_{S^1\setminus\{f(t)\}}\right)$ is connected.

Since f is a homeomorphism, we have by Task E7.1.20 that there is a homeomorphism

$$I \setminus \{t\} \longrightarrow S^1 \setminus \{f(t)\}.$$

By Task E7.3.2, we deduce that there is a homeomorphism

$$S^1 \setminus \{f(t)\} \longrightarrow I \setminus \{t\}.$$

By Corollary 10.5.2, since

$$\left(S^1 \setminus \{f(t)\}, \mathcal{O}_{S^1 \setminus \{f(t)\}}\right)$$

is connected, we deduce that

$$\left(I \setminus \{t\}, \mathcal{O}_{I \setminus \{t\}}\right)$$

is connected. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$I \longrightarrow S^1.$$

In other words, $(I, \mathcal{O}_I)$ is not homeomorphic to $\left(S^1, \mathcal{O}_{S^1}\right)$.

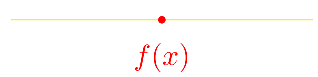
Remark 11.2.6. To prove the assertion that $\left(I \setminus \{t\}, \mathcal{O}_{I \setminus \{t\}}\right)$ is not connected, and the assertion that $\left(S^1 \setminus \{f(t)\}, \mathcal{O}_{S^1 \setminus \{f(t)\}}\right)$ is connected, is the topic of Task E11.2.11.

Remark 11.2.7. There exists a bijection between I and S^1. This is the topic of Task E11.4.3. Hence I and S^1 are 'the same' from the point of view of set theory. Thus, just as in Remark 11.1.3, a topological argument, such as that of Example 11.2.5, must be given to prove that $(I, \mathcal{O}_I)$ is not homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

Example 11.2.8. Suppose that $n > 1$ belongs to $\mathbb{N}$. Let $\mathbb{R}$ be equipped with the standard topology $\mathcal{O}_{\mathbb{R}}$. Let $\mathbb{R}^n$ be equipped with the product topology $\mathcal{O}_{\mathbb{R}^n}$ of Notation E3.3.8. Suppose that x belongs to $\mathbb{R}^n$. Suppose that

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}$$

is a homeomorphism. Let $\mathcal{O}_{\mathbb{R} \setminus \{f(x)\}}$ be the subspace topology on $\mathbb{R} \setminus \{f(x)\}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



By Task E11.2.5, we have that $(\mathbb{R} \setminus \{f(x)\}, \mathcal{O}_{\mathbb{R} \setminus \{f(x)\}})$ is not connected.

Let $\mathcal{O}_{\mathbb{R}^n \setminus \{x\}}$ be the subspace topology on $\mathbb{R}^n \setminus \{x\}$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. By Task E11.3.1, we have that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n \setminus \{x\}})$ is connected. Since f is a homeomorphism, we have by Task E7.1.20 that there is a homeomorphism

$$\mathbb{R}^n \setminus \{x\} \longrightarrow \mathbb{R} \setminus \{f(x)\}.$$

By Corollary 10.5.2, we deduce that

$$\left(\mathbb{R} \setminus \{f(x)\}, \mathcal{O}_{\mathbb{R} \setminus \{f(x)\}}\right)$$

is connected. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$\mathbb{R}^n \longrightarrow \mathbb{R}.$$

In other words, $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not homeomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

Remark 11.2.9. Example 11.2.8 is intuitively evident. We cannot bend or squash ourselves in such a way that we become a line! However, there is a bijection between $\mathbb{R}$ and $\mathbb{R}^n$, for any $n \geq 1$! This is the topic of Task E11.4.4.

Moreover, to prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ when $m \neq n$, $m \geq 2$, and $n \geq 2$, is much harder. One needs more powerful techniques.

11.3. Connected components

Terminology 11.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Then A is a *connected subset* of X with respect to $\mathcal{O}_X$ if $(A, \mathcal{O}_A)$ is a connected.

Terminology 11.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Let A be a connected subset of X with respect to $\mathcal{O}_X$ such that the following hold.

(1) We have that x belongs to A.

(2) For every connected subset B of X with respect to $\mathcal{O}_X$ to which x belongs, we have that B is a subset of A.

We refer to A as the *largest* connected subset of X with respect to $\mathcal{O}_X$ to which x belongs.

We do not yet know whether, for a given x which belongs to X, there is a subset A of X which has the property that it is the largest connected subset of X with respect to $\mathcal{O}_X$ to which x belongs. We shall now demonstrate this.

Remark 11.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Let A_0 and A_1 be connected subsets of X with respect to $\mathcal{O}_X$ which both satisfy (1) and (2) of Terminology 11.3.2. Then $A_0 = A_1$. To check that you understand this is the topic of Task E11.2.6.

Definition 11.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. The *connected component* of x in $(X, \mathcal{O}_X)$ is the union of all connected subsets of X with respect to $\mathcal{O}_X$ to which x belongs.

Notation 11.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. We denote the connected component of x in $(X, \mathcal{O}_X)$ by $\Gamma^x_{(X,\mathcal{O}_X)}$.

Remark 11.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Then $\{x\}$ is a connected subset of X with respect to $\mathcal{O}_X$. This is the topic of Task E11.2.7. Thus x belongs to $\Gamma^x_{(X,\mathcal{O}_X)}$.

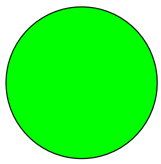
Proposition 11.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Then $\Gamma^x_{(X,\mathcal{O}_X)}$ is a connected subset of X.

Proof. Let $\{A_i\}_{i\in I}$ be the set of connected subsets of X with respect to $\mathcal{O}_X$ to which x belongs. We have that $\{x\}$ is a subset of $\bigcap_{i\in I} A_i$. By Task E10.3.9, we deduce that $\Gamma^x_{(X,\mathcal{O}_X)} = \bigcup_{i\in I} A_i$ is a connected subset of X with respect to $\mathcal{O}_X$. $\square$

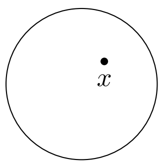
Remark 11.3.8. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Let A be a connected subset of X with respect to $\mathcal{O}_X$ to which x belongs. By definition of $\Gamma^x_{(X,\mathcal{O}_X)}$, we have that A is a subset of $\Gamma^x_{(X,\mathcal{O}_X)}$. By Proposition 11.3.7, we conclude that $\Gamma^x_{(X,\mathcal{O}_X)}$ is the largest connected subset of X with respect to $\mathcal{O}_X$ to which x belongs.

11.4. Examples of connected components

Example 11.4.1. Let $(X, \mathcal{O}_X)$ be a connected topological space. For example, we can take $(X, \mathcal{O}_X)$ to be $(D^2, \mathcal{O}_{D^2})$.



Suppose that x belongs to X.



We have that X is a connected subset of X with respect to $\mathcal{O}_X$, for which (1) and (2) of Terminology 11.3.2 hold. By Remark 11.3.3 and Remark 11.3.8, we conclude that $\Gamma^x_{(X,\mathcal{O}_X)} = X$.

Example 11.4.2. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. Suppose that x belongs to X. Let A be a subset of X to which x belongs. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Then $\mathcal{O}_A$ is the discrete topology on A. To verify this is Task E11.2.8.

Suppose that A has more than one element, so that $A \setminus \{x\}$ is not empty. We have that

$$A = \{x\} \sqcup (A \setminus \{x\}) .$$

Since $\mathcal{O}_A$ is the discrete topology on A, every subset of A belongs to $\mathcal{O}_A$. In particular, both $\{x\}$ and $A \setminus \{x\}$ belong to $\mathcal{O}_A$, Thus $(A, \mathcal{O}_A)$ is not connected. In other words, A is not a connected subset of X with respect to $\mathcal{O}_X$.

We conclude that $\Gamma^x_{(X,\mathcal{O}_X)} = \{x\}$.

Example 11.4.3. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

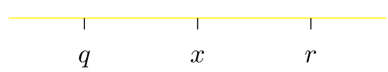
$$\{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\} .$$

Table 11.1 lists the connected subsets of X with respect to $\mathcal{O}_X$. By inspecting Table 11.1, and by Remark 11.3.3 and Remark 11.3.8, we conclude that the connected components in $(X, \mathcal{O}_X)$ of the elements of X are as follows.

Element	Connected component
a	$\{a\}$
b	$\{b, c\}$
c	$\{b, c\}$
d	$\{d\}$

Example 11.4.4. Let $\mathbb{Q}$ be the set of rational numbers. Let $\mathcal{O}_\mathbb{Q}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_\mathbb{R})$. Suppose that q belongs to $\mathbb{Q}$. Let A be a subset of $\mathbb{Q}$ to which q belongs. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(\mathbb{Q}, \mathcal{O}_\mathbb{Q})$. By Task E2.3.1, we have that $\mathcal{O}_\mathbb{Q}$ is the subspace topology on A with respect to $\mathcal{O}_\mathbb{R}$.

Suppose that r belongs to A, and that r is not equal to q. There is an irrational number x with $q < x < r$.



The following hold.

(1) Since x is irrational, and thus does not belong to A, we have that

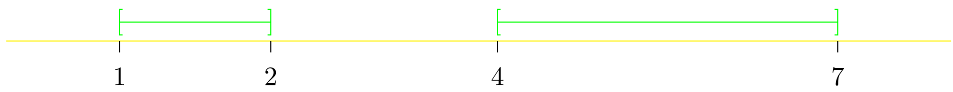
$$A = (A \cap \,]-\infty, x[) \sqcup (A \cap \,]x, \infty[) \,.$$

(2) We have that q belongs to $A \cap \,]-\infty, x[$, and that r belongs to $A \cap \,]x, \infty[$. In particular, neither $A \cap \,]-\infty, x[$ nor $A \cap \,]x, \infty[$ is empty.

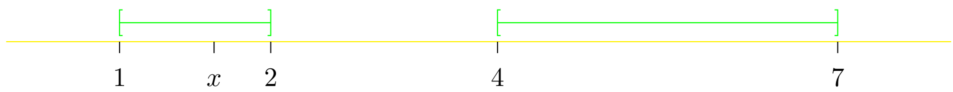
(3) Since both $]-\infty, x[$ and $]x, \infty[$ belong to $\mathcal{O}_{\mathbb{R}}$, and since $\mathcal{O}_A$ is the subspace topology on A with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$, we have that both $A \cap \,]-\infty, x[$ and $A \cap \,]x, \infty[$ belong to $\mathcal{O}_A$.

Thus $(A, \mathcal{O}_A)$ is not connected. In other words, A is not a connected subset of $\mathbb{Q}$ with respect to $\mathcal{O}_{\mathbb{Q}}$. We conclude that $\Gamma^q_{(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})}$ is $\{q\}$.

Example 11.4.5. Let $X = [1, 2] \cup [4, 7]$.

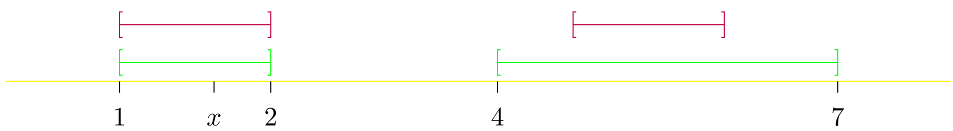


Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that x belongs to $[1, 2]$.



By Task E2.3.1 and Task E10.3.5, we have that $[1, 2]$ is a connected subset of $(X, \mathcal{O}_X)$.

Suppose that A is a subset of X to which x belongs, and which has the property that $A \cap [4, 7]$ is not empty.



Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. The following hold.

(1) We have that $A = (A \cap [1, 2]) \sqcup (A \cap [4, 7])$.

(2) We have that x belongs to $A \cap [1, 2]$. In particular, $A \cap [1, 2]$ is not empty. By assumption, we also have that $A \cap [4, 7]$ is not empty.

(3) As demonstrated in Example 9.6.2, both $[1, 2]$ and $[4, 7]$ belong to $\mathcal{O}_X$. Thus both $A \cap [1, 2]$ and $A \cap [4, 7]$ belong to $\mathcal{O}_A$.

Thus $(A, \mathcal{O}_A)$ is not connected. In other words, A is not a connected subset of X with respect to $\mathcal{O}_X$. By Remark 11.3.3 and Remark 11.3.8, we conclude that $\Gamma^x_{(X,\mathcal{O}_X)}$ is $[1, 2]$.

A similar argument demonstrates that if x belongs to $[4, 7]$, then $\Gamma^x_{(X,\mathcal{O}_X)}$ is $[4, 7]$. To fill in the details is the topic of Task E11.2.9.

11.5. Number of distinct connected components as an invariant

Remark 11.5.1. If two topological spaces are homeomorphic, then they have the same number of distinct connected components. This is the topic of Task E11.3.18.

Therefore, to prove that two topological spaces are not homeomorphic, it suffices to count their respective numbers of distinct connected components, and to observe that they are different. This is a gigantic simplification! It is so much of a simplification that it is only useful to a certain extent, as we shall see.

In particular, it is most definitely not the case that two topological spaces are homeomorphic if and only if they have the same number of distinct connected components. There are many connected topological spaces which are not homeomorphic!

Nevertheless, the idea that we can associate to complicated gadgets, such as topological spaces, simpler *invariants*, which we can calculate with more easily, is of colossal importance in mathematics. These invariants might be: numbers; algebraic gadgets such as groups, vector spaces, or rings; or other structures.

Thus the number of distinct connected components of a topological space is the beginning of a fascinating story!

Example 11.5.2. Let T be the subset of $\mathbb{R}^2$ given by the union of

$$\{(0, y) \mid -1 \leq y \leq 1\}$$

and

$$\{(x, 1) \mid -1 \leq x \leq 1\}.$$

Let $\mathcal{O}_{\mathsf{T}}$ be the subspace topology on T with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let I be the subset of $\mathbb{R}^2$ given by

$$\{(0, y) \mid 0 \leq y \leq 1\}.$$

Let $\mathcal{O}_{\mathsf{I}}$ be the subspace topology on I with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Suppose that

$$\mathsf{T} \xrightarrow{f} \mathsf{I}$$

is a homeomorphism. Let x be the point $(0,1)$ of T.

Let $\mathcal{O}_{\mathsf{T}\setminus\{x\}}$ be the subspace topology on $\mathsf{T}\setminus\{x\}$ with respect to $(\mathsf{T},\mathcal{O}_{\mathsf{T}})$. Then

$$\left(\mathsf{T}\setminus\{x\},\mathcal{O}_{\mathsf{T}\setminus\{x\}}\right)$$

has three distinct connected components.

Let $\mathcal{O}_{\mathsf{I}\setminus\{f(x)\}}$ be the subspace topology on $\mathsf{I}\setminus\{f(x)\}$ with respect to $(\mathsf{I},\mathcal{O}_{\mathsf{I}})$. Suppose that $f(x)$ is $(0,0)$ or $(0,1)$.

Then $\left(\mathsf{I}\setminus\{f(x)\},\mathcal{O}_{\mathsf{I}\setminus\{f(x)\}}\right)$ is connected. Suppose that $f(x)$ is not $(0,0)$ or $(0,1)$.

Then $\left(\mathsf{I} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x)\}}\right)$ has two distinct connected components.

Since f is a homeomorphism, we have by Task E7.1.20 that there is a homeomorphism

$$\mathsf{T} \setminus \{x\} \longrightarrow \mathsf{I} \setminus \{f(x)\}.$$

By Corollary E11.3.19, since

$$\left(\mathsf{T} \setminus \{x\}, \mathcal{O}_{\mathsf{T} \setminus \{x\}}\right)$$

has three distinct connected components, we deduce that

$$\left(\mathsf{I} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x)\}}\right)$$

has three distinct connected components. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$\mathsf{T} \longrightarrow \mathsf{I}.$$

In other words, $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$.

Remark 11.5.3. To fill in the details of the three calculations of numbers of distinct connected components in Example 11.5.2 is the topic of Task E12.2.1.

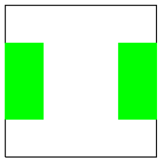
Subset A	Connected?	Reason
$\emptyset$	✓	
$\{a\}$	✓	
$\{b\}$	✓	
$\{c\}$	✓	
$\{d\}$	✓	
$\{a,b\}$	✗	$A = \{a\} \sqcup \{b\}$, and both $\{a\} = A \cap \{a\}$ and $\{b\} = A \cap \{b\}$ belong to $\mathcal{O}_A$.
$\{a,c\}$	✗	$A = \{a\} \sqcup \{c\}$, and both $\{a\} = A \cap \{a\}$ and $\{c\} = A \cap \{b,c,d\}$ belong to $\mathcal{O}_A$.
$\{a,d\}$	✗	$A = \{a\} \sqcup \{d\}$, and both $\{a\} = A \cap \{a\}$ and $\{d\} = A \cap \{d\}$ belong to $\mathcal{O}_A$.
$\{b,c\}$	✓	
$\{b,d\}$	✗	$A = \{b\} \sqcup \{d\}$, and both $\{b\} = A \cap \{b\}$ and $\{d\} = A \cap \{d\}$ belong to $\mathcal{O}_A$.
$\{c,d\}$	✗	$A = \{c\} \sqcup \{d\}$, and both $\{c\} = A \cap \{b,c\}$ and $\{d\} = A \cap \{b,d\}$ belong to $\mathcal{O}_A$.
$\{a,b,c\}$	✗	$A = \{a\} \sqcup \{b,c\}$, and both $\{a\} = A \cap \{a\}$ and $\{b,c\} = A \cap \{b,c\}$ belong to $\mathcal{O}_A$.
$\{a,b,d\}$	✗	$A = \{a\} \cup \{b,d\}$, and both $\{a\} = A \cap \{a\}$ and $\{b,d\} = A \cap \{b,d\}$ belong to $\mathcal{O}_A$.
$\{a,c,d\}$	✗	$A = \{a\} \cup \{c,d\}$, and both $\{a\} = A \cap \{a\}$ and $\{c,d\} = A \cap \{b,c,d\}$ belong to $\mathcal{O}_A$.
$\{b,c,d\}$	✗	$A = \{b,c\} \cup \{d\}$, and both $\{b,c\} = A \cap \{b,c\}$ and $\{d\} = A \cap \{d\}$ belong to $\mathcal{O}_A$.
X	✗	$X = \{a\} \cup \{b,c,d\}$, and both $\{a\}$ and $\{b,c,d\}$ belong to $\mathcal{O}_X$.

Table 11.1.: Connected subsets of the topological space $(X, \mathcal{O}_X)$ of Example 11.4.3. For each subset A of X, we denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$ by $\mathcal{O}_A$.

E11. Exercises for Lecture 11

E11.1. Exam questions

Task E11.1.1. Let A be the subset of I^2 given by the union of $\left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right] \times \left[\frac{1}{4}, \frac{3}{4}\right]$.

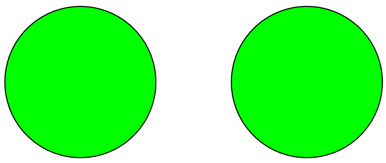


Let

$$I^2 \xrightarrow{\pi} T^2$$

be the quotient map. Let $\mathcal{O}_A$ be the subspace topology on $\pi(A)$ with respect to $(T^2, \mathcal{O}_{T^2})$. Let X be the subset of $\mathbb{R}^2$ given by the union of D^2 and

$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x-3, y)\| \leq 1\right\}.$$



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Is $\left(\pi(A), \mathcal{O}_{\pi(A)}\right)$ homeomorphic to $(X, \mathcal{O}_X)$?

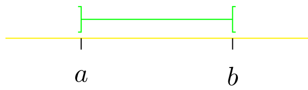
Task E11.1.2. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{[a,b[}$ be the subspace topology on $[a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $\left([a, b[\,, \mathcal{O}_{[a,b[}\right)$ is not homeomorphic to $\left(]a, b[\,, \mathcal{O}_{]a,b[}\right)$.

Task E11.1.3. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{]a,b]}$ be the subspace topology on $]a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Prove that $\left(]a, b]\,, \mathcal{O}_{]a,b]}\right)$ is not homeomorphic to $\left(]a, b[\,, \mathcal{O}_{]a,b[}\right)$. You may wish to proceed by appealing to Task E11.1.2 and to Task E7.1.4.

Task E11.1.4. Suppose that a and b belong to $\mathbb{R}$, and that $a < b$. Let $\mathcal{O}_{[a,b]}$ be the subspace topology on $[a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $\mathcal{O}_{]a,b]}$ be the subspace topology on $]a, b]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $\left([a, b]\,, \mathcal{O}_{[a,b]}\right)$ is not homeomorphic to $\left(]a, b]\,, \mathcal{O}_{]a,b]}\right)$. You may wish to proceed by appealing to Example 11.2.3 and to Task E7.1.4.

Remark E11.1.5. Suppose that $a < b$ belong to $\mathbb{R}$. Let $\mathcal{O}_{[a,b]}$, $\mathcal{O}_{]a,b[}$, $\mathcal{O}_{[a,b[}$, and $]a, b]$ be the subspace topologies with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ on $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$ respectively. Assembling Example 11.2.1, Example 11.2.3, Task E11.1.2, Task E11.1.3, and Task E11.1.4, we have proven that no two of $\left([a, b]\,, \mathcal{O}_{[a,b]}\right)$, $\left(]a, b[\,, \mathcal{O}_{]a,b[}\right)$, $\left([a, b[\,, \mathcal{O}_{[a,b[}\right)$, and $\left(]a, b]\,, \mathcal{O}_{]a,b]}\right)$ are homeomorphic.

Task E11.1.6. Let X be a figure of eight, viewed as a subset of $\mathbb{R}^2$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Prove that $(X, \mathcal{O}_X)$ is not homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

Can you find an argument which does not involve removing the junction point of the figure of eight, depicted below?

Task E11.1.7. Let $(X, \mathcal{O}_X)$ be the figure of eight of Task E11.1.6. Prove that $(X, \mathcal{O}_X)$ is not homeomorphic to the unit interval $(I, \mathcal{O}_I)$.

You may wish to appeal to Task E11.3.17.

Task E11.1.8. Let $X = \{a, b, c\}$ be a set with three elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}.$$

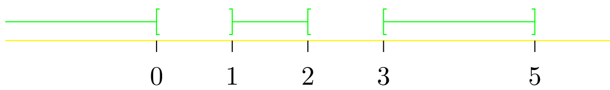
List all the subset of X, and determine whether each is a connected subset of X with respect to $\mathcal{O}_X$. For each which is not, explain why not. Find the connected component in $(X, \mathcal{O}_X)$ of each element of X.

Task E11.1.9. Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{b\}, \{e\}, \{a, b\}, \{b, e\}, \{c, d\}, \{a, b, e\}, \{b, c, d\}, \{c, d, e\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}.$$

Find the distinct connected components of $(X, \mathcal{O}_X)$. To save yourself a little work, you may wish to glance at Corollary E11.3.15 before proceeding.

Task E11.1.10. Let $X =]-\infty, 0[\cup]1, 2[\cup [3, 5]$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

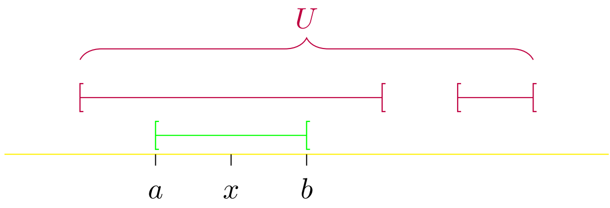


Prove that $(X, \mathcal{O}_X)$ has three distinct connected components.

Task E11.1.11. Let $X = I^2 \cup ([3, 4] \times [0, 1])$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $\sim$ be the equivalence relation on X which you defined in Task E6.1.8. Prove that $(X/\sim, \mathcal{O}_{X/\sim})$ has two distinct connected components.

Task E11.1.12. Let $\mathcal{O}_{\mathsf{Sorg}}$ be the set of subsets U of $\mathbb{R}$ such that if x belongs to U, then there is a half open interval $[a, b[$ such that x belongs to $[a, b[$, and such that $[a, b[$ is a subset of U.



Check that $\mathcal{O}_{\mathsf{Sorg}}$ defines a topology on $\mathbb{R}$. Suppose that x belongs to $\mathbb{R}$. Prove that the connected component of x in $(\mathbb{R}, \mathcal{O}_{\mathsf{Sorg}})$ is $\{x\}$.

Remark E11.1.13. The topological space $(\mathbb{R}, \mathcal{O}_{\mathsf{Sorg}})$ is known as the *Sorgenfrey line*. The topology $\mathcal{O}_{\mathsf{Sorg}}$ is also known as the *lower limit topology* on $\mathbb{R}$.

Task E11.1.14. Let $(X, \mathcal{O}_X)$ be a topological space. Let A_0 and A_1 be connected subsets of X with respect to $\mathcal{O}_X$. Is it necessarily the case that $A_0 \cap A_1$ is a connected subset of X? You may find it helpful to take $(X, \mathcal{O}_X)$ to be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

E11.2. In the lecture notes

Task E11.2.1. In the notation of Example 11.1.2, define a map

$$[0,5] \xrightarrow{f} X$$

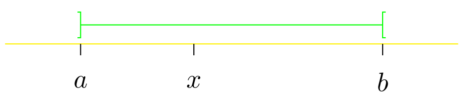
which captures the idea of 'snapping off' $]2,5]$ and 'moving it' to $]4,7]$. Prove that f is a bijection. Prove that f is not continuous.

Task E11.2.2. In the notation of Example 11.2.3, prove that if (I) holds and $f(a) > f(b)$, then

$$\big([a,b[\setminus \{f(a), f(b)\}, \mathcal{O}_{[a,b[\setminus\{f(a),f(b)\}}\big)$$

is not connected.

Task E11.2.3. Suppose that $a < x < b$ belong to $\mathbb{R}$. Let $\mathcal{O}_{]a,b[}$ denote the subspace topology on $]a,b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $\mathcal{O}_{]a,b[\setminus\{x\}}$ denote the subspace topology on $]a,b[\setminus \{x\}$ with respect to $\big(]a,b[, \mathcal{O}_{]a,b[}\big)$.



Suppose that $a_0 < a_1 < b_0 < b_1$ belong to $\mathbb{R}$. Let X be the union of $]a_0, a_1[$ and $]b_0, b_1[$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Prove that $\big(]a,b[\setminus\{x\}, \mathcal{O}_{]a,b[\setminus\{x\}}\big)$ is homeomorphic to $(X, \mathcal{O}_X)$. You may wish to proceed as follows.

(1) Let $\mathcal{O}_{]a,x[}$ denote the subspace topology on $]a,x[$ with respect to

$$\big(]a,b[\setminus\{x\}, \mathcal{O}_{]a,b[\setminus\{x\}}\big)\,.$$

Let $\mathcal{O}_{]a_0,a_1[}$ denote the subspace topology on $]a_0,a_1[$ with respect to $(X, \mathcal{O}_X)$.

By Task E2.3.1 and Example 7.3.3, observe that there is a homeomorphism

$$]a,x[\xrightarrow{f_0}]a_0,a_1[\,.$$

(2) Let $\mathcal{O}_{]x,b[}$ denote the subspace topology on $]x,b[$ with respect to

$$\big(]a,b[\setminus\{x\}, \mathcal{O}_{]a,b[\setminus\{x\}}\big)\,.$$

Let $\mathcal{O}_{]b_0,b_1[}$ denote the subspace topology on $]b_0,b_1[$ with respect to $(X, \mathcal{O}_X)$.

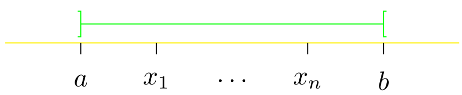
By Task E2.3.1 and Example 7.3.4, observe that there is a homeomorphism

$$]x,b[\xrightarrow{f_1}]b_0,b_1[\,.$$

(3) By Task E7.3.5, deduce from (1) and (2) that there is a homeomorphism

$$]a, b[\setminus \{x\} \longrightarrow X.$$

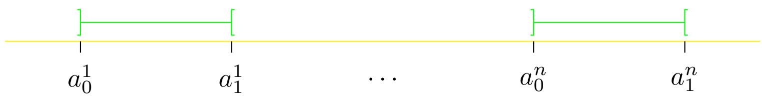
Task E11.2.4. Suppose that $a < x_1 < \ldots < x_n < b$ belong to $\mathbb{R}$. Let $\mathcal{O}_{]a,b[}$ denote the subspace topology on $]a, b[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $\mathcal{O}_{]a,b[\setminus\{x_1,\ldots,x_n\}}$ denote the subspace topology on $]a, b[\setminus \{x_1, \ldots, x_n\}$ with respect to $\big(]a, b[\,, \mathcal{O}_{]a,b[}\big)$.



Suppose that $a_0^1 < a_1^1 < \ldots < a_0^n < a_1^n$ belong to $\mathbb{R}$. Let X be

$$\bigcup_{1 \leq i \leq n} \left] a_0^i, a_1^i \right[.$$

Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Prove that $\big(]a, b[\setminus \{x_1, \ldots x_n\}, \mathcal{O}_{]a,b[\setminus\{x_1,\ldots,x_n\}}\big)$ is homeomorphic to $(X, \mathcal{O}_X)$. You may wish to proceed by induction, appealing to Task E11.2.3 and to Task E7.3.5.

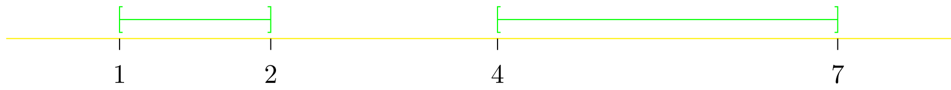
Task E11.2.5. Let X be a subset of $\mathbb{R}$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that x belongs to $\mathbb{R}$. Let $\mathcal{O}_{X\setminus\{x\}}$ be the subspace topology on $X \setminus \{x\}$ with respect to $(X, \mathcal{O}_X)$. Prove that $(X \setminus \{x\}, \mathcal{O}_{X\setminus\{x\}})$ is not connected.

Task E11.2.6. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Let A_0 and A_1 be connected subsets of X with respect to $\mathcal{O}_X$ which both satisfy (1) and (2) of Terminology 11.3.2. Prove that $A_0 = A_1$.

Task E11.2.7. Let $X = \{x\}$ be a set with one element. As discussed in Example **??**, the unique topology $\mathcal{O}_X$ on X is given by $\{\emptyset, X\}$. Then $(X, \mathcal{O}_X)$ is connected. Check that you understand why!

Task E11.2.8. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. Let A be a subset of X. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Prove that $\mathcal{O}_A$ is the discrete topology on A.

Task E11.2.9. Let $(X, \mathcal{O}_X)$ be as in Example 11.4.5.



Prove that if x belongs to $[4,7]$, then $\Gamma^x_{(X,\mathcal{O}_X)} = [4,7]$.

Task E11.2.10. Prove carefully the three assertions concerning numbers of connected components in Example 11.5.2.

Task E11.2.11. In the notation of Example 11.2.5, prove that $\left(I \setminus \{t\}, \mathcal{O}_{I\setminus\{t\}}\right)$ has two distinct connected components, and that $\left(S^1 \setminus \{f(t)\}, \mathcal{O}_{S^1\setminus\{f(t)\}}\right)$ is connected.

E11.3. For a deeper understanding

Task E11.3.1. Suppose that n belongs to $\mathbb{N}$, and that $n > 1$. Suppose that x belongs to $\mathbb{R}^n$. Let $\mathcal{O}_{\mathbb{R}^n\setminus\{x\}}$ be the subspace topology on $\mathbb{R}^n \setminus \{x\}$ with respect to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. Prove that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n\setminus\{x\}})$ is connected. You may wish to proceed as follows.

(1) Observe that $\mathbb{R}^n \setminus \{x\}$ is the union of $]-\infty, x[\times \mathbb{R}^{n-1}$, $]x, \infty[\times \mathbb{R}^{n-1}$, $\mathbb{R}^{n-1} \times]x, \infty[$, and $\mathbb{R}^{n-1} \times]-\infty, x[$.

(2) By Task E10.3.5 and Proposition 10.7.1, observe each of these four sets is a connected subset of $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

(3) By Task E2.3.1, deduce that each is a connected subset of $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n\setminus\{x\}})$.

(4) By Task E10.3.9, deduce that $(\mathbb{R}^n \setminus \{x\}, \mathcal{O}_{\mathbb{R}^n\setminus\{x\}})$ is connected.

Task E11.3.2. Let $\mathcal{O}_{[0,1[}$ be the subspace topology on $[0,1[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

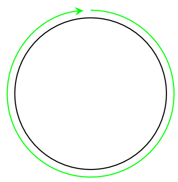
In Task E7.3.7, you were asked to prove that the map

$$[0,1[\xrightarrow{f} S^1$$

given by $t \mapsto \phi(t)$, where

$$\mathbb{R} \xrightarrow{\phi} S^1$$

is the map of Task E5.3.27. Prove that f is not a homeomorphism.



Task E11.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x_0 and x_1 belong to X. Prove that either $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X,\mathcal{O}_X)}$, or that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cap \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is empty. You may wish to proceed as follows.

(1) Suppose that x_0 and x_1 belong to X, and that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cap \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is not empty. By Proposition 11.3.7, we have that $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$ and $\Gamma^{x_1}_{(X,\mathcal{O}_X)}$ are connected subsets of X with respect to $\mathcal{O}_X$. By Task E10.3.9, deduce that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cup \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is a connected subset of X with respect to $\mathcal{O}_X$.

(2) By Remark 11.3.6, we have that x_0 belongs to $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$. Thus x_0 belongs to $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cup \Gamma^{x_1}_{(X,\mathcal{O}_X)}$. By (1) and the definition of $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$, deduce that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cup \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is a subset of $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$.

(3) Deduce that $\Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is a subset of $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$.

(4) Arguing as in (2) and (3), demonstrate that $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$ is a subset of $\Gamma^{x_1}_{(X,\mathcal{O}_X)}$.

(5) Conclude that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X,\mathcal{O}_X)}$.

Terminology E11.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x_0 and x_1 belong to X. Then $\Gamma^{x_0}_{(X,\mathcal{O}_X)}$ and $\Gamma^{x_1}_{(X,\mathcal{O}_X)}$ are *distinct* if $\Gamma^{x_0}_{(X,\mathcal{O}_X)} \cap \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ is empty.

Remark E11.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. By Remark 11.3.6, we have that x belongs to Γ_x. Thus $X = \bigcup_{x \in X} \Gamma^{x}_{(X,\mathcal{O}_X)}$.

Terminology E11.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that n belongs to $\mathbb{N}$. Then $(X, \mathcal{O}_X)$ has n *distinct connected components* if there is a set $\{x_j\}_{1 \leq j \leq n}$ of elements of X such that the following hold.

(1) We have that $X = \bigcup_{1 \leq j \leq n} \Gamma^{x_j}_{(X,\mathcal{O}_X)}$.

(2) For every $1 \leq j < k \leq n$, we have that $\Gamma^{x_j}_{(X,\mathcal{O}_X)}$ and $\Gamma^{x_k}_{(X,\mathcal{O}_X)}$ are distinct.

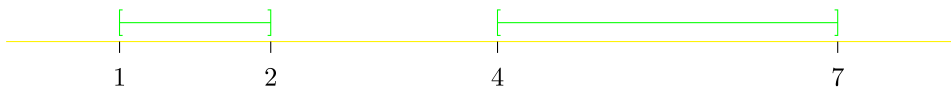
Remark E11.3.7. By Task E11.3.3, a topological space $(X, \mathcal{O}_X)$ has n distinct connected components if the set $\left\{ \Gamma^{x}_{(X,\mathcal{O}_X)} \right\}_{x \in X}$ has exactly n elements (remember that all equal elements of a set count as one!).

Remark E11.3.8. In particular, there is at most one n such that $(X, \mathcal{O}_X)$ has n distinct connected components.

Terminology E11.3.9. Let $(X, \mathcal{O}_X)$ be a topological space. Then $(X, \mathcal{O}_X)$ has *finitely many distinct connected components* if there is an $n \in \mathbb{N}$ such that $(X, \mathcal{O}_X)$ has n distinct connected components.

Example E11.3.10. Suppose that $(X, \mathcal{O}_X)$ is connected. By Example 11.4.1, we then have that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X,\mathcal{O}_X)}$ for all x_0 and x_1 which belong to X. Thus $(X, \mathcal{O}_X)$ has one distinct connected component.

Example E11.3.11. Let $X = [1,2] \cup [4,7]$.



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By 11.4.5, $(X, \mathcal{O}_X)$ has two distinct connected components.

Example E11.3.12. Let $\mathbb{Q}$ be the set of rational numbers. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Example 11.4.4, $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ has infinitely many distinct connected components.

Task E11.3.13. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Prove that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is closed with respect to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) By Proposition 11.3.7, we have that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is a connected subset of X with respect to $\mathcal{O}_X$. By Corollary E10.3.4, deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a connected subset of X with respect to $\mathcal{O}_X$.

(2) By Remark 11.3.6, we have that x belongs to $\Gamma^{x}_{(X,\mathcal{O}_X)}$. By Remark 8.3.3, deduce that x belongs to $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$. By (1) and the definition of $\Gamma^{x}_{(X,\mathcal{O}_X)}$, deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a subset of $\Gamma^{x}_{(X,\mathcal{O}_X)}$.

(3) By Remark 8.5.4, we have that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$. Deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is equal to $\Gamma^{x}_{(X,\mathcal{O}_X)}$.

(4) By Proposition 9.1.1, conclude that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is closed in X with respect to $\mathcal{O}_X$.

Task E11.3.14. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that $(X, \mathcal{O}_X)$ has finitely many distinct connected components. Prove that every connected component belongs to $\mathcal{O}_X$. You may wish to proceed as follows.

(1) Since $(X, \mathcal{O}_X)$ has finitely many distinct connected components, there is an $n \in \mathbb{N}$, and a set $\{x_j\}_{1 \leq j \leq n}$, such that $X = \bigcup_{1 \leq j \leq n} \Gamma^{x_j}_{(X,\mathcal{O}_X)}$, and $\Gamma^{x_j}_{(X,\mathcal{O}_X)}$ and $\Gamma^{x_k}_{(X,\mathcal{O}_X)}$ are distinct for every $1 \leq j < k \leq n$.

(2) Suppose that x belongs to X. By (1) and Task E11.3.3, observe that $\Gamma^{x}_{(X,\mathcal{O}_X)} = \Gamma^{x_k}_{(X,\mathcal{O}_X)}$ for some $1 \leq k \leq n$.

(3) By Task E11.3.13, we have that $\Gamma^{x_j}_{(X,\mathcal{O}_X)}$ is closed in X with respect to $\mathcal{O}_X$ for every $1 \leq j \in n$ such that $j \neq k$. By Remark E1.3.2, deduce that $X \setminus \Gamma^{x}_{(X,\mathcal{O}_X)} = \bigcup_{1 \,\leq\, j \,\leq\, n \text{ and } j \,\neq\, k} \Gamma^{x_j}_{(X,\mathcal{O}_X)}$ is closed in X with respect to $\mathcal{O}_X$.

(4) Conclude that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ belongs to $\mathcal{O}_X$.

Corollary E11.3.15. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that X is finite. Then every connected component of $(X, \mathcal{O}_X)$ belongs to $\mathcal{O}_X$.

Proof. If X is finite, then X has only finitely many subsets. Thus $(X, \mathcal{O}_X)$ has only finitely many distinct connected components. By Task E11.3.14, we deduce that every connected component of $(X, \mathcal{O}_X)$ belongs to $\mathcal{O}_X$. $\square$

Task E11.3.16. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a continuous map. Suppose that x belongs to X. Prove that $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a subset of $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$. You may wish to proceed as follows.

(1) By Proposition 11.3.7, we have that $\Gamma^{x}_{(X,\mathcal{O}_X)}$ is a connected subset of X with respect to $\mathcal{O}_X$. By Task E10.3.2, deduce that $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a connected subset of Y with respect to $\mathcal{O}_Y$.

(2) We have that $f(x)$ belongs to $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$. By definition of $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$, deduce that $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a subset of $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$.

Task E11.3.17. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Suppose that x belongs to X. Let $\mathcal{O}_{\Gamma^{x}_{(X,\mathcal{O}_X)}}$ be the subspace topology on $\Gamma^{x}_{(X,\mathcal{O}_X)}$ with respect to $(X, \mathcal{O}_X)$. Let $\mathcal{O}_{\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}}$ be the subspace topology on $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$ with respect to $(Y, \mathcal{O}_Y)$. Prove that $\left(\Gamma^{x}_{(X,\mathcal{O}_X)}, \mathcal{O}_{\Gamma^{x}_{(X,\mathcal{O}_X)}}\right)$ is homeomorphic to $\left(\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}, \mathcal{O}_{\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}}\right)$. You may wish to proceed as follows.

(1) By Task E11.3.16, we have that $f\left(\Gamma^{x}_{(X,\mathcal{O}_X)}\right)$ is a subset of $\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$. Since f is continuous, deduce by Task E5.1.8 and Task E5.1.9 that the map

$$\Gamma^{x}_{(X,\mathcal{O}_X)} \xrightarrow{f'} \Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$$

given by $y \mapsto f(y)$ is continuous.

(2) Since f is a homeomorphism, there is a continuous map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. By Task E11.3.16, we have that $g\left(\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}\right)$ is a subset of $\Gamma^{g(f(x))}_{(X,\mathcal{O}_X)}$. Since $g \circ f = id_X$, we have that $g\left(f(x)\right) = x$. Thus $g\left(\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}\right)$ is a subset of $\Gamma^{x}_{(X,\mathcal{O}_X)}$. Since g is continuous, deduce by Task E5.1.9 and Task E5.1.9 that the map

$$\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)} \xrightarrow{g'} \Gamma^{x}_{(X,\mathcal{O}_X)}$$

given by $y \mapsto g(y)$ is continuous.

(3) Observe that $g' \circ f' = id_{\Gamma^{x}_{(X,\mathcal{O}_X)}}$, and that $f' \circ g' = id_{\Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}}$. Conclude that f' is a homeomorphism.

Task E11.3.18. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Prove that there is a bijection between the set $\Gamma_{(X,\mathcal{O}_X)} = \left\{\Gamma^{x}_{(X,\mathcal{O}_X)}\right\}_{x \in X}$ and the set $\Gamma_{(Y,\mathcal{O}_Y)} = \left\{\Gamma^{y}_{(Y,\mathcal{O}_Y)}\right\}_{y \in Y}$. You may wish to proceed as follows.

(1) Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Suppose that x_0 and x_1 belong to X, and that $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X;\mathcal{O}_X)}$. As a corollary of (3) of Task E11.3.17, we have that $\Gamma^{f(x_0)}_{(Y,\mathcal{O}_Y)} = f\left(\Gamma^{x_0}_{(X,\mathcal{O}_X)}\right)$, and that $\Gamma^{f(x_1)}_{(Y,\mathcal{O}_Y)} = f\left(\Gamma^{x_1}_{(X,\mathcal{O}_X)}\right)$. Since $\Gamma^{x_0}_{(X,\mathcal{O}_X)} = \Gamma^{x_1}_{(X,\mathcal{O}_X)}$, we have that $f\left(\Gamma^{x_0}_{(X,\mathcal{O}_X)}\right) = f\left(\Gamma^{x_1}_{(X,\mathcal{O}_X)}\right)$. Deduce that $\Gamma^{f(x_0)}_{(Y,\mathcal{O}_Y)} = \Gamma^{f(x_1)}_{(Y,\mathcal{O}_Y)}$.

(2) By Task E7.3.2, there is a homeomorphism

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. Suppose that y_0 and y_1 belong to Y, and that $\Gamma^{y_0}_{(Y,\mathcal{O}_Y)} = \Gamma^{y_1}_{(Y,\mathcal{O}_Y)}$. Arguing as in (1), demonstrate that $\Gamma^{g(y_0)}_{(X,\mathcal{O}_X)} = \Gamma^{g(y_1)}_{(X,\mathcal{O}_X)}$.

(3) Let

$$\Gamma_X \xrightarrow{f'} \Gamma_Y$$

be the map given by $\Gamma^{x}_{(X,\mathcal{O}_X)} \mapsto \Gamma^{f(x)}_{(Y,\mathcal{O}_Y)}$. By (1), this map is well-defined. Let

$$\Gamma_Y \xrightarrow{g'} \Gamma_X$$

be the map given by $\Gamma^{y}_{(Y,\mathcal{O}_Y)} \mapsto \Gamma^{g(y)}_{(X,\mathcal{O}_X)}$. By (2), this map is well-defined. Observe that $g' \circ f' = id_{\Gamma_{(X,\mathcal{O}_X)}}$, and that $f' \circ g' = id_{\Gamma_{(Y,\mathcal{O}_Y)}}$,

Corollary E11.3.19. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Suppose that there is an $n \in \mathbb{N}$ such that $(X, \mathcal{O}_X)$ has n distinct connected components. Then $(Y, \mathcal{O}_Y)$ has n distinct connected components.

Proof. Follows immediately from Task E11.3.18 and Remark E11.3.7. □

Notation E11.3.20. Let J be a set. For every j which belongs to J, let X_j be a set. Let $\bigsqcup_{j\in J} X_j$ be the corresponding coproduct, in the sense of Definition A.3.3. We denote by

$$X_j \xrightarrow{i_j} \bigsqcup_{j\in J} X_j$$

the map given by $x \mapsto (x, j)$.

Task E11.3.21. Let J be a set. For every j which belongs to J, let $(X_j, \mathcal{O}_{X_j})$ be a topological space. Let $\mathcal{O}_{\bigsqcup_{j\in J} X_j}$ be the set of subsets U of the coproduct $\bigsqcup_{j\in J} X_j$ such that $i_j^{-1}(U)$ belongs to $\mathcal{O}_{X_j}$. Prove that $\left(\bigsqcup_{j\in J} X_j, \mathcal{O}_{\bigsqcup_{j\in J} X_j}\right)$ is a topological space. You may wish to look back at the proof of Proposition 6.1.5.

Terminology E11.3.22. We refer to $\mathcal{O}_{\bigsqcup_{j\in J} X_j}$ as the *coproduct topology* on $\bigsqcup_{j\in J} X_j$.

Task E11.3.23. Let J be a set. For every j which belongs to J, let $(X_j, \mathcal{O}_{X_j})$ be a topological space. Let $\bigsqcup_{j\in J} X_j$ be equipped with the coproduct topology $\mathcal{O}_{\bigsqcup_{j\in J} X_j}$. Observe that

$$X_j \xrightarrow{i_j} \bigsqcup_{j\in J} X_j$$

is continuous, for every j which belongs to J.

Task E11.3.24. For every pair of integers j and n such that $0 \leq j \leq n$, let $(X_j, \mathcal{O}_{X_j})$ be a topological space. How many connected components does $\left(\bigsqcup_{0\leq j\leq n} X_j, \mathcal{O}_{\bigsqcup_{0\leq j\leq n} X_j}\right)$ have? Prove that your guess holds!

E11.4. Exploration — bijections

Task E11.4.1. Suppose that $a < b$ and $a_0 < a_1 < b_0 < b_1$ belong to $\mathbb{R}$. Prove that there is a bijection

$$]a, b[\longrightarrow]a_0, a_1[\cup]b_0, b_1[\,.$$

You may wish to proceed as follows.

(1) A homeomorphism is in particular a bijection. By Example 7.3.4, we thus have that there is a bijection

$$]a, b[\xrightarrow{f}]a_0, a_1[\,.$$

By Task E7.2.1, deduce that f is an injection.

(2) As observed in Remark A.2.3, the inclusion map

$$]a_0, a_1[\xrightarrow{i}]a_0, a_1[\cup]b_0, b_1[$$

is an injection. By Proposition A.2.2, deduce that the map

$$]a, b[\xrightarrow{f \circ i}]a_0, a_1[\cup]b_0, b_1[$$

is an injection.

(3) By Example 7.3.10 and Task E7.3.2, there is a bijection

$$\mathbb{R} \xrightarrow{g}]a, b[\,.$$

By Task E7.2.1, deduce that f is an injection.

(4) The inclusion map

$$]a_0, a_1[\cup]b_0, b_1[\xrightarrow{j} \mathbb{R}$$

is an injection. By Proposition A.2.2, deduce that the map

$$]a_0, a_1[\cup]b_0, b_1[\xrightarrow{g \circ j}]a, b[$$

is an injection.

(5) By (2), (4), and Proposition A.2.5, conclude that there is a bijection

$$]a, b[\longrightarrow]a_0, a_1[\cup]b_0, b_1[\,.$$

Task E11.4.2. Find a bijection

$$[1, 2] \cup [4, 7] \xrightarrow{f} [1, 5]$$

You may wish to proceed as follows.

(1) Let f be the identity on $[1, 2]$.

(2) Send 4 to 3, and send 7 to 5.

(3) Appealing to Task E11.4.1, let f map $]4, 7[$ bijectively to $]2, 3[\cup]3, 5[$.

Task E11.4.3. Find a bijection between I and S^1. You may wish to proceed as follows.

(1) Map 0 to $(0, 1)$, and map 1 to $(0, -1)$.

(2) By Task E11.4.1, observe that there is a bijection from $]0, 1[$ to $\left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$.

(3) Use the bijection of (2) and the map ϕ of Task E5.3.27to map $]0, 1[$ bijectively to the union of

$$\left\{(x, y) \in S^1 \mid y > 0\right\}$$

and

$$\left\{(x, y) \in S^1 \mid y < 0\right\}.$$

Task E11.4.4. Find a bijection between $\mathbb{R}$ and $\mathbb{R}^n$. You may wish to proceed as follows.

(1) Observe that the map

$$\mathbb{R} \xrightarrow{f} \mathbb{R}^n$$

given by $x \mapsto (x, 0, \ldots, 0)$ is an injection.

(2) By Example 7.3.10 and Task E7.3.2, there is a bijection

$$\mathbb{R} \xrightarrow{g_1} \left]1, \tfrac{3}{2}\right[.$$

By Task E7.2.1, we have that g_1 is an injection. Let

$$\mathbb{R} \xrightarrow{g_n} \left]n, n + \tfrac{1}{2}\right[$$

be the map given by $x \mapsto g_1(x) + n - 1$. Since g_1 is an injection, deduce that g_n is an injection.

(3) Deduce from (2) that the map

$$\mathbb{R}^n \longrightarrow \mathbb{R}$$

given by $(x_1, \ldots, x_n) \mapsto (g_1(x_1), \ldots, g_n(x_n))$ is an injection.

(4) By (1), (3), and Proposition A.2.5, deduce that there is a bijection between $\mathbb{R}$ and $\mathbb{R}^n$.

E11.5. Exploration — totally disconnected topological spaces

Definition E11.5.1. A topological space $(X, \mathcal{O}_X)$ is *totally disconnected* if, for every x which belongs to X, the connected component of x in $(X, \mathcal{O}_X)$ is $\{x\}$.

Example E11.5.2. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. By Example 11.4.2, we have that $(X, \mathcal{O}_X)$ is totally disconnected.

Example E11.5.3. Let $\mathbb{Q}$ be the set of rational numbers. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Example 11.4.4, we have that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is totally disconnected.

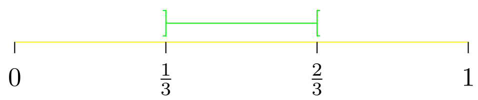
Example E11.5.4. By Task E11.1.12, the Sorgenfrey line $(\mathbb{R}, \mathcal{O}_{\mathsf{Sorg}})$ is totally disconnected.

Notation E11.5.5. Let Cantor be the subset of I given by

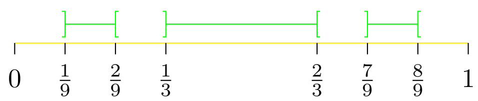
$$I \setminus \left(\bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N},\ 1 \leq n \leq 3^{m-1}} \left] \frac{3(n-1)+1}{3^m}, \frac{3(n-1)+2}{3^m} \right[\right).$$

Remark E11.5.6. In other words, Cantor is obtained as follows.

(1) Delete $\left] \frac{1}{3}, \frac{2}{3} \right[$ from I.



(2) Delete $\left] \frac{1}{9}, \frac{2}{9} \right[$ and $\left] \frac{7}{9}, \frac{8}{9} \right[$ from I.



(3) Delete $\left] \frac{1}{27}, \frac{2}{27} \right[$, $\left] \frac{7}{27}, \frac{8}{27} \right[$, $\left] \frac{19}{27}, \frac{20}{27} \right[$, and $\left] \frac{25}{27}, \frac{26}{27} \right[$ from I.

(4) Continue this pattern of deletions of open intervals for all 3^n, where n belongs to $\mathbb{N}$.

Terminology E11.5.7. We refer to Cantor as the *Cantor set*.

Task E11.5.8. Let $\mathcal{O}_{\mathsf{Cantor}}$ be the subspace topology on Cantor with respect to $(I, \mathcal{O}_I)$. Prove that $(\mathsf{Cantor}, \mathcal{O}_{\mathsf{Cantor}})$ is totally disconnected.

12. Tuesday 11th February

12.1. Further examples of the number of distinct connected components as an invariant

Example 12.1.1. Let K be the subset of $\mathbb{R}^2$ given by the union of

$$\left\{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\right\}$$

and

$$\left\{(x, y) \in \mathbb{R}^2 \mid x = y \text{ and } -1 \leq y \leq 1\right\}.$$

Let $\mathcal{O}_{\mathsf{K}}$ be the subspace topology on K with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$ be as in Example 11.5.2.

Suppose that

$$\mathsf{K} \xrightarrow{f} \mathsf{T}$$

is a homeomorphism. Let x be the point $(0, 0)$ of K.

Let $\mathcal{O}_{\mathsf{K}\setminus\{x\}}$ be the subspace topology on $\mathsf{K} \setminus \{x\}$ with respect to $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$. Then

$$\left(\mathsf{K} \setminus \{x\}, \mathcal{O}_{\mathsf{K}\setminus\{x\}}\right)$$

has four distinct connected components.

Let $\mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}$ be the subspace topology on $\mathsf{T} \setminus \{f(x)\}$ with respect to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$. Suppose that $f(x)$ is $(0, 1)$.

Then

$$\left(\mathsf{T} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}\right)$$

has three distinct connected components.

Suppose that $f(x) = (x', y')$. Suppose that $0 < |x'| < 1$.

Then

$$\left(\mathsf{T} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}\right)$$

has two distinct connected components.

Suppose that $-1 < y' < 1$.

Then

$$\left(\mathsf{T} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{T} \setminus \{f(x)\}}\right)$$

has two distinct connected components.

Suppose that $f(x)$ is $(-1, 1)$, $(1, 1)$, or $(0, -1)$.

Then

$$\left(\mathsf{T} \setminus \{f(x)\}, \mathcal{O}_{\mathsf{T} \setminus \{f(x)\}}\right)$$

is connected.

Thus

$$\left(\mathsf{T} \setminus \{f(x)\}\,, \mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}\right)$$

has at most three distinct connected components. Since f is a homeomorphism, we have, by Task E7.1.20, that there is a homeomorphism

$$\mathsf{K} \setminus \{x\} \longrightarrow \mathsf{T} \setminus \{f(x)\}\,.$$

By Corollary E11.3.19, since

$$\left(\mathsf{K} \setminus \{x\}, \mathcal{O}_{\mathsf{K}\setminus\{x\}}\right)$$

has four distinct connected components, we deduce that

$$\left(\mathsf{T} \setminus \{f(x)\}\,, \mathcal{O}_{\mathsf{T}\setminus\{f(x)\}}\right)$$

has four distinct connected components. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$\mathsf{K} \longrightarrow \mathsf{T}.$$

In other words, $(\mathsf{K}, \mathcal{O}_{\mathsf{K}})$ is not homeomorphic to $(\mathsf{T}, \mathcal{O}_{\mathsf{T}})$.

Remark 12.1.2. To fill in the details of the calculations of numbers of distinct connected components in Example 12.1.1 is the topic of Task **??**.

Example 12.1.3. Let $\varnothing$ be the subset of $\mathbb{R}^2$ given by the union of S^1 and

$$\{(x, y) \mid -1 \leq x \leq 1 \text{ and } x = y\}\,.$$

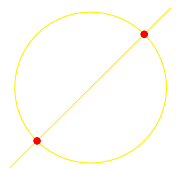
Let $\mathcal{O}_{\varnothing}$ be the subspace topology on $\varnothing$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

Let $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ be as in Example 11.5.2.

Suppose that

$$\varnothing \xrightarrow{f} \mathsf{I}$$

is a homeomorphism. Let x be the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ of $\varnothing$. Let y be the point $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ of $\varnothing$.



Let $\mathcal{O}_{\varnothing \setminus \{x,y\}}$ be the subspace topology on $\varnothing \setminus \{x, y\}$ with respect to $(\varnothing, \mathcal{O}_{\varnothing})$. Then

$$\left(\varnothing \setminus \{x, y\}, \mathcal{O}_{\varnothing \setminus \{x,y\}}\right)$$

has five distinct connected components.

Suppose that neither $f(x)$ nor $f(y)$ is $(0, 0)$ or $(0, 1)$.

Then

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x), f(y)\}}\right)$$

has three distinct connected components.

Suppose that one of $f(x)$ or $f(y)$ is $(0,0)$ or $(0,1)$, and that the other is neither $(0,0)$ nor $(0,1)$.

Then

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I}\setminus\{f(x),f(y)\}}\right)$$

has two distinct connected components.

Suppose that one of $f(x)$ or $f(y)$ is $(0,0)$, and that the other is $(0,1)$.

Then

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x), f(y)\}}\right)$$

is connected.

Thus

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x), f(y)\}}\right)$$

has at most three distinct connected components. Since f is a homeomorphism, we have by Task E7.1.20 that there is a homeomorphism

$$\varnothing \setminus \{x, y\} \longrightarrow \mathsf{I} \setminus \{f(x), f(y)\}.$$

By Corollary E11.3.19, since

$$\left(\varnothing \setminus \{x, y\}, \mathcal{O}_{\varnothing \setminus \{x,y\}}\right)$$

has five distinct connected components, we deduce that

$$\left(\mathsf{I} \setminus \{f(x), f(y)\}, \mathcal{O}_{\mathsf{I} \setminus \{f(x), f(y)\}}\right)$$

has five distinct connected components. Thus we have a contradiction. We conclude that there does not exist a homeomorphism

$$\varnothing \longrightarrow \mathsf{I}.$$

In other words, $(\varnothing, \mathcal{O}_{\varnothing})$ is not homeomorphic to $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$.

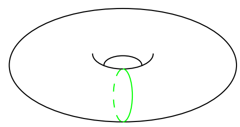
Remark 12.1.4. We cannot distinguish $(\varnothing, \mathcal{O}_{\mathsf{O}})$ from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ by removing just one point from each topological space and counting the resulting numbers of distinct connected components. To check that you understand why is the topic of Task E12.2.2.

Remark 12.1.5. To fill in the details of the calculations of numbers of distinct connected components in Example 12.1.3 is the topic of Task E12.2.3.

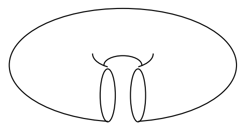
12.2. Can we take our technique further?

Remark 12.2.1. In all of our examples of distinguishing a pair of topological spaces by means of connectedness, at least one of the two has been 'one dimensional': built out of lines. Can our technique distinguish between 'higher dimensional' topological spaces?

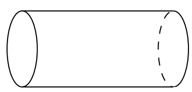
Remark 12.2.2. Let us try to distinguish $(T^2, \mathcal{O}_{T^2})$ from $(S^2, \mathcal{O}_{S^2})$. Let X be a subset of T^2 such that $(X, \mathcal{O}_X)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, where $\mathcal{O}_X$ is the subspace topology on X with respect to $(T^2, \mathcal{O}_{T^2})$.



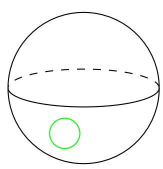
For the X depicted above, we have that $T^2 \setminus X$ is as depicted below.



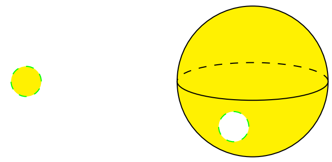
Let $\mathcal{O}_{T^2 \setminus X}$ be the subspace topology on $T^2 \setminus X$ with respect to $(T^2, \mathcal{O}_{T^2})$. We have that $(T^2 \setminus X, \mathcal{O}_{T^2 \setminus X})$ is homeomorphic to a cylinder.



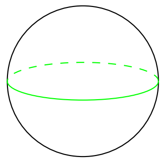
In particular, we have that $(T^2 \setminus X, \mathcal{O}_{T^2 \setminus X})$ is connected. Let Y be a subset of S^2 such that $(Y, \mathcal{O}_Y)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, where $\mathcal{O}_Y$ is the subspace topology on Y with respect to $(S^2, \mathcal{O}_{S^2})$.



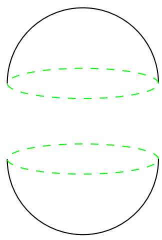
For any such Y, it seems intuitively that $S^2 \setminus Y$ has exactly two distinct connected components. In the example depicted above, we obtain the open disc enclosed by the circle, and the open subset of S^2 which remains after cutting out the closed disc enclosed by the circle.



Suppose that Y is the equator.



Then $S^2 \setminus Y$ consists of the northern hemisphere and the southern hemisphere.



Suppose that

$$T^2 \xrightarrow{f} S^2$$

is a homeomorphism. Let $(X, \mathcal{O}_X)$ be as above, with the property that $(T^2 \setminus X, \mathcal{O}_{T^2 \setminus X})$ is connected. Let $S^2 \setminus f(X)$ be equipped with the subspace topology $\mathcal{O}_{S^2 \setminus f(X)}$ with respect to $(S^2, \mathcal{O}_{S^2})$. Since f is a homeomorphism, we have, by Task E7.1.20, that there is a homeomorphism

$$T^2 \setminus \{X\} \longrightarrow S^2 \setminus f(X).$$

By Corollary 10.5.2, we deduce that $\left(S^2 \setminus f(X), \mathcal{O}_{S^2 \setminus f(X)}\right)$ is connected. Let $\mathcal{O}_{f(X)}$ be the subspace topology on $f(X)$ with respect to $(S^2, \mathcal{O}_{S^2})$.

Since f is a homeomorphism, we have, by Task **??**, that $\left(f(X), \mathcal{O}_{f(X)}\right)$ is homeomorphic to $(S^1, \mathcal{O}_{S^1})$. If our intuition is correct, we deduce that $\left(S^2 \setminus f(X), \mathcal{O}_{S^2 \setminus f(X)}\right)$ has exactly two distinct connected components. Thus we have a contradiction. We deduce that there does not exist a homeomorphism

$$T^2 \longrightarrow S^2.$$

In other words, $(T^2, \mathcal{O}_{T^2})$ and $(S^2, \mathcal{O}_{S^2})$ are not homeomorphic.

Remark 12.2.3. This argument *does* prove that $(T^2, \mathcal{O}_{T^2})$ is not homeomorphic to $(S^2, \mathcal{O}_{S^2})$. However, we have to be very careful! We must rigorously prove that $(S^2 \setminus Y, \mathcal{O}_{S^2 \setminus Y})$ has exactly two distinct connected components.

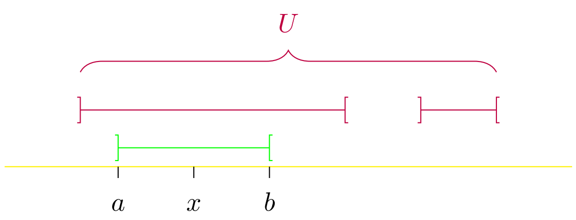
This is not at all an easy matter! Homeomorphism is a very flexible notion, and Y could be very wild. How do we know that the two examples we considered in Remark 12.2.2 are representative of all possible Y? We need to be sure that the requirement that we have a homeomorphism, as opposed to only a continuous surjection, excludes examples which are as wild as the Peano curve of Task **??**.

The fact that $(S^2 \setminus Y, \mathcal{O}_{S^2 \setminus Y})$ has exactly two distinct connected components, for any $(Y, \mathcal{O}_Y)$ which is homeomorphic to $(S^1, \mathcal{O}_{S^1})$, is known as the *Jordan curve theorem*.

12.3. Locally connected topological spaces

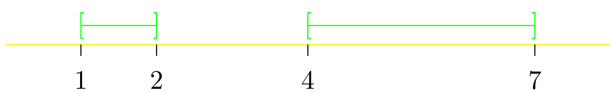
Definition 12.3.1. A topological space $(X, \mathcal{O}_X)$ is *locally connected* if, for every x which belongs to X, and every neighbourhood U of x in X with respect to $\mathcal{O}_X$, there is a neighbourhood W of x in X with respect to $\mathcal{O}_X$ which is both a connected subset of X with respect to $\mathcal{O}_X$, and a subset of U.

Example 12.3.2. Suppose that x belongs to $\mathbb{R}$. Let U be a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ to which x belongs, and which is a subset of U.

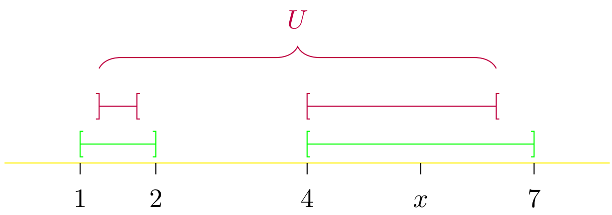


By Task E10.3.5, we have that $]a, b[$ is a connected subset of $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. We conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally connected.

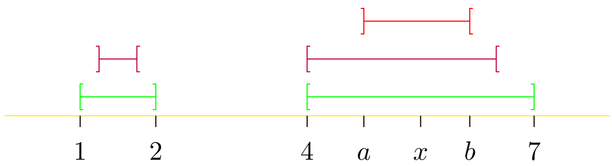
Example 12.3.3. Let $X = [1, 2] \cup [4, 7]$.



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that x belongs to $[4, 7]$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$.



By definition of $\mathcal{O}_X$ and $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$, to which x belongs, such that $X \cap]a, b[$ is a subset of U.



The following hold.

(1) By Task E1.3.5, we have that $[4, 7] \cap]a, b[$ is an interval. By Task E10.3.5, we deduce that $[4, 7] \cap]a, b[$ is a connected subset of $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

(2) By definition of $\mathcal{O}_X$, we have that $X \cap]a, b[$ belongs to $\mathcal{O}_X$. As was demonstrated in Example 9.6.2, we also have that $[4, 7]$ belongs to $\mathcal{O}_X$. Thus

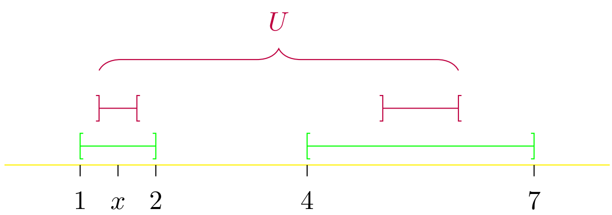
$$[4, 7] \cap]a, b[= [4, 7] \cap (X \cap]a, b[)$$

belongs to $\mathcal{O}_X$.

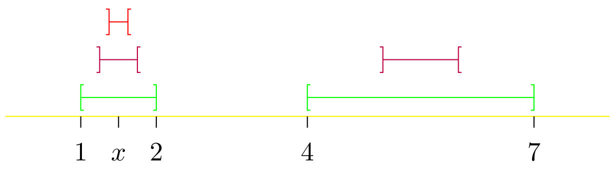
By Task E12.3.2, we deduce from (1) and (2) that $[4, 7] \cap]a, b[$ is a connected subset of X with respect to $\mathcal{O}_X$.

In addition, we have that x belongs to $[4, 7] \cap]a, b[$. By (2), we thus have that $[4, 7] \cap]a, b[$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$. Moreover, since $[4, 7]$ is a subset of X, and since $X \cap]a, b[$ is a subset of U.

Suppose now that x belongs to $[1, 2]$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$.



By an analogous argument to that which we gave in the case that x belongs to $[4,7]$, there is an open interval $]a',b'[$ such that $[1,2] \cap]a',b'[$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$, is a connected subset of X with respect to $\mathcal{O}_X$, and is a subset of U.



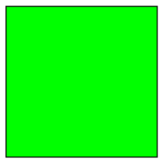
To fill in the details is the topic of Task E12.2.5. We conclude that $(X, \mathcal{O}_X)$ is locally connected.

Remark 12.3.4. The ingredients of this argument can be organised into a more general method for proving that a topological space is locally connected. By Task E2.3.1 and Task E12.3.9, both $[1,2]$ and $[4,7]$ are connected subsets of X with respect to $\mathcal{O}_X$. Moreover, as was demonstrated in Example 9.6.2, both $[1,2]$ and $[4,7]$ belong to $\mathcal{O}_X$. By Task E12.3.8, we conclude that $(X, \mathcal{O}_X)$ is locally connected.

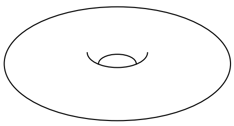
Example 12.3.5. By Example 12.3.2, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is locally connected. By Task E12.1.7, we deduce that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is locally connected.

Example 12.3.6. By Task E12.3.9, we have that $(I, \mathcal{O}_I)$ is locally connected.

By Task E12.1.7, we deduce that $(I^2, \mathcal{O}_{I^2})$ is locally connected.



Example 12.3.7. By Example 12.3.6, we have that $(I^2, \mathcal{O}_{I^2})$ is locally connected. By Task E12.3.10, we deduce that $(T^2, \mathcal{O}_{T^2})$ is locally connected.

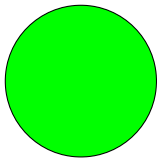


Remark 12.3.8. By a similar argument, $(M^2, \mathcal{O}_{M^2})$ and $(K^2, \mathcal{O}_{K^2})$ are locally connected. To check that you understand this is the topic of Task E12.2.6.

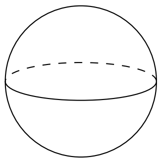
Example 12.3.9. By Example 12.3.6, we have that $(I^2, \mathcal{O}_{I^2})$ is locally connected. By Task E7.2.9, there is a homeomorphism

$$I^2 \longrightarrow D^2.$$

By Task E12.1.8, we deduce that $(D^2, \mathcal{O}_{D^2})$ is connected.



Example 12.3.10. By Example 12.3.9, we have that $(D^2, \mathcal{O}_{D^2})$ is locally connected. By Task E12.3.10, we deduce that $(S^2, \mathcal{O}_{S^2})$ is locally connected.



Example 12.3.11. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that q belongs to $\mathbb{Q}$. By Example 11.4.4, we have that $\Gamma^q_{(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})}$ is $\{q\}$. In other words, $\{q\}$ is the only connected subset of $\mathbb{Q}$ to which q belongs. However, the set $\{q\}$ does not belong to $\mathcal{O}_{\mathbb{Q}}$. To check this is the topic of Task E12.2.4. Thus there is no neighbourhood of q in $\mathbb{Q}$ with respect to $\mathcal{O}_{\mathbb{Q}}$ which is a connected subset of $\mathbb{Q}$ with respect to $\mathcal{O}_{\mathbb{Q}}$. We conclude that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally connected.

Remark 12.3.12. We could also argue as follows. By Example 11.4.4, we have that $\Gamma^q_{(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})}$ is $\{q\}$. The set $\{q\}$ does not belong to $\mathcal{O}_{\mathbb{Q}}$, as you are asked to check in Task E12.2.4. By Corollary E12.3.4, we deduce that $(\mathbb{Q}, \mathcal{O}_{\mathbb{Q}})$ is not locally connected.

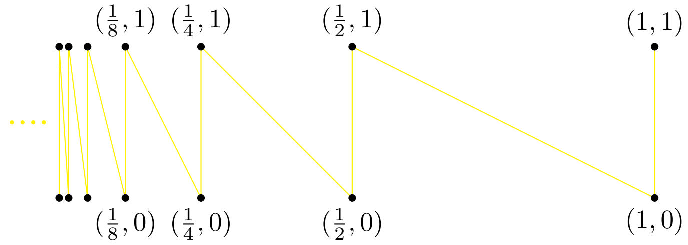
12.4. A topological space which is connected but not locally connected

Example 12.4.1. Let A be the subset of $\mathbb{R}^2$ given by the union of the sets

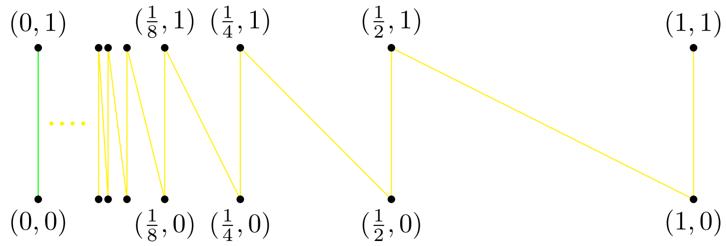
$$\bigcup_{n \in \mathbb{N}} \left\{ \left(\frac{1}{2^{n-1}}, y \right) \mid y \in [0, 1] \right\}$$

and

$$\bigcup_{n \in \mathbb{N}} \left\{ (x, -2^n x + 2) \mid x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \right\}.$$



Let X be the closure of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. By Task E8.1.7, we have that X is the union of A and the line $\{0\} \times [0, 1]$.

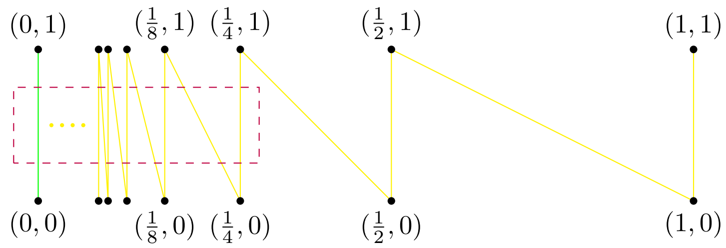


Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. By Task E10.1.6, we have that $(X, \mathcal{O}_X)$ is connected.

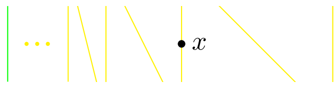
Let U be the neighbourhood of $\left(0, \frac{1}{2}\right)$ in X with respect to $\mathcal{O}_X$ given by

$$X \cap \left(\left] -1, \tfrac{9}{32} \right[\times \left] \tfrac{1}{4}, \tfrac{3}{4} \right[\right).$$

Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X, \mathcal{O}_X)$.



Suppose that (x, y) belongs to U. Let B be a subset of U to which both $(0, \frac{1}{2})$ and (x, y) belong. Let $\mathcal{O}_B$ be the subspace topology on B with respect to $(U, \mathcal{O}_U)$. Suppose that $x = \frac{1}{2^n}$, where n belongs to $\mathbb{N}$.



Let c be a real number with the property that $\frac{7}{2^{n+2}} < c < \frac{1}{2^n}$. Then B is the union of, for example,

$$B \cap \left(]-1, c[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

and

$$B \cap \left(]c, 2[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right),$$

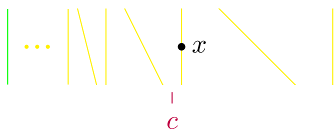
and this union is disjoint. The only significance in the choice of -1 and 2 is that $-1 < 0$, and $2 > 1$. Both

$$B \cap \left(]-1, c[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

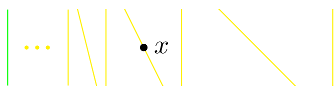
and

$$B \cap \left(]c, 2[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

belong to $\mathcal{O}_B$. Thus B is not a connected subset of U with respect to $\mathcal{O}_U$.



Suppose instead that $\frac{5}{2^{n+2}} < x < \frac{7}{2^{n+2}}$, where n belongs to $\mathbb{N}$.



Let c be a real number with the property that $\frac{1}{2^{n+1}} < c < \frac{5}{2^{n+2}}$. Then B is the union of, for example,

$$B \cap \left(]-1, c[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

and

$$B \cap \left(]c, 2[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right),$$

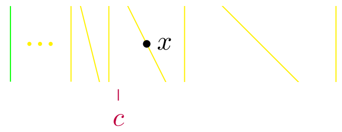
and this union is disjoint. Moreover, both

$$B \cap \left(]-1, c[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

and

$$B \cap \left(]c, 2[\times]\tfrac{1}{4}, \tfrac{3}{4}[\right)$$

belong to $\mathcal{O}_B$. Thus B is not a connected subset of U with respect to $\mathcal{O}_U$.



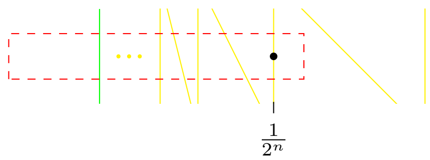
We have now demonstrated that if $x > 0$, then there is no connected subset of U with respect to $\mathcal{O}_U$ to which both $(0, \frac{1}{2})$ and (x, y) belong. Thus $\Gamma^{(0,\frac{1}{2})}_{(U,\mathcal{O}_U)}$ is a subset of $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$. We have that $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ is a connected subset of U with respect to $\mathcal{O}_U$. To check this is the topic of Task E12.2.7. We conclude that $\Gamma^{(0,\frac{1}{2})}_{(U,\mathcal{O}_U)}$ is $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$.

Suppose that $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ belongs to $\mathcal{O}_U$. By Task E2.3.1 and the definition of $\mathcal{O}_{\mathbb{R}^2}$, there are real numbers $a_0 < 0 < a_1$ and $\frac{1}{4} \leq b_0 < \frac{1}{2} < b_1 \leq \frac{3}{4}$ such that

$$U \cap (]a_0, a_1[\times]b_0, b_1[)$$

is a subset of $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$. Let n be a natural number such that $0 < \frac{1}{2^n} < a_1$. Then $(\frac{1}{2^n}, \frac{1}{2})$ belongs to

$$U \cap (]a_0, a_1[\times]b_0, b_1[).$$



Since $(\frac{1}{2^n}, \frac{1}{2})$ does not belong to $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$, we have a contradiction. We conclude that $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ does not belong to $\mathcal{O}_U$.

Putting everything together, we have demonstrated that $\Gamma^{(0,\frac{1}{2})}_{(U,\mathcal{O}_U)}$ does not belong to $\mathcal{O}_U$. By Task E12.3.3, we conclude that $(X, \mathcal{O}_X)$ is not locally connected.

Remark 12.4.2. The topological space $(X, \mathcal{O}_X)$ is a variant of a topological space known as the *topologist's sine curve*.

Remark 12.4.3. We could have proven that $(X, \mathcal{O}_X)$ is not locally connected by working with any $(0, y)$ such that $0 \leq y \leq 1$ in place of $(0, \frac{1}{2})$. To check that you understand this is the topic of Task E12.2.8.

Remark 12.4.4. In a nutshell, the reason that $(X, \mathcal{O}_X)$ is not locally connected is that, for any particular (x, y) which belongs to X with $x > 0$, there is a 'gap' between (x, y) and the y-axis, which is detected when we explore connectedness 'locally' around (x, y).

When we work 'globally', namely when we consider $(X, \mathcal{O}_X)$ as a whole, there is no 'gap' between the y-axis and the rest of X, because the intervals zig-zag infinitely closely towards the y-axis.

E12. Exercises for Lecture 12

E12.1. Exam questions

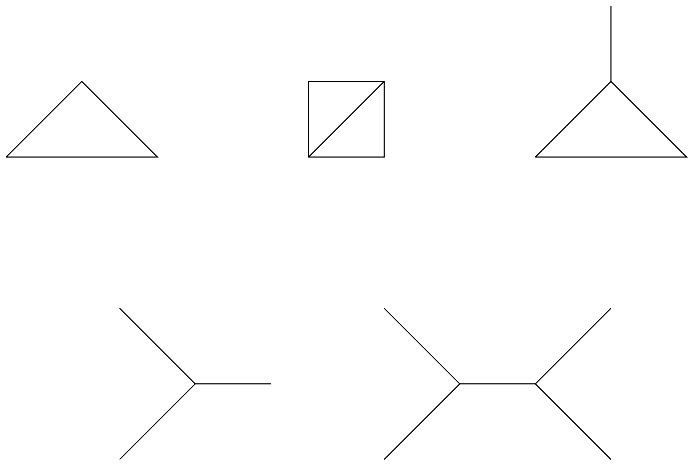
Task E12.1.1 (Continuation Exam, August 2013)**.** Let X be the subset of $\mathbb{R}^2$ depicted below.

Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let Y be the subset of $\mathbb{R}^2$ depicted below.

Let $\mathcal{O}_Y$ be the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Is $(X, \mathcal{O}_X)$ homeomorphic to $(Y, \mathcal{O}_Y)$?

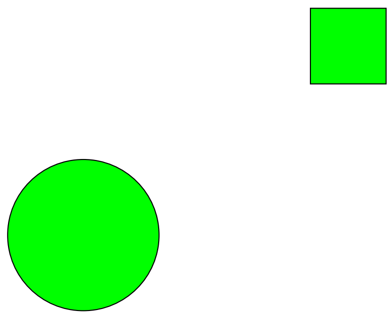
Task E12.1.2. View the letters B, C, D, E, F, G, H as subsets of $\mathbb{R}^2$. Let each be equipped with the subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Which of the letters are homeomorphic, and which are not?

Task E12.1.3. View each of the following shapes as a subset of $\mathbb{R}^2$. Each consists of intervals glued together. In particular, all of the shapes have no 'inside'.



Let each shape be equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that no two of the shapes are homeomorphic.

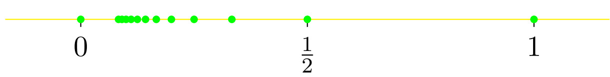
Task E12.1.4. Let X be the union of D^2 and $[3,4] \times [2,3]$.



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that $(X, \mathcal{O}_X)$ is locally connected. You may wish to appeal to Task E12.3.8.

Task E12.1.5. Let X be a set. Let $\mathcal{O}_X$ be the discrete topology on X. Prove that $(X, \mathcal{O}_X)$ is locally connected.

Task E12.1.6. Let X be the subset of $\mathbb{R}$ given by the union of $\{0\}$ and $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $(X, \mathcal{O}_X)$ is not locally connected.



Task E12.1.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally connected topological spaces. Prove that $(X \times Y, \mathcal{O}_{X \times Y})$ is locally connected.

Task E12.1.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is locally connected. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Prove that $(Y, \mathcal{O}_Y)$ is locally connected.

E12.2. In the lecture notes

Task E12.2.1. Prove carefully the assertions concerning numbers of distinct connected components in Example 12.1.1.

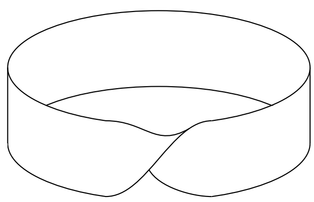
Task E12.2.2. How many distinct connected components can we obtain by removing one point from O? Explain why your answer means that we cannot distinguish $(\varnothing, \mathcal{O}_{\mathsf{O}})$ from $(\mathsf{I}, \mathcal{O}_{\mathsf{I}})$ by removing just one point from each topological space, and counting the resultings numbers of distinct connected components.

Task E12.2.3. Prove carefully the assertions concerning numbers of distinct connected components in Example 12.1.3.

Task E12.2.4. Let $\mathcal{O}_{\mathbb{Q}}$ be the subspace topology on $\mathbb{Q}$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Suppose that q belongs to $\mathbb{Q}$. Prove that $\{q\}$ does not belong to $\mathcal{O}_{\mathbb{Q}}$.

Task E12.2.5. Let $(X, \mathcal{O}_X)$ be as in Example 12.3.3. Suppose that x belongs to $[1, 2]$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$. Prove that there is an open interval $]a', b'[$ such that $[1, 2] \cap]a', b'[$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$, is a connected subset of X with respect to $\mathcal{O}_X$, and is a subset of U.

Task E12.2.6. Prove that $(M^2, \mathcal{O}_{M^2})$ is locally connected.



Task E12.2.7. Let $(U, \mathcal{O}_U)$ be as in Example 12.4.1. Prove that $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ is a connected subset of U with respect to $\mathcal{O}_U$. You may wish to proceed as follows.

(1) Let $\mathcal{O}_{\{0\}\times]\frac{1}{4},\frac{3}{4}[}$ be the subspace topology on $\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[$ with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $\mathcal{O}_{]\frac{1}{4},\frac{3}{4}[}$ be the subspace topology on $\left]\frac{1}{4}, \frac{3}{4}\right[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $\left(\{0\} \times \left]\frac{1}{4}, \frac{3}{4}\right[, \mathcal{O}_{\{0\}\times]\frac{1}{4},\frac{3}{4}[}\right)$ is homeomorphic to $\left(\left]\frac{1}{4}, \frac{3}{4}\right[, \mathcal{O}_{]\frac{1}{4},\frac{3}{4}[}\right)$. You may wish to look back at your argument for Task E7.1.8.

(2) Appeal to Task E2.3.1, and to Corollary 10.5.2.

Task E12.2.8. Let $(X, \mathcal{O}_X)$ be as in Example 12.4.1. Prove that $(X, \mathcal{O}_X)$ is not locally connected by working with $(0, 1)$ rather than $(0, \frac{1}{2})$. Can you furthermore see how to adapt the argument of Example 12.4.1 to any $(0, y)$ such that $0 \leq y \leq 1$?

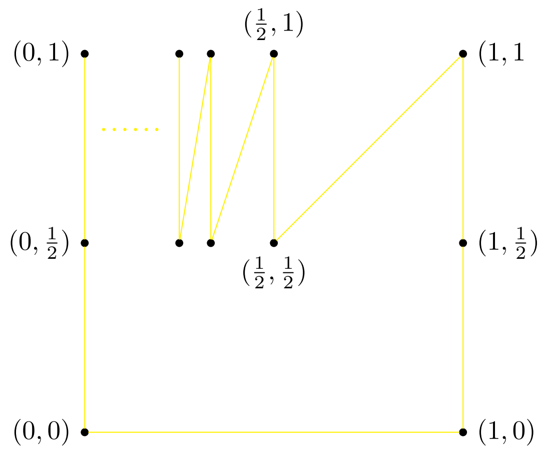
Task E12.2.9. Let X be the subset of $\mathbb{R}^2$ given by the union of the sets $\{0, 1\} \times [0, \frac{1}{2}]$, $I \times \{0\}$,

$$\bigcup_{n \geq 0} \left\{ \left(\frac{1}{2^n}, y \right) \mid y \in \left[\frac{1}{2}, 1 \right] \right\},$$

and

$$\bigcup_{n \geq 0} \left\{ (x, 2^n x) \mid x \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n} \right] \right\},$$

where n is an integer.



Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Check that you understand Example 12.4.1 by proving that $(X, \mathcal{O}_X)$ is not locally connected.

Remark E12.2.10. The topological space $(X, \mathcal{O}_X)$ is a variant of a topological space known as the *Warsaw circle*.

E12.3. For a deeper understanding

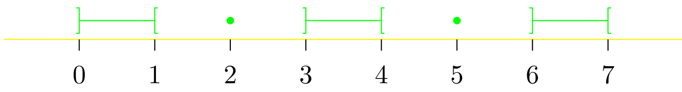
Task E12.3.1. Let X be the subset of $\mathbb{R}$ given by

$$]0, 1[\cup \{2\} \cup]3, 4[\cup \{5\} \cup]6, 7[\cup \{8\} \cdots .$$

In other words, X is given by

$$\bigcup_{n \in \mathbb{N}}]3n - 3, 3n - 2[\cup \{3n - 1\}.$$

Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



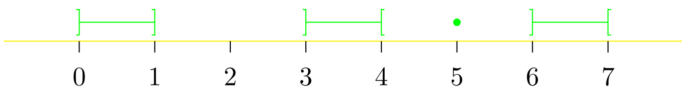
Let Y be the subset of $\mathbb{R}$ given by

$$]0,1] \cup]3,4[\cup \{5\} \cup]6,7[\cup \{8\} \cup \cdots .$$

In other words, Y is given by

$$]0,1] \cup \left(\bigcup_{n \in \mathbb{N}}]3n, 3n+1[\cup \{3n+2\} \right).$$

Let $\mathcal{O}_Y$ be the subspace topology on $\mathcal{O}_Y$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.



Prove that there is a continuous bijection

$$X \xrightarrow{f} Y$$

and a continuous bijection

$$Y \xrightarrow{g} X,$$

but that $(X, \mathcal{O}_X)$ is not homeomorphic to $(Y, \mathcal{O}_Y)$. You may wish to proceed to as follows.

(1) Let

$$X \xrightarrow{f} Y$$

be given by

$$f(x) = \begin{cases} x & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

Observe that f is a bijection. By Task E5.3.14 and Task E5.3.23 (1), observe that f is continuous.

(2) Let

$$Y \xrightarrow{g} X$$

be given by

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \in \,]0,1], \\ \frac{x-2}{2} & \text{if } x \in \,]3,4[, \\ x-3 & \text{otherwise.} \end{cases}$$

Observe that g is a bijection. By Task E5.3.14 and Task E5.3.23 (1), observe that g is continuous.

(3) Suppose that y belongs to $]0,1]$. Demonstrate that Γ_y is $]0,1]$.

(4) Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are homeomorphic. Let $\mathcal{O}_{]0,1]}$ be the subspace topology on $]0,1]$ with respect to $(Y, \mathcal{O}_Y)$. By Task E11.3.17, deduce from (3) that there is an x which belongs to X with the property that $\big(]0,1]\,, \mathcal{O}_{]0,1]}\big)$ is homeomorphic to $(\Gamma_x, \mathcal{O}_{\Gamma_x})$, where $\mathcal{O}_{\Gamma_x}$ is the subspace topology on Γ_x with respect to $(X, \mathcal{O}_X)$.

(5) Suppose that n belongs to $\mathbb{N}$. Demonstrate that if x belongs to $]3n-3, 3n-2[$, then Γ_x is $]3n-3, 3n-2[$. Demonstrate that if x is $3n-1$, then Γ_x is $\{3n-1\}$.

(6) Let $\mathcal{O}_{]3n-3,3n-2[}$ be the subspace topology on $]3n-3, 3n-2[$ with respect to $(X, \mathcal{O}_X)$. By Task E2.3.1 and Task E11.1.3 we have that $\big(]0,1]\,, \mathcal{O}_{]0,1]}\big)$ is not homeomorphic to $\big(]3n-3, 3n-2[\,, \mathcal{O}_{]3n-3,3n-2[}\big)$.

(7) Let $\mathcal{O}_{\{3n-1\}}$ be the subspace topology on $\{3n-1\}$ with respect to $(X, \mathcal{O}_X)$. Observe that $\big(]0,1]\,, \mathcal{O}_{]0,1]}\big)$ is not homeomorphic to $\big(\{3n-1\}, \mathcal{O}_{\{3n-1\}}\big)$, since there cannot be a bijection between a set with one element and $]0,1]$. To check that you understand this was the topic of Task E7.2.2.

(8) Observe that (6) and (7) together contradict (4) and (5). Conclude that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are not homeomorphic.

Task E12.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let U be a subset of X which belongs to $\mathcal{O}_X$. Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X, \mathcal{O}_X)$. Let A be a connected subset of X with respect to $\mathcal{O}_X$. Suppose that A is a subset of U. Prove that A is a connected subset of U with respect to $\mathcal{O}_U$. You may wish to proceed as follows.

(1) Let U_0 and U_1 be subsets of A such that $A = U_0 \sqcup U_1$, and such that both U_0 and U_1 belong to $\mathcal{O}_U$. By Task E2.3.3 (1), observe that U_0 and U_1 belong to $\mathcal{O}_X$.

(2) Since A is a connected subset of X with respect to $\mathcal{O}_X$, deduce that at least one of U_0 and U_1 is empty. Conclude that A is a connected subset of U with respect to $\mathcal{O}_U$.

Task E12.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Prove that $(X, \mathcal{O}_X)$ is locally connected if and only if, for every subset U of X which belongs to $\mathcal{O}_X$, we have that $\Gamma^x_{(U,\mathcal{O}_U)}$ belongs to $\mathcal{O}_X$, where $\mathcal{O}_U$ is the subspace topology on U with respect to $(X, \mathcal{O}_X)$. You may wish to proceed as follows.

(1) Suppose that $(X, \mathcal{O}_X)$ is locally connected. Let U be a subset of X which belongs to $\mathcal{O}_X$. Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X, \mathcal{O}_X)$. Suppose that x belongs to U. Since $(X, \mathcal{O}_X)$ is locally connected, there is a neighbourhood W of x in X with respect to $\mathcal{O}_X$ such that W is a connected subset of X with respect to $\mathcal{O}_X$, and such that W is a subset of U. By Task E12.3.2, we have that W is a connected subset of U with respect to $\mathcal{O}_U$. Deduce that W is a subset of $\Gamma^x_{(U,\mathcal{O}_U)}$.

(2) By Task E8.3.1, deduce that $\Gamma^x_{(U,\mathcal{O}_U)}$ belongs to $\mathcal{O}_X$.

(3) Conversely, suppose that, for every subset U of X which belongs to $\mathcal{O}_X$, we have that $\Gamma^x_{(U,\mathcal{O}_U)}$ belongs to $\mathcal{O}_X$. Suppose that x belongs to X. Let U_x be a neighbourhood of x in X with respect to $\mathcal{O}_X$. Then $\Gamma^x_{(U_x,\mathcal{O}_{U_x})}$ is a connected subset of U_x with respect to $\mathcal{O}_{U_x}$. By assumption, we have that $\Gamma^x_{(U_x,\mathcal{O}_{U_x})}$ belongs to $\mathcal{O}_X$. Conclude that $(X, \mathcal{O}_X)$ is locally connected.

Corollary E12.3.4. Let $(X, \mathcal{O}_X)$ be a locally connected topological space. Suppose that x belongs to $(X, \mathcal{O}_X)$. Then $\Gamma^x_{(X,\mathcal{O}_X)}$ belongs to $\mathcal{O}_X$.

Proof. Follows immediately from Task E12.3.3, since X belongs to $\mathcal{O}_X$. □

Task E12.3.5. Let $(X, \mathcal{O}_X)$ be a locally connected topological space. Let U be a subset of X which belongs to $\mathcal{O}_X$. Let $\mathcal{O}_U$ be the subspace topology on U with respect to $(X, \mathcal{O}_X)$. Prove that $(U, \mathcal{O}_U)$ is locally connected. You may wish to appeal to Task E12.3.3, Task E2.3.3 (1), and Task E2.3.1.

Task E12.3.6. Prove that a topological space $(X, \mathcal{O}_X)$ is both totally disconnected and locally connected if and only if $\mathcal{O}_X$ is the discrete topology on X. You may wish to appeal to Task E12.1.5 and to Corollary E12.3.4.

Task E12.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that X is finite. Prove that $(X, \mathcal{O}_X)$ is locally connected. You may wish to appeal to Task E11.3.15 and to Task E12.3.3.

Task E12.3.8. Let $(X, \mathcal{O}_X)$ be a topological space. Let X_0 and X_1 be subsets of X which belong to $\mathcal{O}_X$. Suppose that $X = X_0 \sqcup X_1$. Let $\mathcal{O}_{X_0}$ be the subspace topology on X_0 with respect to $(X, \mathcal{O}_X)$, and let $\mathcal{O}_{X_1}$ be the subspace topology on X_1 with respect to $(X, \mathcal{O}_X)$. Prove that $(X, \mathcal{O}_X)$ is locally connected if and only if both $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ are locally connected. You may wish to proceed as follows.

(1) Suppose that $(X, \mathcal{O}_X)$ is locally connected. By Task E12.3.5, deduce that $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ are locally connected.

(2) Suppose that $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ are locally connected. Suppose that x belongs to X. Since $X = X_0 \sqcup X_1$, observe that either x belongs to X_0 or that x belongs to X_1.

(3) Suppose that x belongs to X_0. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$. Then $X_0 \cap U$ is a neighbourhood of x in X_0 with respect to $\mathcal{O}_{X_0}$. Since $(X_0, \mathcal{O}_{X_0})$ is locally connected, deduce that there is a neighbourhood U_x of x in X_0 with respect to $\mathcal{O}_{X_0}$ such that U_x is both a connected subset of X_0 with respect to $\mathcal{O}_{X_0}$, and a subset of $X_0 \cap U$.

(4) Since $X_0 \cap U$ is a subset of U, observe that U_x is a subset of U.

(5) By Task E2.3.3 (1), since X_0 belongs to $\mathcal{O}_X$ and U_x is a neighbourhood of x in X_0 with respect to $\mathcal{O}_{X_0}$, deduce that U_x is a neighbourhood of x in X with respect to $\mathcal{O}_X$.

(6) By Task E2.3.1, since U_x is a connected subset of X_0 with respect to $\mathcal{O}_{X_0}$, deduce that U_x is a connected subset of X with respect to $\mathcal{O}_X$.

(7) By an analogous argument, observe that if x belongs to X_1, then there is a neighbourhood U_x of x in X with respect to $\mathcal{O}_X$ which is both a connected subset of X with respect to $\mathcal{O}_X$, and a subset of U.

(8) Conclude from (4) – (7) that $(X, \mathcal{O}_X)$ is locally connected.

Task E12.3.9. Let X be an interval. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Prove that $(X, \mathcal{O}_X)$ is locally connected. You may wish to appeal to Task E1.3.5 and to Task E10.3.5.

Task E12.3.10. Let $(X, \mathcal{O}_X)$ be a locally connected topological space. Let $\sim$ be an equivalence relation on X. Prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is locally connected. You may wish to proceed as follows.

(1) Suppose that $[x]$ belongs to $X/\sim$. Let U be a neighbourhood of $[x]$ in $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. By Remark 6.1.9, we have that π is continuous. Thus, observe that $\pi^{-1}(U)$ belongs to $\mathcal{O}_X$.

(2) Let $\mathcal{O}_{\pi^{-1}(U)}$ be the subspace topology on $\pi^{-1}(U)$ with respect to $(X, \mathcal{O}_X)$. By Corolllary E12.3.4, observe that

$$\Gamma^x_{\left(\pi^{-1}(U), \mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a neighbourhood of x in X with respect $\mathcal{O}_X$, and is a connected subset of X with respect to $\mathcal{O}_X$.

(3) Since

$$\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a connected subset of X with respect to $\mathcal{O}_X$, deduce, by Task E10.3.2, that

$$\pi\left(\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}\right)$$

is a connected subset of $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$.

(4) Let $\mathcal{O}_U$ be the subspace topology on U with respect to $\left(X/\sim, \mathcal{O}_{X/\sim}\right)$. By definition of $\Gamma^{[x]}_{(U,\mathcal{O}_U)}$, deduce that

$$\pi\left(\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}\right)$$

is a subset of $\Gamma^{[x]}_{(U,\mathcal{O}_U)}$.

(5) Deduce that

$$\pi^{-1}\left(\pi\left(\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}\right)\right)$$

is a subset of $\pi^{-1}\left(\Gamma^{[x]}_{(U,\mathcal{O}_U)}\right)$.

(6) We have that

$$\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a subset of

$$\pi^{-1}\left(\pi\left(\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}\right)\right).$$

Deduce that

$$\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a subset of

$$\pi^{-1}\left(\Gamma^{[x]}_{(U,\mathcal{O}_U)}\right).$$

(7) As observed in (2), we have that

$$\Gamma^x_{\left(\pi^{-1}(U),\mathcal{O}_{\pi^{-1}(U)}\right)}$$

is a neighbourhood of x in X with respect to $\mathcal{O}_X$. By Task E8.3.1, deduce that

$$\pi^{-1}\left(\Gamma^{[x]}_{(U,\mathcal{O}_U)}\right)$$

belongs to $\mathcal{O}_X$. Thus $\Gamma^{[x]}_{(U,\mathcal{O}_U)}$ belongs to $\mathcal{O}_{X/\sim}$.

(8) By Task E12.3.3, conclude that $\left(X/{\sim}, \mathcal{O}_{X/\sim}\right)$ is locally connected.

Task E12.3.11. Let $\mathcal{O}_{\mathbb{N}}$ be the discrete topology on $\mathbb{N}$. Let $(X, \mathcal{O}_X)$ be the topological space of Task E12.1.6. Let

$$\mathbb{N} \xrightarrow{f} X$$

be the map given by

$$n \mapsto \begin{cases} 0 & \text{if } n = 1, \\ \frac{1}{n-1} & \text{if } n > 1. \end{cases}$$

Prove that f is a continuous surjection. You may wish to appeal to Task E5.1.14.

Task E12.3.12. Let $\mathcal{O}_{]0,1]}$ be the subspace topology on $]0,1]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$.

Let $(X, \mathcal{O}_X)$ be the Warsaw circle. Construct a continuous surjection

$$]0,1] \longrightarrow X.$$

You may wish to appeal to (2) of Task E5.3.23.

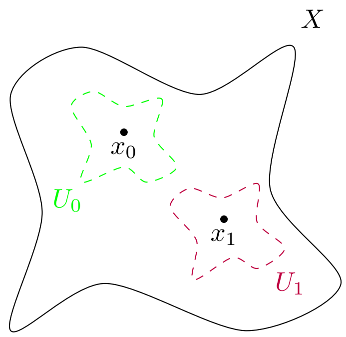
Remark E12.3.13. Let $\mathcal{O}_{\mathbb{N}}$ be the discrete topology on $\mathbb{N}$. By Task E12.1.5, we have that $\mathbb{N}$ is locally connected. By Task E12.1.6, the topological space $(X, \mathcal{O}_X)$ of Task E12.3.11 is not locally connected. Thus Task E12.3.11 demonstrates that an analogue of Proposition 10.5.1 does not hold for locally connected topological spaces. It is for this reason that the proof needed for Task E12.3.10 is more involved than the proof of Corollary 10.5.3.

Let $\mathcal{O}_{]0,1]}$ be the subspace topology on $]0,1]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. By Task E12.3.9, we have that $\left(]0,1], \mathcal{O}_{]0,1]}\right)$ is locally connected. By Task E12.2.9, the Warsaw circle is not locally connected. Thus Task E12.3.12 gives a a second demonstration that an analogue of Proposition 10.5.1 does not hold for locally connected topological spaces.

13. Monday 17th February

13.1. Hausdorff topological spaces

Definition 13.1.1. A topological space $(X, \mathcal{O}_X)$ is *Hausdorff* if, for all x_0 and x_1 which belong to X such that $x_0 \neq x_1$, there is a neighbourhood U_0 of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of x_1 in X with respect to $\mathcal{O}_X$, such that $U_0 \cap U_1$ is empty.



13.2. Examples and non-examples of Hausdorff topological spaces

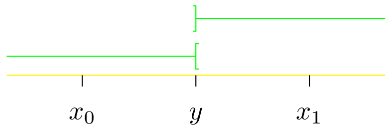
Example 13.2.1. Suppose that x_0 and x_1 belong to $\mathbb{R}$, and that $x_0 \neq x_1$. Relabelling x_0 and x_1 if necessary, we may assume that $x_0 < x_1$.

Let$y$ be a real number such that $x_0 < y < x_1$. The following hold.

(1) We have that x_0 belongs to $]-\infty, y[$.

(2) We have that x_1 belongs to $]y, \infty[$.

(3) We have that $]-\infty, y[\cap]y, \infty[$ is empty.



Both $]-\infty, y[$ and $]y, \infty[$ belong to $\mathcal{O}_{\mathbb{R}}$. To verify this is the topic of Task E13.2.1. We conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff.

Example 13.2.2. Let X be a set. Let $\mathcal{O}_X^{\mathsf{indisc}}$ be the indiscrete topology on X. Suppose that x_0 and x_1 belong to X, and that $x_0 \neq x_1$. The only neighbourhood of x_0 in X with respect to $\mathcal{O}_X^{\text{indisc}}$ is X, and x_1 belongs to X. Thus there is no neighbourhood of x_0 in X with respect to $\mathcal{O}_X^{\mathsf{indisc}}$ which does not contain x_1. In particular, $(X, \mathcal{O}_X^{\mathsf{indisc}})$ is not Hausdorff.

Example 13.2.3. Let X be a set. Let $\mathcal{O}_X^{\mathsf{disc}}$ be the discrete topology on X. Suppose that x_0 and x_1 belong to X, and that $x_0 \neq x_1$. The following hold.

(1) We have that $\{x_0\}$ belongs to $\mathcal{O}_X^{\mathsf{disc}}$.

(2) We have that $\{x_1\}$ belongs to $\mathcal{O}_X^{\mathsf{disc}}$.

(3) We have that $\{x_0\} \cap \{x_1\}$ is empty.

We conclude that $(X, \mathcal{O}_X^{\mathsf{disc}})$ is Hausdorff.

Example 13.2.4. Let X be the set $\{a, b, c\}$. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{a, b\}, \{a, c\} X\}.$$

Every neighbourhood of b in X with respect to $\mathcal{O}_X$ also contains c. Thus $(X, \mathcal{O}_X)$ is not Hausdorff.

Example 13.2.5. Let X be the set $\{a, b, c\}$. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}.$$

Every neighbourhood of b in X with respect to $\mathcal{O}_X$ also contains a. Thus $(X, \mathcal{O}_X)$ is not Hausdorff.

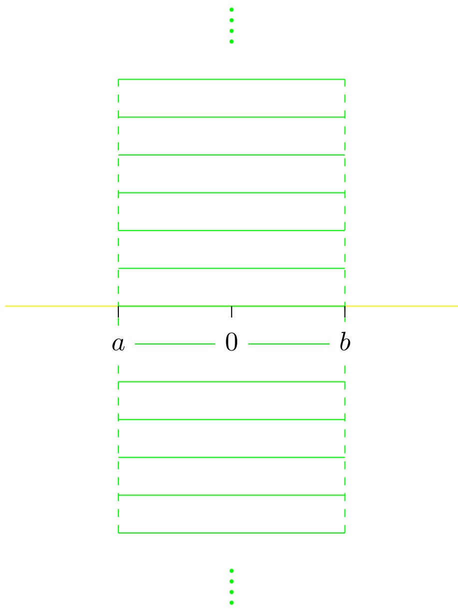
Remark 13.2.6. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that X is finite, or more generally that $\mathcal{O}_X$ is finite. Then $(X, \mathcal{O}_X)$ is Hausdorff if and only if $\mathcal{O}_X$ is the discrete topology. This is Corollary E13.3.7.

Example 13.2.7. Let $\mathcal{O}$ be the topology on $\mathbb{R}^2$ given by

$$\{U \times \mathbb{R} \mid U \text{ belongs to } \mathcal{O}_{\mathbb{R}}\}.$$

To verify that $\mathcal{O}$ defines a topology is Task E13.2.2. Suppose that x_0 and x_1 belong to $\mathbb{R}$, and that $x_0 \neq x_1$. Let W be a neighbourhood of $(0, x_0)$ in X with respect to $\mathcal{O}$.

By definition of $\mathcal{O}$, there is a neighbourhood U of 0 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ such that $W = U \times \mathbb{R}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ with $a < 0 < b$ such that $]a, b[$ is a subset of U. Thus $]a, b[\times \mathbb{R}$ is a subset of W.



We have that $(0, x_1)$ belongs to $]a, b[\times \mathbb{R}$. Thus $(0, x_1)$ belongs to W.

We have demonstrated that every neighbourhood of $(0, x_0)$ in $\mathbb{R}^2$ with respect to $\mathcal{O}$ contains $(0, x_1)$. We conclude that $(\mathbb{R}^2, \mathcal{O})$ is not Hausdorff.

13.3. Canonical methods to prove that a topological space is Hausdorff

Proposition 13.3.1. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let A be a subset of X. Let $\mathcal{O}_A$ be the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Then $(A, \mathcal{O}_A)$ is Hausdorff.

Proof. Suppose that a_0 and a_1 belong to A, and that $a_0 \neq a_1$. Since $(X, \mathcal{O}_X)$ is Hausdorff, there is a neighbourhood U_0 of a_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of a_1 in X with respect to $\mathcal{O}_X$, such that $U_0 \cap U_1$ is empty. The following hold.

(1) By definition of $\mathcal{O}_A$, we have that $A \cap U_0$ belongs to $\mathcal{O}_A$. Thus $A \cap U_0$ is a neighbourhood of a_0 in A with respect to $\mathcal{O}_A$.

(2) By definition of $\mathcal{O}_A$, we have that $A \cap U_1$ belongs to $\mathcal{O}_A$. Thus $A \cap U_1$ is a neighbourhood of a_1 in A with respect to $\mathcal{O}_A$.

(3) We have that $(A \cap U_0) \cap (A \cap U_1)$ is a subset of $U_0 \cap U_1$. Since $U_0 \cap U_1$ is empty, we deduce that $(A \cap U_0) \cap (A \cap U_1)$ is empty.

We conclude that $(A, \mathcal{O}_A)$ is Hausdorff.

□

Example 13.3.2. By Example 13.2.1, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. By Proposition 13.3.1, we deduce that $(I, \mathcal{O}_I)$ is Hausdorff.

Proposition 13.3.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be Hausdorff topological spaces. Then $(X \times Y, \mathcal{O}_{X \times Y})$ is Hausdorff.

Proof. Suppose that (x_0, y_0) and (x_1, y_1) belong to $X \times Y$, and that $(x_0, y_0) \neq (x_1, y_1)$. Then either $x_0 \neq x_1$ or $y_0 \neq y_1$, or both $x_0 \neq x_1$ and $y_0 \neq y_1$.

Suppose that $x_0 \neq x_1$. Since $(X, \mathcal{O}_X)$ is Hausdorff, there is a neighbourhood U_0^X of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1^X of x_1 in X with respect to $\mathcal{O}_X$, such that $U_0^X \cap U_1^X$ is empty. The following hold.

(1) We have that $U_0^X \times Y$ belongs to $\mathcal{O}_{X \times Y}$. Thus $U_0^X \times Y$ is a neighbourhood of (x_0, y_0) in $X \times Y$ with respect to $\mathcal{O}_{X \times Y}$.

(2) We have that $U_1^X \times Y$ belongs to $\mathcal{O}_{X \times Y}$. Thus $U_1^X \times Y$ is a neighbourhood of (x_1, y_1) in $X \times Y$ with respect to $\mathcal{O}_{X \times Y}$.

(3) We have that $(U_0^X \times Y) \cap (U_1^X \times Y) = (U_0^X \cap U_1^X) \times Y$. Since $U_0^X \cap U_1^X$ is empty, we deduce that $(U_0^X \times Y) \cap (U_1^X \times Y)$ is empty.

Suppose instead that $y_0 \neq y_1$. By an analogous argument, there is a neighbourhood U_0^Y of y_0 in Y with respect to $\mathcal{O}_Y$, and a neighbourhood U_1^Y of y_1 in Y with respect to $\mathcal{O}_Y$, such that the following hold.

(1 bis) We have that $X \times U_0^Y$ is a neighbourhood of (x_0, y_0) in $X \times Y$ with respect to $\mathcal{O}_{X \times Y}$.

(2 bis) We have that $X \times U_1^Y$ is a neighbourhood of (x_1, y_1) in $X \times Y$ with respect to $\mathcal{O}_{X \times Y}$.

(3 bis) We have that $(X \times U_0^Y) \cap (X \times U_1^Y)$ is empty.

We conclude that $(X \times Y, \mathcal{O}_{X \times Y})$ is Hausdorff. □

Example 13.3.4. By Example 13.2.1, we have that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is Hausdorff. By Proposition 13.3.3, we deduce that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is Hausdorff, for any $n \geq 1$.

Example 13.3.5. By Example 13.3.2, we have that $(I, \mathcal{O}_I)$ is Hausdorff. By Proposition 13.3.3, we deduce that $(I^2, \mathcal{O}_{I^2})$ is Hausdorff.

Alternatively, by Example 13.3.4 we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. We can deduce from this that $(I^2, \mathcal{O}_{I^2})$ is Hausdorff by Proposition 13.3.1.

Example 13.3.6. By Example 13.3.4 we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. By Proposition 13.3.1, we deduce that $(S^1, \mathcal{O}_{S^1})$ is Hausdorff.

Proposition 13.3.7. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is Hausdorff. Let

$$X \xrightarrow{f} Y$$

be a bijection. Suppose that f is open, in the sense of Definition E7.1.15. Then $(Y, \mathcal{O}_Y)$ is Hausdorff.

Proof. Since f is a bijection, there is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$. Suppose that y_0 and y_1 belong to Y, and that $y_0 \neq y_1$. Since $y_0 \neq y_1$, we have that $g(y_0) \neq g(y_1)$. To check that you understand this is the topic of Task E13.2.3 (1).

Since $(X, \mathcal{O}_X)$ is Hausdorff, there is a neighbourhood U_0 of $g(y_0)$ in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of $g(y_1)$ in X with respect to $\mathcal{O}_X$, such that $U_0 \cap U_1$ is empty. The following hold.

(1) Since $U_0 \cap U_1$ is empty, we have that $f(U_0) \cap f(U_1)$ is empty. To verify this is the topic of Task E13.2.3 (2).

(2) Since f is open, we have that $f(U_0)$ belongs to $\mathcal{O}_Y$. Since $f \circ g = id_Y$, we have that $f(g(y_0)) = y_0$. Thus we have that $f(U_0)$ is a neighbourhood of y_0 in Y with respect to $\mathcal{O}_Y$.

(3) Since f is open, we have that $f(U_1)$ belongs to $\mathcal{O}_Y$. Since $f \circ g = id_Y$, we have that $f(g(y_1)) = y_1$. Thus we have that $f(U_1)$ is a neighbourhood of y_1 in Y with respect to $\mathcal{O}_Y$.

We conclude that $(Y, \mathcal{O}_Y)$ is Hausdorff.

□

Corollary 13.3.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that $(X, \mathcal{O}_X)$ is Hausdorff. Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are homeomorphic. Then $(Y, \mathcal{O}_Y)$ is Hausdorff.

Proof. Follows immediately from Proposition 13.3.7 since, by Task E7.3.1, a homeomorphism is in particular bijective and open. □

Example 13.3.9. By Example 13.3.5, we have that $(I^2, \mathcal{O}_{I^2})$ is Hausdorff. By Task E7.2.9, there is a homeomorphism

$$I^2 \longrightarrow D^2.$$

By Corollary 13.3.8, we deduce that $(D^2, \mathcal{O}_{D^2})$ is Hausdorff.

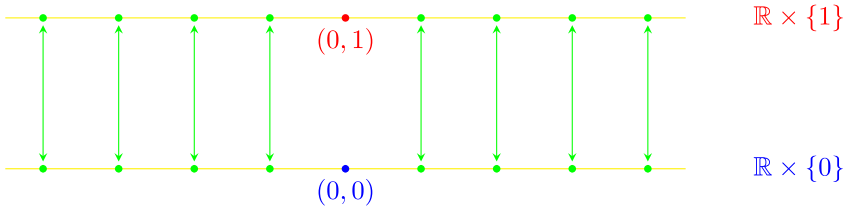
Alternatively, by Example 13.3.4 we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. We can thus deduce from Proposition 13.3.1 that $(D^2, \mathcal{O}_{D^2})$ is Hausdorff.

13.4. Example of a quotient of a Hausdorff topological space which is not Hausdorff

Example 13.4.1. Let X be the subset of $\mathbb{R}^2$ given by the union of $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$.

Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. By Example 13.3.4, we have that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is Hausdorff. By Proposition 13.3.1, we deduce that $(X, \mathcal{O}_X)$ is Hausdorff.

Let $\sim$ be the equivalence relation on X generated by $(x, 0) \sim (x, 1)$, for all $x \in \mathbb{R}$ such that $x \neq 0$.

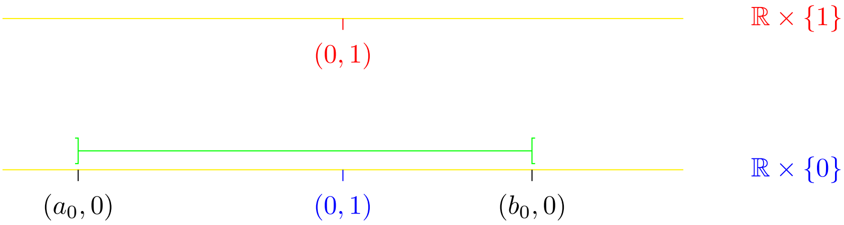


We shall demonstrate that $(X/\sim, \mathcal{O}_{X/\sim})$ is not Hausdorff. Let

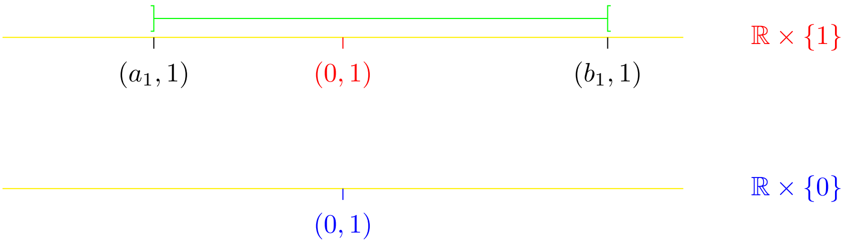
$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. Let U_0 be a neighbourhood of $\pi((0, 0))$ in $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$. Let U_1 be a neighbourhood of $\pi((0, 1))$ in $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$.

By definition of $\mathcal{O}_{X/\sim}$, we have that $\pi^{-1}(U_0)$ belongs to $\mathcal{O}_X$. By definition of $\mathcal{O}_X$ and $\mathcal{O}_{\mathbb{R}^2}$, we deduce that there is an open interval $]a_0, b_0[$, with $a_0 < 0 < b_0$, such that $]a_0, b_0[\times \{0\}$ is a subset of $\pi^{-1}(U_0)$. To check that you understand this is the topic of Task E13.2.4.



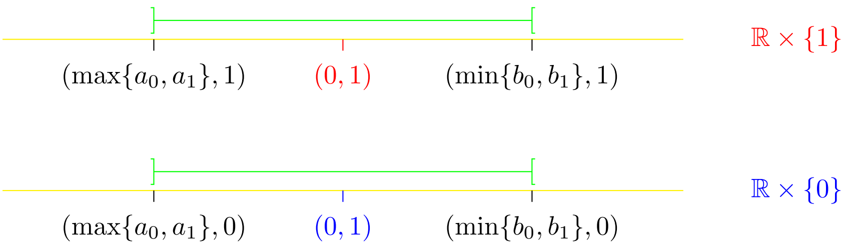
By an analogous argument, there is an open interval $]a_1, b_1[$, with $a_1 < 0 < b_1$, such that $]a_1, b_1[\times \{1\}$ is a subset of $\pi^{-1}(U_1)$.



The following hold.

(1) We have that $]\max\{a_0, a_1\}, \min\{b_0, b_1\}[\times \{0\}$ is a subset of $\pi^{-1}(U_0)$.

(2) We have that $]\max\{a_0, a_1\}, \min\{b_0, b_1\}[\times \{1\}$ is a subset of $\pi^{-1}(U_1)$.



We deduce that

$$\pi\left((]\max(a_0, a_1), \min(b_0, b_1)[\setminus \{0\}) \times \{0\} \right)$$

is a subset of both U_0 and U_1. In particular, $U_0 \cap U_1$ is not empty. We conclude that $(X/\sim, \mathcal{O}_{X/\sim})$ is not Hausdorff.

Remark 13.4.2. The topological space $(X/\sim, \mathcal{O}_{X/\sim})$ is sometimes known as the *real line with two origins*.

Remark 13.4.3. Example 13.4.1 demonstrates that a quotient of a Hausdorff topological space is not necessarily Hausdorff. Thus we do not yet have a 'canonical method' to prove that $(M^2, \mathcal{O}_{M^2})$, $(K^2, \mathcal{O}_{K^2})$, and our other examples of quotients of topological spaces, are Hausdorff.

We shall see later that if $(X, \mathcal{O}_X)$ and $\sim$ satisfy certain conditions, then $(X/\sim, \mathcal{O}_{X/\sim})$ can be proven by a 'canonical method' to be Hausdorff.

Remark 13.4.4. We can intuitively believe that a quotient of a Hausdorff topological space might not be Hausdorff. In a Hausdorff topological space, every two points can be 'separated' by subsets belonging to the topology: the points are 'not too close together'.

When we take a quotient, however, we may identify many points. Thus points which were not 'close together' before taking the quotient may be 'close together' afterwards. So much so that we may no longer be able to 'separate' every two points.

E13. Exercises for Lecture 13

E13.1. Exam questions

Task E13.1.1. Let $X = \{a, b, c, d\}$ be a set with four elements. Let $\mathcal{O}_X$ be the topology on X given by

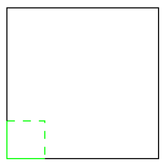
$$\{\emptyset, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

Demonstrate that $(X, \mathcal{O}_X)$ is not Hausdorff.

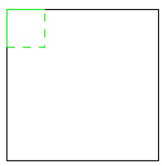
Task E13.1.2. Prove that the Sorgenfrey line of Task E11.1.12 is Hausdorff.

Task E13.1.3. Let $\mathcal{O}$ be the topology on I^2 given by the set of subsets U of I^2 such that, for every x which belongs to U, we have either that $x = 0$, or else that one of the following holds.

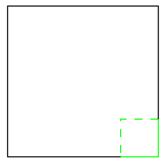
(1) We have that x belongs to $[0, y[\times [0, y[$ for some $0 < y < \frac{1}{2}$, and this set is a subset of U.



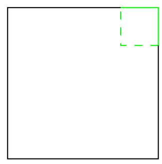
(2) We have that x belongs to $[0, y[\times]1 - y, 1]$ for some $0 < y < \frac{1}{2}$, and this set is a subset of U.



(3) We have that x belongs to $]1 - y, 1] \times [0, y[$ for some $0 < y < \frac{1}{2}$, and this set is a subset of U.

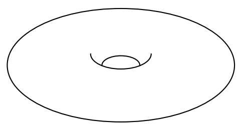


(4) We have that x belongs to $]1-y,1] \times]1-y,1]$ for some $0 < y < \frac{1}{2}$, and this set is a subset of U.



Is $(I^2, \mathcal{O})$ homeomorphic to $(I^2, \mathcal{O}_{I^2})$?

Task E13.1.4. Prove that $(T^2, \mathcal{O}_{T^2})$ is Hausdorff.



Remark E13.1.1. The intention in Task E13.1.4 is for you to give a proof from first principles. In a later lecture, we shall see how to prove that $(T^2, \mathcal{O}_{T^2})$ is Hausdorff by a 'canonical method'.

It is also possible to give a proof by appealing to Corollary 13.3.8 and the fact, discussed in Example 8.1.4, that $(T^2, \mathcal{O}_{T^2})$ is homeomorphic to $(S^1 \times S^1, \mathcal{O}_{S^1 \times S^1})$. Since $(S^1, \mathcal{O}_{S^1})$ is Hausdorff by Example 13.3.6, we have that $(S^1 \times S^1, \mathcal{O}_{S^1 \times S^1})$ is Hausdorff by Proposition 13.3.3.

E13.2. In the lecture notes

Task E13.2.1. Suppose that x belongs to $\mathbb{R}$. Prove that $]-\infty, x[$ and $]x, \infty[$ belong to $\mathcal{O}_{\mathbb{R}}$.

Task E13.2.2. Prove that the set $\mathcal{O}$ of Example 13.2.7 defines a topology on $\mathbb{R}^2$.

Task E13.2.3. Let X and Y be sets, and let

$$X \xrightarrow{f} Y$$

be a bijection. Thus there is a map

$$Y \xrightarrow{g} X$$

such that $g \circ f = id_X$ and $f \circ g = id_Y$.

(1) Suppose that y_0 and y_1 belong to Y, and that $y_0 \neq y_1$. Prove that $g(y_0) \neq g(y_1)$. You may wish to appeal to the fact that $f \circ g = id_Y$.

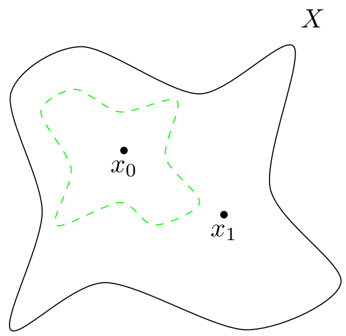
(2) Suppose that U_0 and U_1 are subsets of X, and that $U_0 \cap U_1$ is empty. Prove that $f(U_0) \cap f(U_1)$ is empty. You may wish to appeal to the fact that $g \circ f = id_X$.

Task E13.2.4. In the notation of Example 13.4.1, prove that, for any neighbourhood U of $\pi((0,0))$ in $X/\sim$ with respect to $\mathcal{O}_{X/\sim}$, there is an open interval $]a,b[$ with $a < 0 < b$ such that $]a,b[\times \{0\}$ is a subset of $\pi^{-1}(U)$.

E13.3. For a deeper understanding

Task E13.3.1. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let $\mathcal{O}'_X$ be a topology on X such that $\mathcal{O}_X$ is a subset of $\mathcal{O}'_X$. Prove that $(X, \mathcal{O}'_X)$ is Hausdorff.

Definition E13.3.2. A topological space $(X, \mathcal{O}_X)$ is T1 if, for every ordered pair (x_0, x_1) such that x_0 and x_1 belong to X and $x_0 \neq x_1$, there is a neighbourhood of x_0 in X with respect to $\mathcal{O}_X$ which does not contain x_1.



Remark E13.3.3. . Suppose that $(X, \mathcal{O}_X)$ is a Hausdorff topological space. Then $(X, \mathcal{O}_X)$ is a T1 topological space.

Task E13.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that x belongs to X. Prove that $\{x\}$ is closed in X with respect to $\mathcal{O}_X$ if and only if $(X, \mathcal{O}_X)$ is a T1 topological space. You may wish to proceed as follows.

(1) Suppose that $(X, \mathcal{O}_X)$ is a T1 topological space. Suppose that y belongs to X, and that $x \neq y$. Since $(X, \mathcal{O}_X)$ is a T1 topological space, there is a neighbourhood U_y of y in X with respect to $\mathcal{O}_X$ such that x does not belong to U_y. Deduce that y is not a limit point of $\{x\}$ in X with respect to $\mathcal{O}_X$.

(2) By Proposition 9.1.1, deduce from (1) that $\{x\}$ is closed in X with respect to $\mathcal{O}_X$..

(3) Suppose instead that $\{x\}$ is closed in X with respect to $\mathcal{O}_X$ for every x which belongs to X. Suppose that x_0 and x_1 belong to X, and that $x_0 \neq x_1$. Since $\{x_0\}$ is closed in X with respect to $\mathcal{O}_X$, observe have that $X \setminus \{x_1\}$ belongs to $\mathcal{O}_X$.

(4) Moreover, observe that x_0 belongs to $X \setminus \{x_1\}$. Conclude that $(X, \mathcal{O}_X)$ is T1.

Corollary E13.3.5. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Suppose that x belongs to X. Then $\{x\}$ is closed in X with respect to $\mathcal{O}_X$.

Proof. Follows immediately from Task E13.3.4 and Remark E13.3.3. □

Task E13.3.6. Let $(X, \mathcal{O}_X)$ be a T1 topological space. Suppose that $\mathcal{O}_X$ is finite. Prove that $\mathcal{O}_X$ is the discrete topology on X. You may wish to proceed as follows.

(1) Suppose that x belongs to X. Since $(X, \mathcal{O}_X)$ is T1, there is, for every y which belongs to X such that $x \neq y$, a neighbourhood U_y of x in X with respect to $\mathcal{O}_X$ such that y does not belong to U_y. Observe that

$$\bigcap_{y \in Y \setminus \{x\}} U_y$$

is $\{x\}$.

(2) Since $\mathcal{O}_X$ is finite, observe that

$$\bigcap_{y \in Y \setminus \{x\}} U_y$$

belongs to $\mathcal{O}_X$.

(3) Deduce that $\{x\}$ belongs to $\mathcal{O}_X$. Conclude that $\mathcal{O}_X$ is the discrete topology on X.

Corollary E13.3.7. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Suppose that $\mathcal{O}_X$ is finite. Then $\mathcal{O}_X$ is the discrete topology on X.

Proof. Follows immediately from Task E13.3.6 and Remark E13.3.3. □

Task E13.3.8. Let $(X/\sim, \mathcal{O}_{X/\sim})$ be the real line with two origins of Example 13.4.1. Prove that $(X/\sim, \mathcal{O}_{X/\sim})$ is T1. You may wish to appeal to the fact that for any open interval $]a, b[$ such that $a < 0 < b$, we have that $\pi\left(]a, b[\times \{0\}\right)$ belongs to $\mathcal{O}_{X/\sim}$, but does not contain $\pi\left((0, 1)\right)$.

Remark E13.3.9. Example 13.4.1 and Task E13.3.8 demonstrate that a T1 topological space is not necessarily Hausdorff.

E13.4. Exploration — Hausdorffness for metric spaces

Definition E13.4.1. Let X be a set. A metric d on X is *separating* if, for any x_0 and x_1 which belong to X with the property that $d(x_0, x_1) = 0$, we have that $x_0 = x_1$.

Definition E13.4.2. A metric space (X, d) is *separated* if d is separating.

Task E13.4.3. Let (X, d) be a separated, symmetric metric space. Let $\mathcal{O}_d$ be the topology on X corresponding to d of Task E3.4.9. Prove that $(X, \mathcal{O}_d)$ is Hausdorff. You may wish to proceed as follows.

(1) Suppose that x_0 and x_1 belong to X, and that $x_0 \neq x_1$. Since (X, d) is separated, deduce that $d(x_0, x_1) > 0$.

(2) Let $\epsilon = \frac{d(x_0,x_1)}{2}$. Appealing to Task E4.3.2, observe that $B_\epsilon(x_0)$ is a neighbourhood of x_0 in X with respect to $\mathcal{O}_d$, and that $B_\epsilon(x_1)$ is a neighbourhood of x_1 in X with respect to $\mathcal{O}_d$-

(3) Suppose that y belongs to $B_\epsilon(x_0)$. By definition of d, we have that

$$\begin{aligned} d(x_0, x_1) &\leq d(x_0, y) + d(y, x_1) \\ &< \tfrac{d(x_0,x_1)}{2} + d(y, x_1). \end{aligned}$$

Thus we have that

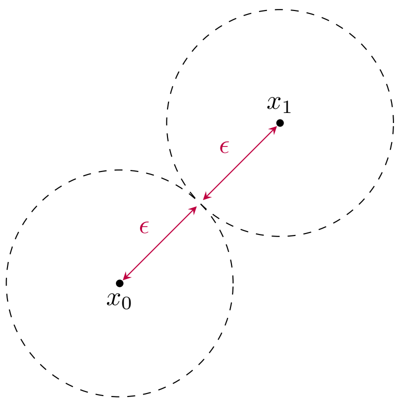
$$d(y, x_1) > \tfrac{d(x_0,x_1)}{2}.$$

Since (X, d) is symmetric, deduce that

$$d(x_1, y) > \tfrac{d(x_0,x_1)}{2}.$$

(4) Deduce from (3) that y does not belong to $B_\epsilon(x_1)$, and thus that $B_\epsilon(x_0) \cap B_\epsilon(x_1)$ is empty.

(5) Conclude from (2) and (4) that (X, d) is Hausdorff.



Definition E13.4.4. A topological space $(X, \mathcal{O}_X)$ is *perfectly normal* if, for every ordered pair of subsets A_0 and A_1 of subsets of X which are closed in X with respect to $\mathcal{O}_X$, which have the property that $A_0 \cap A_1$ is empty, and which are both not empty, there is a continuous map

$$X \xrightarrow{f} I$$

such that $f^{-1}(\{0\}) = A_0$ and $f^{-1}(\{1\}) = A_1$.

Task E13.4.5. Let $(X, \mathcal{O}_X)$ be a perfectly normal topological space. Prove that $(X, \mathcal{O}_X)$ is Hausdorff. You may wish to appeal to Corollary E13.3.5.

Task E13.4.6. Let (X, d) be a separated, symmetric metric space. Let $\mathcal{O}_d$ be the topology on X corresponding to d of Task E3.4.9. Prove that $(X, \mathcal{O}_d)$ is perfectly normal. You may wish to proceed as follows.

(1) Since $A_0 \cap A_1$ is empty, deduce, by Task E9.4.2, that $d(x, A_0) + d(x, A_1) > 0$ for every x which belongs to X.

(2) Since (X, d) is symmetric, we have by Task E4.3.8 that the map

$$X \xrightarrow{d(-,A_0)} \mathbb{R}$$

given by $x \mapsto d(x, A_0)$ is continuous, and that the map

$$X \xrightarrow{d(-,A_1)} \mathbb{R}$$

given by $x \mapsto d(x, A_1)$ is continuous. By (1), Task E5.3.6, Task E5.3.10, and Task E5.1.9, deduce that the map

$$X \xrightarrow{f} I$$

given by $x \mapsto \frac{d(x,A_0)}{d(x,A_0)+d(x,A_1)}$ is continuous.

(3) By Remark E4.3.1 and Task E9.4.2, observe that $f^{-1}(\{0\}) = A_0$, and that $f^{-1}(\{1\}) = A_1$.

(4) Conclude from (2) and (3) that (X, d) is perfectly normal.

Remark E13.4.1. Task E13.4.6 and Task E13.4.5 give a second proof that the topological space arising from every metric space is Hausdorff.

14. Tuesday 18th February

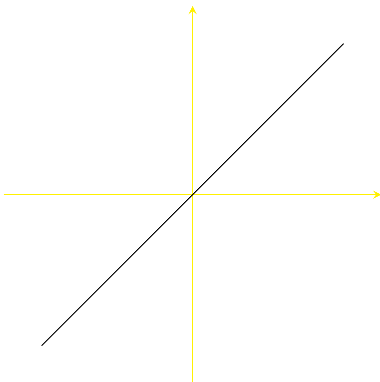
14.1. Characterisation of Hausdorff topological spaces

Notation 14.1.1. Let X be a set. We denote the subset

$$\{(x, x) \in X \times X \mid x \in X\}$$

of $X \times X$ by $\Delta(X)$.

Example 14.1.2. Let X be $\mathbb{R}$. Then $\Delta(X)$ is the line in $\mathbb{R}^2$ defined by $y = x$.



Proposition 14.1.3. A topological space $(X, \mathcal{O}_X)$ is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.

Proof. We consider the following assertions.

(1) We have that $\Delta(X)$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.

(2) Every limit point of $\Delta(X)$ in $X \times X$ with respect to $\mathcal{O}_{X \times X}$ belongs to $\Delta(X)$.

(3) For every (x_0, x_1) which belongs to $X \times X$, there is a neighbourhood W of (x_0, x_1) in $X \times X$ with respect to $\mathcal{O}_{X \times X}$ such that $W \cap \Delta(X)$ is empty.

(4) For every (x_0, x_1) which belongs to $X \times X$, there is a neighbourhood U_0 of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of x_1 in X with respect to $\mathcal{O}_X$, such that $(U_0 \times U_1) \cap \Delta(X)$ is empty.

(5) There is a neighbourhood U_0 of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of x_1 in X with respect to $\mathcal{O}_X$, such that $U_0 \cap U_1$ is empty.

By Proposition 9.1.1, we have that (1) holds if and only if (2) holds. By definition of a limit point of $\Delta(X)$ in $X \times X$ with respect to $\mathcal{O}_{X \times X}$, we have that (2) holds if and only if (3) holds. By Task E14.2.1, we have that (3) holds if and only if (4) holds. By Task E14.2.2, we have that (4) holds if and only if (5) holds. We conclude that (1) holds if and only if (5) holds, as required.

$\square$

14.2. A necessary condition for a quotient of a Hausdorff topological space to be Hausdorff

Remark 14.2.1. Let X be a set. As discussed in §A.4, a relation on X is formally a subset R of $X \times X$. When we write that $x_0 \sim x_1$, we formally mean that (x_0, x_1) belongs to R.

By extension, when we write that $\sim$ is a relation on X, this is shorthand for: we have a subset R of $X \times X$, and shall write $x_0 \sim x_1$ when (x_0, x_1) belongs to R. When we adopt this shorthand, we shall denote R by $R_\sim$. Tautologically, we thus have that

$$R_\sim = \{(x_0, x_1) \in X \times X \mid x_0 \sim x_1\}.$$

Proposition 14.2.2. Let $(X, \mathcal{O}_X)$ be a Hausdorff topological space. Let $\sim$ be an equivalence relation on X. Suppose that $(X/\sim, \mathcal{O}_{X/\sim})$ is a Hausdorff topological space. Then $R_\sim$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.

Proof. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. Let

$$X \times X \xrightarrow{\pi \times \pi} (X/\sim) \times (X/\sim)$$

be the map given by $(x_0, x_1) \mapsto (\pi(x_0), \pi(x_1))$. By Remark 6.1.9 we have that π is continuous. By Task E5.3.17, we deduce that $\pi \times \pi$ is continuous.

Since $(X/\sim, \mathcal{O}_{X/\sim})$ is a Hausdorff topological space, we have, by Proposition 14.1.3, that $\Delta(X/\sim)$ is closed in $(X/\sim) \times (X/\sim)$ with respect to $\mathcal{O}_{(X/\sim)\times(X/\sim)}$. Since $\pi \times \pi$ is continuous, we deduce, by Task E5.1.13, that $(\pi \times \pi)^{-1}(\Delta(X/\sim))$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$.

We have that $R_\sim = (\pi \times \pi)^{-1}(\Delta(X/\sim))$. To verify this is the topic of Task E14.2.3. We conclude that $R_\sim$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$. $\square$

Example 14.2.3. Let X and $\sim$ be as in Example 13.4.1. Then $R_\sim$ is the union of the following four sets.

(1) $\Delta\left((\mathbb{R} \setminus \{0\}) \times \{0\}\right)$.

(2) $\Delta\left((\mathbb{R} \setminus \{0\}) \times \{1\}\right)$.

(3) $\left((\mathbb{R} \setminus \{0\}) \times \{0\}\right) \times \left((\mathbb{R} \setminus \{0\}) \times \{1\}\right)$.

(4) $\left((\mathbb{R} \setminus \{0\}) \times \{1\}\right) \times \left((\mathbb{R} \setminus \{0\}) \times \{0\}\right)$.

By Task E14.2.4. we have that $((0,0),(0,0))$ is a limit point of $R_\sim$ in $X \times X$ with respect to $\mathcal{O}_{X \times X}$. Since $((0,0),(0,0))$ does not belong to $R_\sim$, we deduce, by Proposition 9.1.1, that $R_\sim$ is not closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$. By Proposition 14.2.2, we conclude that $(X/\sim, \mathcal{O}_{X/\sim})$ is not Hausdorff, as we demonstrated directly in Example 13.4.1.

Remark 14.2.4. In general, that $R_\sim$ is closed in $X \times X$ with respect to $\mathcal{O}_{X \times X}$ is not sufficient to ensure that $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff. An example is discussed in Task E14.3.1 – Task E14.3.5.

14.3. Compact topological spaces

Definition 14.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. An *open covering* of X with respect to $\mathcal{O}_X$ is a set $\{U_j\}_{j \in J}$ of subsets of X such that the following hold.

(1) We have that U_j belongs to $\mathcal{O}_X$ for every j which belongs to J.

(2) We have that $X = \bigcup_{j \in J} U_j$.

Definition 14.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open covering of X with respect to $\mathcal{O}_X$. Let K be a subset of J. Then $\{U_k\}_{k \in K}$ is a *finite subcovering* of $\mathcal{U}$ if the following hld.

(1) We have that $\{U_k\}_{k \in K}$ is finite.

(2) We have that $X = \bigcup_{k \in K} U_k$.

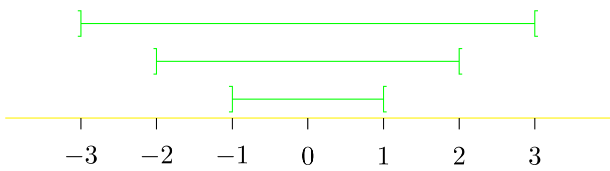
Definition 14.3.3. A topological space $(X, \mathcal{O}_X)$ is *compact* if, for every open covering $\mathcal{U} = \{U_j\}_{j \in J}$ of X with respect to $\mathcal{O}_X$, there is a subset K of J such that $\{U_k\}_{k \in K}$ is a finite subcovering of $\mathcal{U}$.

Example 14.3.4. Let $(X, \mathcal{O}_X)$ be a topological space. Suppose that $\mathcal{O}_X$ is finite. Then every set $\{U_j\}_{j \in J}$ such that U_j belongs to $\mathcal{O}_X$ for all j which belong to J is finite. Thus $(X, \mathcal{O}_X)$ is compact.

Remark 14.3.5. In particular, if X is finite, then $(X, \mathcal{O}_X)$ is compact.

14.4. Examples of topological spaces which are not compact

Example 14.4.1. The set $\mathcal{U} = \{]-n, n[\}_{n \in \mathbb{N}}$ is an open covering of $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.



Let K be a subset of $\mathbb{N}$ such that $\{]-n, n[\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\mathbb{N}$. Then

$$\bigcup_{n \in K}]-n, n[=]-m, m[\,,$$

where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K}]-n, n[= \mathbb{R}.$$

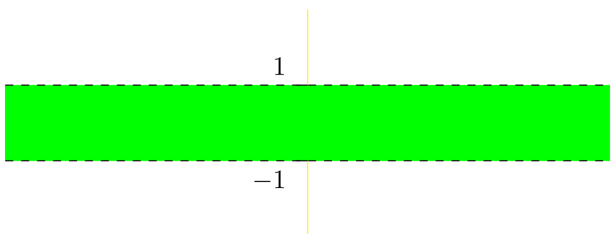
Thus $\{]-n, n[\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

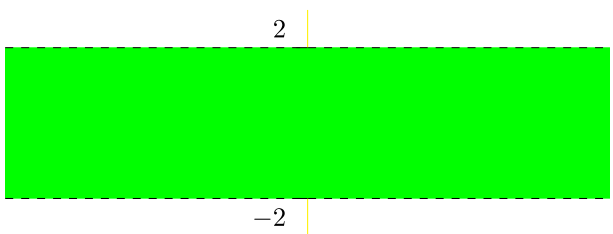
This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$ is not compact.

Example 14.4.2. The set

$$\mathcal{U} = \{\mathbb{R} \times]-n, n[\}_{n \in \mathbb{N}}$$

is an open covering of $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.





Let K be a subset of $\mathbb{N}$ such that $\{\mathbb{R} \times]-n,n[\}_{n\in K}$ is finite. This is the same as to say that K is a finite subset of $\mathbb{N}$. Then

$$\bigcup_{n\in K} (\mathbb{R} \times]-n,n[) = \mathbb{R} \times]-m,m[\,,$$

where $m = \max K$. In particular, we do not have that

$$\bigcup_{n\in K} (\mathbb{R} \times]-n,n[) = \mathbb{R}^2.$$

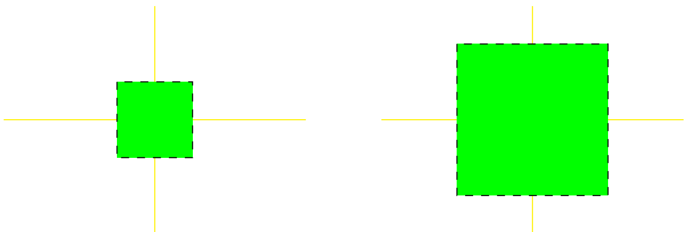
Thus $\{\mathbb{R} \times]-n,n[\}_{n\in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is not compact.

Example 14.4.3. The set

$$\mathcal{U} = \{]-n,n[\times]-n,n[\}_{n\in\mathbb{N}}$$

is an open covering of $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.



Let K be a subset of $\mathbb{N}$ such that $\{]-n,n[\times]-n,n[\}_{n\in K}$ is finite. This is the same as to say that K is a finite subset of $\mathbb{N}$. Then

$$\bigcup_{n\in K} (]-n,n[\times]-n,n[) =]-.m,m[\times]-m,m[\,,$$

where $m = \max K$. In particular, we do not have that

$$\bigcup_{n\in K} (]-n,n[\times]-n,n[) = \mathbb{R}^2.$$

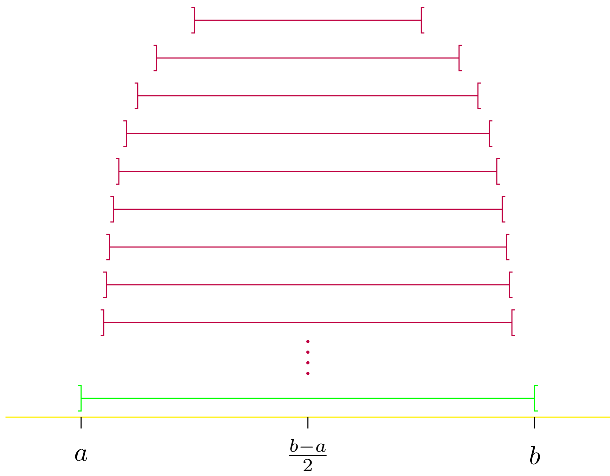
Thus $\{]-n,n[\times]-n,n[\}_{n\in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. Thereby it gives a second proof that $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is not compact.

Example 14.4.4. Suppose that a and b belong to $\mathbb{R}$. Let $\mathcal{O}_{]a,b[}$ be the subspace topology on $]a,b[$ with respect to $\mathcal{O}_{]a,b[}$. The set

$$\mathcal{U} = \left\{ \left]a + \tfrac{1}{n}, b - \tfrac{1}{n}\right[\right\}_{n\,\in\,\mathbb{N} \text{ and } \frac{1}{n} < \frac{b-a}{2}}$$

is an open covering of $]a, b[$ with respect to $\mathcal{O}_{]a,b[}$.



Let K be a subset of $\left\{n \in \mathbb{N} \mid \frac{1}{n} < \frac{b-a}{2}\right\}$ such that $\left\{\left]a + \frac{1}{n}, b - \frac{1}{n}\right[\right\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\left\{n \in \mathbb{N} \mid \frac{1}{n} < \frac{b-a}{2}\right\}$. Then

$$\bigcup_{n \in K} \left]a + \tfrac{1}{n}, b - \tfrac{1}{n}\right[= \left]a + \tfrac{1}{m}, b - \tfrac{1}{m}\right[,$$

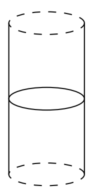
where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K} \left]a + \tfrac{1}{n}, b - \tfrac{1}{n}\right[=]a, b[\,.$$

Thus $\left\{\left]a + \frac{1}{n}, b - \frac{1}{n}\right[\right\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $\left(]a, b[\,, \mathcal{O}_{]a,b[}\right)$ is not compact.

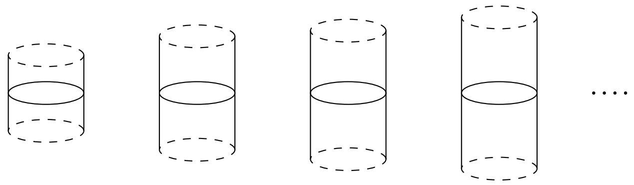
Example 14.4.5. Let us think of $S^1 \times]0, 1[$ as a cylinder with the two circles at its ends removed.



The set

$$\left\{S^1 \times \left]\tfrac{1}{n}, 1 - \tfrac{1}{n}\right[\right\}_{n \,\in\, \mathbb{N} \text{ and } n > 2}$$

is an open covering of $S^1 \times]0, 1[$ with respect to $\mathcal{O}_{S^1 \times]0,1[}$.



Let K be a subset of $\{n \in \mathbb{N} \mid n > 2\}$ such that $\left\{S^1 \times \left]\frac{1}{n}, 1 - \frac{1}{n}\right[\right\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\{n \in \mathbb{N} \mid n > 2\}$. Then

$$\bigcup_{n \in K} \left(S^1 \times \left]\tfrac{1}{n}, 1 - \tfrac{1}{n}\right[\right) = S^1 \times \left]\tfrac{1}{m}, 1 - \tfrac{1}{m}\right[,$$

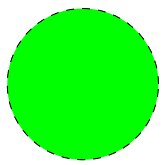
where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K} \left(S^1 \times \left]\tfrac{1}{n}, 1 - \tfrac{1}{n}\right[\right) = S^1 \times \left]0, 1\right[.$$

Thus $\left\{S^1 \times \left]\frac{1}{n}, 1 - \frac{1}{n}\right[\right\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $\left(S^1 \times \left]0, 1\right[, \mathcal{O}_{S^1 \times]0,1[}\right)$ is not compact.

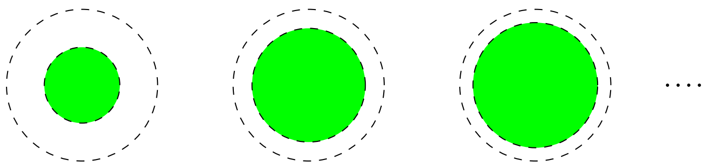
Example 14.4.6. Let $\mathcal{O}_{D^2 \setminus S^1}$ be the subspace topology on $D^2 \setminus S^1$ with respect to $(D^2, \mathcal{O}_{D^2})$.



Let U_n be the subset of $D^2 \setminus S^1$ given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1 - \tfrac{1}{n}\right\}.$$

The set $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ is an open covering of $D^2 \setminus S^1$ with respect to $\mathcal{O}_{D^2 \setminus S^1}$.



Let K be a subset of $\mathbb{N}$ such that $\{U_n\}_{n \in K}$ is finite. This is the same as to say that K is a finite subset of $\mathbb{N}$. Then

$$\bigcup_{n \in K} U_n = U_m,$$

where $m = \max K$. In particular, we do not have that

$$\bigcup_{n \in K} U_n = D^2 \setminus S^1.$$

Thus $\{U_n\}_{n \in K}$ is not a finite subcovering of $\mathcal{U}$.

This demonstrates that $\mathcal{U}$ does not admit a finite subcovering. We conclude that $\left(D^2 \setminus S^1, \mathcal{O}_{D^2 \setminus S^1}\right)$ is not compact.

E14. Exercises for Lecture 14

E14.1. Exam questions

Task E14.1.1. Give a counterexample to the following assertion: the set

$$\{(x, x) \in \mathbb{R}^2 \mid x \in X\}$$

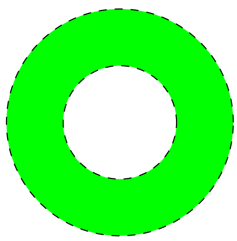
is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ for every subset X of $\mathbb{R}$. Give an example of a topological property which can be imposed upon X to ensure that the assertion correct. Justify your answer.

Task E14.1.2. Let $\mathcal{O}_{[0,1[}$ be the subspace topology on $[0,1[$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Is $\left([0,1[\,, \mathcal{O}_{[0,1[}\right)$ compact?

Task E14.1.3. Find an open covering of $I^2 \times \mathbb{R}$ with respect to $\mathcal{O}_{I^2 \times \mathbb{R}}$ which does not admit a finite subcovering. Conclude that $(I^2 \times \mathbb{R}, \mathcal{O}_{I^2 \times \mathbb{R}})$ is not compact.

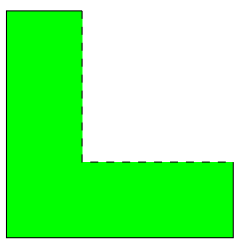
Task E14.1.4. Let X be the 'open annulus' given by

$$\left\{(x, y) \in \mathbb{R}^2 \mid \tfrac{1}{2} < \|(x, y)\| < 1\right\}.$$



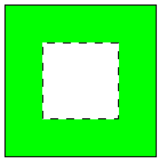
Let $\mathcal{O}_X$ the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Give an example of an open covering of $(X, \mathcal{O}_X)$ which does not admit a finite subcovering. Deduce that $(X, \mathcal{O}_X)$ is not compact.

Task E14.1.5. Let X be the union of $[0,1[\, \times [0,3]$ and $[0,3] \times [0,1[$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



Find an open covering of X which does not admit a finite subcovering. Conclude that $(X, \mathcal{O}_X)$ is not compact.

Task E14.1.6. Let X be the set given by $I^2 \setminus \left(\left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right]\right)$.



Let

$$I^2 \xrightarrow{\pi} T^2$$

be the quotient map. Let $\mathcal{O}_{\pi(X)}$ be the subspace topology on $\pi(X)$ with respect to $(T^2, \mathcal{O}_{T^2})$. Demonstrate that $\left(\pi(X), \mathcal{O}_{\pi(X)}\right)$ is not compact.

Task E14.1.7. Prove that the Sorgenfrey line of Task E11.1.12 is not compact.

Task E14.1.8. Give an example of an equivalence relation $\sim$ on $\mathbb{R}$ such that $(\mathbb{R}/\sim, \mathcal{O}_{\mathbb{R}/\sim})$ is compact.

E14.2. In the lecture notes

Task E14.2.1. Let $(X_0, \mathcal{O}_{X_0})$ and $(X_1, \mathcal{O}_{X_1})$ be topological spaces. Let A be a subset of $X_0 \times X_1$. Suppose that (x_0, x_1) belongs to $X_0 \times X_1$. Prove that the following assertions are equivalent.

(1) There is a neighbourhood W of (x_0, x_1) in $X \times X$ with respect to $\mathcal{O}_{X \times X}$ such that $W \cap A$ is empty.

(2) There is a neighbourhood U_0 of x_0 in X with respect to $\mathcal{O}_X$, and a neighbourhood U_1 of x_1 in X with respect to $\mathcal{O}_X$ such that $(U_0 \times U_1) \cap A$ is empty.

Task E14.2.2. Let X_0 and X_1 be sets. Let A be a subset of $X_0 \times X_1$. Let U_0 be a subset of X_0, and let U_1 be a subset of X_1. Prove that $(U_0 \times U_1) \cap A = U_0 \cap U_1$.

Task E14.2.3. Let X be a set, and let $\sim$ be an equivalence relation on X. Let

$$X \xrightarrow{\pi} X/\sim$$

be the quotient map. Let

$$X \times X \xrightarrow{\pi \times \pi} (X/\sim) \times (X/\sim)$$

be the map given by $(x_0, x_1) \mapsto (\pi(x_0), \pi(x_1))$. Prove that $R_\sim = (\pi \times \pi)^{-1} (\Delta (X/\sim))$.

Task E14.2.4. Let X and $\sim$ be as in Example 13.4.1. Prove that $((0,0), (0,0))$ is a limit point of $R_\sim$ in $X \times X$ with respect to $\mathcal{O}_X$.

E14.3. For a deeper understanding

Task E14.3.1. Let Σ be the set given by

$$\left\{ \tfrac{1}{n} \mid n \in \mathbb{N} \right\}.$$

Let $\mathcal{O}^K$ be the set of subsets U with the property that, for every x which belongs to U, there are real numbers a and b such that one of the following holds.

(1) We have that x belongs to $]a,b[$, and that $]a,b[$ is a subset of U.

(2) We have that x belongs to $]a,b[\setminus (]a,b[\cap \Sigma)$, and that $]a,b[\setminus (]a,b[\cap \Sigma)$ is a subset of U.

Prove that $\mathcal{O}^K$ defines a topology on $\mathbb{R}$.

Terminology E14.3.2. The topology $\mathcal{O}^K$ is known as the *K-topology* on $\mathbb{R}$.

Task E14.3.3. Prove that $(\mathbb{R}, \mathcal{O}^K)$ is Hausdorff. You may wish to proceed as follows.

(1) Observe that $\mathcal{O}_{\mathbb{R}}$ is a subset of $\mathcal{O}^K$.

(2) Appeal to Example 13.2.1 and to Task E13.3.1.

Task E14.3.4. Let $\sim$ be the equivalence relation on $\mathbb{R}$ generated by $1 \sim \frac{1}{n}$ for every n which belongs to $\mathbb{N}$. Prove that $(\mathbb{R}/\sim, \mathcal{O}_{\mathbb{R}/\sim})$ is not Hausdorff. You may wish to proceed as follows.

(1) Let

$$\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\sim$$

be the quotient map. Let U_0 be a neighbourhood of $\pi(0)$ in $\mathbb{R}/\sim$ with respect to $\mathcal{O}_{\mathbb{R}/\sim}$. Let U_1 be a neighbourhood of $\pi(1)$ in $\mathbb{R}/\sim$ with respect to $\mathcal{O}_{\mathbb{R}/\sim}$. By Remark 6.1.9, we have that π is continuous. Deduce that $\pi^{-1}(U_0)$ and $\pi^{-1}(U_1)$ belong to $\mathcal{O}_{\mathbb{R}}$.

(2) Since $\pi(1)$ belongs to U_1, observe that, by definition of $\sim$, the set Σ is a subset of $\pi^{-1}(U_1)$.

(3) Suppose that n belongs to $\mathbb{N}$. Since $\frac{1}{n}$ belongs to $\pi^{-1}(U_1)$, and since $\pi^{-1}(U_1)$ belongs to $\mathcal{O}^K$, observe that, by definition of $\mathcal{O}^K$ and the fact that $\frac{1}{n}$ belongs to Σ, there are real numbers a_n and b_n such that $a_n < \frac{1}{n} < b_n$, and such that $]a_n, b_n[$ is a subset of $\pi^{-1}(U_1)$.

(4) Since $\pi^{-1}(U_0)$ belongs to $\mathcal{O}^K$, we have, by definition of $\mathcal{O}^K$, that there are real numbers a and b such that one of the following holds.

(I) We have that 0 belongs to $]a, b[$, and that $]a, b[$ is a subset of $\pi^{-1}(U_0)$.

(II) We have that 0 belongs to $]a, b[\setminus (]a, b[\cap \Sigma)$, and that $]a, b[\setminus (]a, b[\cap \Sigma)$ is a subset of $\pi^{-1}(U_0)$.

In either case, let n be a natural number such that $\frac{1}{n} < b$. Let x be a real number which does not belong to Σ, and which has the property that $a_n < x < \frac{1}{n}$ and that $0 < x$. Observe that x belongs to both $\pi^{-1}(U_0)$ and to $\pi^{-1}(U_1)$.

(5) Deduce from (4) that $\pi(x)$ belongs to both U_0 and U_1. In other words, $U_0 \cap U_1$ is not empty.

(6) Conclude that $(\mathbb{R}/\sim, \mathcal{O}_{\mathbb{R}/\sim})$ is not Hausdorff.

Task E14.3.5. Let $\sim$ be the equivalence relation on $\mathbb{R}$ of Task E14.3.4. Let $\mathcal{O}^{K^2}$ be the product topology on $\mathbb{R}^2$ with respect to two copies of $(\mathbb{R}, \mathcal{O}^K)$. Prove that $R_\sim$ is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}^{K^2}$. You may wish to proceed as follows.

(1) Suppose that x is a limit point of Σ in $\mathbb{R}$ with respect to $\mathcal{O}^K$. By Task E8.3.10, deduce that x is a limit point of Σ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

(2) Demonstrate that the only limit point of Σ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ is 0.

(3) Suppose that a and b belong to $\mathbb{R}$, and that $a < 0 < b$. Observe $]a, b[\setminus \Sigma$ is a neighbourhood of 0 in $\mathbb{R}$ with respect to $\mathcal{O}^K$. Since $\Sigma \cap (]a, b[\setminus \Sigma)$ is empty, deduce that 0 is not a limit point of Σ in $\mathbb{R}$ with respect to $\mathcal{O}^K$.

(4) Deduce from (1)–(3) that Σ is closed in $\mathbb{R}$ with respect to $\mathcal{O}^K$.

(5) By Task E3.3.1, deduce from (4) that $\Sigma \times \Sigma$ is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}^{K^2}$.

(6) Observe that $R_\sim$ is $\Sigma \times \Sigma$.

(7) Conclude that $R_\sim$ is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}^{K^2}$.

Task E14.3.6. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Suppose that either $(X, \mathcal{O}_X)$ or $(Y, \mathcal{O}_Y)$ is not compact. Prove that $(X \times Y, \mathcal{O}_{X \times Y})$ is not compact. You may wish to glance back at Example 14.4.2 and Example 14.4.5.

A. Set theoretic foundations

A.1. Set theoretic equalities and relations

Remark A.1.1. Throughout the course, we shall make use of various set theoretic equalities and relations. Table A.1 and Table A.2 list many of these.

Remark A.1.2. Here is one more set theoretic identity which does not fit into Table A.1! Given a set X, a set Y, a subset A of X, and a subset B of Y, we have that

$$(X \times Y) \setminus (A \times B) = ((X \setminus A) \times B) \cup (A \times (Y \setminus B)) \cup ((X \setminus A) \times (Y \setminus B)).$$

A.2. Injections, surjections, and bijections

Definition A.2.1. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is an *injection*, or *injective*, if, for every x_0 and x_1 which belong to X such that $f(x_0) = f(x_1)$, we have that $x_0 = x_1$.

Proposition A.2.2. Let X, Y, and Z be sets. Let

$$X \xrightarrow{f} Y$$

and

$$Y \xrightarrow{g} Z$$

be injections. Then

$$X \xrightarrow{g \circ f} Z$$

is an injection.

Proof. Suppose that x_0 and x_1 belong to X, and that $g(f(x_0)) = g(f(x_1))$. Since g is injective, we deduce that $f(x_0) = f(x_1)$. Since f is injective, we deduce that $x_0 = x_1$. $\square$

Remark A.2.3. Let X be a set. Let A be a subset of X. Let

$$A \xrightarrow{i} X$$

be the inclusion map of Terminology 5.2.1. Then i is an injection.

Definition A.2.4. Let X and Y be sets. A map

$$X \xrightarrow{f} Y$$

is a *surjection*, or *surjective*, if, for every y which belongs to Y, there is an x which belongs to X such that $f(x) = y$.

Proposition A.2.5. Let X and Y be sets. Suppose that there exists an injection

$$X \xrightarrow{f} Y,$$

and that there exists an injection

$$Y \xrightarrow{g} X.$$

Then there exists a bijection

$$X \longrightarrow Y.$$

Proof. Let A_1 be the subset $X \setminus g(Y)$ of X. For every n which belongs to $\mathbb{N}$, let A_n be the subset of Y given by $g\left(f(A_{n-1})\right)$.

Suppose that x belongs to X, and that x does not belong to A_1. Then x belongs to $g(Y)$. Since g is injective, we deduce that there is a unique $y_x \in Y$ such that $g(y_x) = x$.

Let

$$X \xrightarrow{f'} Y$$

be the map given by

$$x \mapsto \begin{cases} f(x) & \text{if } x \text{ belongs to } A_n \text{ for some } n \in \mathbb{N}, \\ y_x & \text{otherwise.} \end{cases}$$

We shall first prove that f' is injective. Suppose that x_0 and x_1 belong to X, and that $f'(x_0) = f'(x_1)$. Suppose that x_1 does not belong to A_n for any $n \in \mathbb{N}$. Then $f'(x_1) = y_{x_1}$. Suppose that x_0 belongs to A_m for some $m \in \mathbb{N}$. Then $f'(x_0) = f(x_0)$. We deduce that

$$x_1 = g\left(y_{x_1}\right) = g\left(f'(x_1)\right) = g\left(f(x_0)\right).$$

Thus x_1 belongs to A_{m+1}. This contradicts our assumption on x_1. We deduce that x_0 does not belong to A_m for any $m \in \mathbb{N}$. Thus $f'(x_0) = y_{x_0}$. Then

$$x_0 = g(y_{x_0}) = g\left(f'(x_0)\right) = g\left(f'(x_1)\right) = g(y_{x_1}) = x_1.$$

An entirely analogous argument demonstrates that if x_0 does not belong to A_n for any $n \in \mathbb{N}$, then $x_0 = x_1$.

Suppose now that there is an $m \in \mathbb{N}$ such that x_0 belongs to A_n, and that there is an $n \in \mathbb{N}$ such that x_1 belongs to A_n, Then

$$f(x_0) = f'(x_0) = f'(x_1) = f(x_1).$$

Since f is injective, we deduce that $x_0 = x_1$. This completes our proof that f' is injective.

We shall now prove that f is surjective. Suppose that y belongs to Y. We have that $g(y)$ does not belong to A_1. For every $n \in \mathbb{N}$, suppose that y does not belong to $f(A_n)$. Then $g(y)$ does not belong to A_n for every $n \in \mathbb{N}$. Thus $f'\left(g(y)\right) = y$. Suppose instead that there is an $n \in \mathbb{N}$ such that y belongs to $f(A_n)$. Then there is an $x \in A_n$ such that $f(x) = y$. This completes our proof that f' is surjective.

We have demonstrated that f' is both injective and surjective. By Task E7.2.1, we conclude that f' is bijective. $\square$

Proposition A.2.5 does not assert that f and g are inverse to each other. Rather, we used f and g to find a new map

$$X \longrightarrow Y,$$

which we proved to be a bijection.

Remark A.2.6. Proposition A.2.5 is sometimes known as the *Cantor-Bernstein-Schröder theorem.*

A.3. Coproducts

Notation A.3.1. Let J be a set. For every j which belongs to J, let X_j be a set. We denote by $\bigsqcup_{j \in J} X_j$ the set $\bigcup_{j \in J} \left(X_j \times \{j\}\right)$.

Remark A.3.2. Suppose that j_0 and j_1 belong to J. We allow that $X_{j_0} = X_{j_1}$.

Definition A.3.3. Let J be a set. For every j which belongs to J, let X_j be a set. We refer to $\bigsqcup_{j \in J} X_j$ as a *coproduct.*

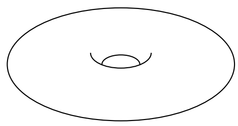
Notation A.3.4. Let J and X be sets. For every j which belongs to J, let X_j be X. We often denote $\bigsqcup_{j \in J} X_j$ by $\bigsqcup_{j \in J} X$.

Notation A.3.5. When J is $\{0,1\}$, we often denote $\bigsqcup_{\{0,1\}} X_j$ by $X_0 \sqcup X_1$. In particular, we often denote $\bigsqcup_{j\in\{0,1\}} X$ by $X \sqcup X$. When J is $\{0, 1, \ldots, n\}$, we similarly often denote $\bigsqcup_{j\in\{0,1,\ldots,n\}} X_j$ by

$$\underbrace{X_0 \sqcup X_1 \sqcup \ldots \sqcup X_n}_{n}.$$

We sometimes also denote $\bigsqcup_{j\in\{0,1,\ldots,n\}} X_j$ by $\bigsqcup_{0\leq j\leq n} X_j$.

It is important to appreciate that $X \sqcup X$ and $X \cup X$ are very different! For $X \cup X$ is X, but $X \sqcup X$ can be thought of as 'two disjoint copies' of X. Think of T^2.



One doughnut is very different from two doughnuts!

Proposition A.3.6. Let X be a set, and let X_0 and X_1 be subsets of X. Suppose that $X = X_0 \cup X_1$. Moreover, suppose that this union is disjoint, in the sense of Terminology 9.5.1. Then there is a bijection between X and the coproduct of X_0 and X_1.

Proof. Let

$$(X_0 \times \{0\}) \cup (X_1 \times \{1\}) \xrightarrow{f} X$$

be the map given by $(x_0, 0) \mapsto x_0$ for every x_0 which belongs to X_0, and by $(x_1, 1) \mapsto x_1$ for every x_1 which belongs to X_1. Let

$$X \xrightarrow{g} (X_0 \times \{0\}) \cup (X_1 \times \{1\})$$

be the map given by

$$x \mapsto \begin{cases} (x, 0) & \text{if } x_0 \text{ belongs to } X_0, \\ (x, 1) & \text{if } x_1 \text{ belongs to } X_1. \end{cases}$$

The fact that X is the disjoint union of X_0 and X_1 exactly ensures that g is well defined. We have that $g \circ f = id_{(X_0\times\{0\})\cup(X_1\times\{1\})}$, and that $f \circ g = id_X$.

□

Remark A.3.7. Proposition A.3.6 justifies our use of the same notation for disjoint unions and coproducts.

A.4. Equivalence relations

Definition A.4.1. Let X be a set. A *relation* on X is a subset of $X \times X$.

Notation A.4.2. Let X be a set, and let R be a relation on X. Suppose that x_0 belongs to X, that x_1 belongs to X, and that (x_0, x_1) belongs to R. We write $x_0 \sim x_1$.

Definition A.4.3. Let X be a set. A relation R on X is an *equivalence relation* if the following hold.

(1) For all $x \in X$, we have that $x \sim x$.

(2) For all $x_0 \in X$ and $x_1 \in X$, such that $x_0 \sim x_1$, we have that $x_1 \sim x_0$.

(3) For all $x_0 \in X$, $x_1 \in X$, and $x_2 \in X$, such that $x_0 \sim x_1$ and $x_1 \sim x_2$, we have that $x_0 \sim x_2$.

Remark A.4.4. Axiom (1) is known as *reflexivity.* Axiom (2) is known as *symmetry.* Axiom (3) is known as *transitivity.*

Example A.4.5. Let $X = \{a, b, c\}$ be a set with three elements. We have the following.

(1) The relation R of X given by

$$\{(a, b), (b, a)\}$$

is not an equivalence relation. Symmetry and transitivity hold, but reflexivity does not.

(2) The relation R of X given by

$$\{(a, a), (b, b), (c, c), (b, c)\}$$

is not an equivalence relation. Reflexivity and transitivity hold, but symmetry does not, since (c, b) does not belong to R.

(3) The relation R of X given by

$$\{(a, a), (b, b), (c, c), (a, c), (c, a), (b, c), (c, b)\}$$

is not an equivalence relation. Reflexivity and symmetry hold, but transitivity does not, since (a, c) and (c, b) belong to R, but (a, b) does not.

(4) The relation R of X given by

$$\{(a, a), (b, b), (c, c)\}$$

is an equivalence relation on R. It is the relation defined by equality.

(5) The relation R of X given by

$$\{(a,a),(b,b),(c,c),(a,c),(c,a)\}$$

is an equivalence relation on R.

Notation A.4.6. Let X be a set, and let R be a relation on X. Let us consider the following subsets of $X \times X$.

(1) Let Δ be the subset of $X \times X$ given by

$$\{(x,x) \in X \times X \mid x \in X\}.$$

(2) Let R_{sym} denote the set

$$\{(x_0,x_1) \in X \times X \mid (x_1,x_0) \in R\}.$$

(3) Let R_{equiv} denote the set of $(x,x') \in X \times X$ such that there are is an integer $n \geq 2$ and an n-tuple $(x_1,\ldots,x_n) \in X \times \ldots \times X$ with the following properties.

   (a) We have that $x = x_1$.

   (b) We have that $x' = x_n$.

   (c) For every $1 \leq i \leq n-1$, we have that $(x_i,x_{i+1}) \in \Delta \cup R \cup R_{\mathsf{sym}}$.

Remark A.4.7. For any $y,y' \in X$, we have that if $(y,y') \in \Delta \cup R \cup R_{\mathsf{sym}}$, then $(y',y) \in \Delta \cup R \cup R_{\mathsf{sym}}$.

Proposition A.4.8. Let X be a set, and let R be a relation on X. Then R_{equiv} defines an equivalence relation on X.

Proof. We verify that the conditions of Definition A.4.3 hold.

(1) Let $x \in X$. Since $(x,x) \in \Delta$, and hence $(x,x) \in \Delta \cup R \cup R_{\mathsf{sym}}$. Thus the pair (x,x) exhibits that $(x,x) \in R_{\mathsf{equiv}}$.

(2) Let $(x,x') \in R_{\mathsf{equiv}}$. By definition of R_{equiv}, there is an integer $n \geq 2$, and an n-tuple $(x_1,\ldots,x_n) \in X \times \ldots \times X$, with the following properties.

   (a) We have that $x = x_1$.

   (b) We have that $x' = x_n$.

   (c) For every $1 \leq i \leq n-1$, we have that $(x_i,x_{i+1}) \in \Delta \cup R \cup R_{\mathsf{sym}}$.

   By Remark A.4.7, we have, for every $1 \leq n-1$, that $(x_{i+1},x_i) \in \Delta \cup R \cup R_{\mathsf{sym}}$. Thus the n-tuple $(x_n,\ldots,x_1)$ exhibits that (x',x) belongs to R_{equiv}.

(3) Let $(x,x') \in R_{\mathsf{equiv}}$, and let $(x',x'') \in R_{\mathsf{equiv}}$.

   By definition of R_{equiv}, there is an integer $m \geq 2$, and an m-tuple $(x_1,\ldots,x_m) \in X \times \ldots \times X$, with the following properties.

(a) We have that $x = x_1$.

(b) We have that $x' = x_n$.

(c) For every $1 \leq i \leq m-1$, we have that $(x_i, x_{i+1}) \in \Delta \cup R \cup R_{\mathsf{sym}}$.

In addition, there is an integer $n \geq 2$, and an n-tuple $(y_1, \ldots, y_n) \in X \times \ldots \times X$, with the following properties.

(a) We have that $x' = y_1$.

(b) We have that $x'' = y_n$.

(c) For every $1 \leq i \leq n-1$, we have that $(y_i, y_{i+1}) \in \Delta \cup R \cup R_{\mathsf{sym}}$.

The $(m+n-1)$-tuple $(x_1, \ldots, x_{m-1}, x_m = y_1, y_2, \ldots, y_n)$ exhibits that (x, x'') belongs to R_{equiv}.

□

Remark A.4.9. Let X be a set, and let R be a relation on X. It is straightforward to prove that if R' is a relation on X such that $R \subset R'$, then $R_{\mathsf{equiv}} \subset R'$. In other words, R_{equiv} is the smallest equivalence relation on X containing R.

Terminology A.4.10. Let X be a set, and let R be a relation on X. We refer to R_{equiv} as the *equivalence relation generated by* R.

Remark A.4.11. In practise, given R, we typically do not determine R_{equiv} by working directly with the definition given in Notation A.4.6. Rather we just 'inductively throw in by hand everything we need to obtain an equivalence relation, but nothing else'!

Example A.4.12. Let $X = \{a, b, c\}$ be a set with three elements. We have the following.

(1) Let R be the relation on X given by

$$\{(a, b), (b, a)\}.$$

Then R_{equiv} is given by

$$\{(a, a), (b, b), (c, c), (a, b), (b, a)\}.$$

(2) Let R be the relation on X given by

$$\{(a, a), (b, b), (c, c), (b, c)\}.$$

Then R_{equiv} is given by

$$\{(a, a), (b, b), (c, c), (b, c), (c, b)\}.$$

(3) Let R be the relation on X given by

$$\{(a,a),(b,b),(c,c),(a,c),(c,a),(b,c),(c,b)\}.$$

Then R_{equiv} is given by

$$\{(a,a),(b,b),(c,c),(a,c),(c,a),(b,c),(c,b),(a,b),(b,a)\}.$$

In other words, R_{equiv} is all of $X \times X$.

(4) Let R be the relation on X given by

$$\{(b,b)\}.$$

Then R_{equiv} is given by

$$\{(a,a),(b,b),(c,c)\}.$$

Equality	Setting
$X \cap \left(\bigcup_{i\in I} Y_i\right) = \bigcup_{i\in I} (X \cap Y_i)$	A set X, and a (possibly infinite) set $\{Y_i\}_{i\in I}$ of sets.
$X \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} (X \setminus A_i)$	A set X, and a (possibly infinite) set $\{A_i\}_{i\in I}$ of subsets of X.
$X \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I} (X \setminus A_i)$	A set X, and a (possibly infinite) set $\{A_i\}_{i\in I}$ of subsets of X.
$f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i)$	A map $$X \xrightarrow{f} Y$$ of sets, and a (possibly infinite) set $\{A_i\}_{i\in I}$ of subsets of Y.
$f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i)$	A map $$X \xrightarrow{f} Y$$ of sets, and a (possibly infinite) set $\{A_i\}_{i\in I}$ of subsets of Y.
$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$	A map $$X \xrightarrow{f} Y$$ of sets, and a subset A of Y.
$f\left(f^{-1}(A)\right) = A$	A surjective map $$X \xrightarrow{f} Y$$ of sets, and a subset A of Y.

Table A.1.: Set theoretic equalities

Relation	Setting
$f(A \cap B) \subset f(A) \cap f(B)$	A map $$X \xrightarrow{f} Y$$ of sets, a subset A of X, and a subset B of X.
$f(A) \subset f(B)$	A map $$X \xrightarrow{f} Y$$ of sets, and subsets A and B of X such that $A \subset B$.
$f^{-1}(A) \subset f^{-1}(B)$	A map $$X \xrightarrow{f} Y$$ of sets, and subsets A and B of Y such that $A \subset B$.
$A \subset f^{-1}(f(A))$	A map $$X \xrightarrow{f} Y$$ of sets, and a subset A of X.
$f\left(f^{-1}(A)\right) \subset A$	A map $$X \xrightarrow{f} Y$$ of sets, and a subset A of Y.

Table A.2.: Set theoretic relations

17 Tuesday 12th March

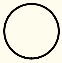
17.1 Knots and links – definitions and examples

Terminology 17.1 A **subspace** of $\mathbb{R}^3$ is a subset X of $\mathbb{R}^3$ equipped with the subspace topology $\mathcal{O}_X$ with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$.

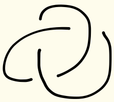
Definition 17.2 A **knot** is a subspace $(K, \mathcal{O}_K)$ of $\mathbb{R}^3$ which is homeomorphic to $(S^1, \mathcal{O}_{S^1})$.

Examples 17.3

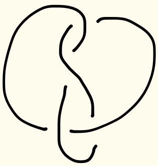
(1) Unknot.



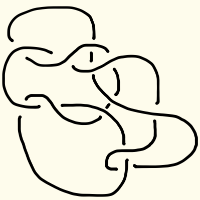
(2) Trefoil.



(3) Figure of eight.



(4)

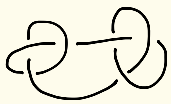


This can in fact be unknotted to

!!!

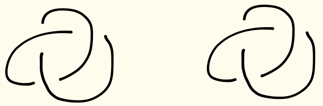
In the language we will introduce shortly, this knot is 'isotopic' to the unknot.

(5) Granny knot.

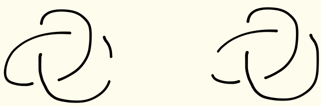


This is the 'connected sum' of two trefoil knots. By a 'connected sum' we mean the following construction.

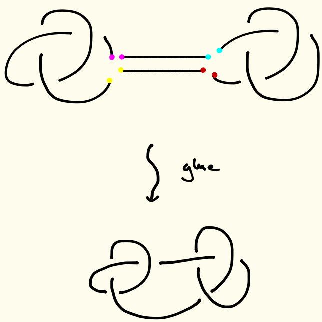
(i) Begin with two trefoils.



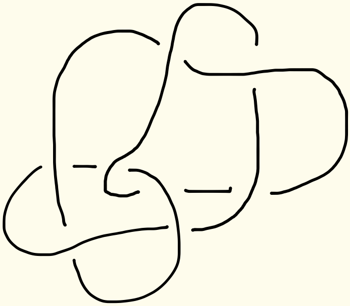
(ii) Cut a small piece out of each trefoil.



(iii) Connect the two trefoils by gluing in a disjoint pair of line segments.



(6) True lovers' knot.



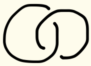
Can you see where the name comes from? Look for two hearts!

Definition 17.4 A **link** is a subspace $(L, \mathcal{T}_L)$ of $\mathbb{R}^3$ which is homeomorphic to a finite disjoint union of copies of $(S^1, \mathcal{T}_{S^1})$.

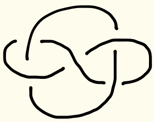
Remark 17.5 In particular, a knot is a link.

Examples 17.6

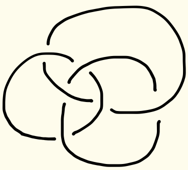
(1) Hopf link.



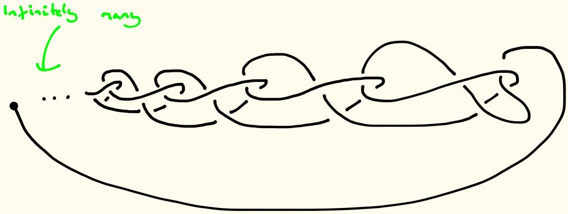
(2) Whitehead link.



(3) Borromean rings.



Remark 17.7 In our study of knots and links we must impose some restrictions to exclude 'wild knots' such as the following.

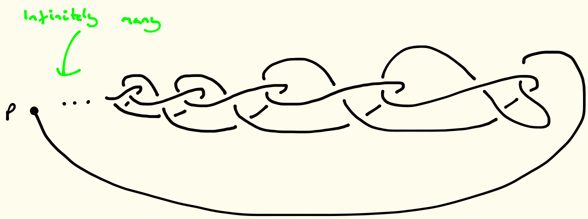


Note this knot would be able to be untied (in the language we are about to introduce, it would be isotopic to the unknot) if it were not for the fact that this would take an infinite length of time!

This kind of issue arises in many places in topology. There are two common ways to exclude this kind of behaviour.

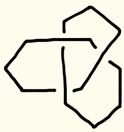
i) We could require that there is a 'smooth' homeomorphism between our knots and $(S^1, \mathcal{O}_{S^1})$, and similarly require that there is a 'smooth' homeomorphism between our links and $(\coprod_n S^1, \mathcal{O}_{\coprod_n S^1})$ for some $n \geq 1$.

This excludes the above example from being a knot, since it is not smooth at the point p indicated below.



Topology under this kind of smoothness assumption is studied in several courses here: MA3402 Analyse på manifoldigheter, TMA4190 Manifoldigheter, MA8402 Lie-grupper og Lie-algebraer.

ii) We could require that our knots and links be able to be obtained by glueing together finitely many line segments. Here are a few examples.



Trefoil

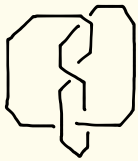
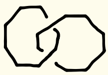


Figure of eight



Hopf link

This excludes our 'wild' example from being a knot, since we can only obtain it by glueing together infinitely many line segments.

Topology under this kind of assumption is known as 'piecewise linear topology'.

I will implicitly take approach (ii), but don't worry about this – just be aware that we have to make some assumption to exclude 'wild' knots and links.

17.2 Isotopy

Definition 17.8 Let (K, θ_K) and $(K', \theta_{K'})$ be knots. Then K is *isotopic* to K' if there exists a continuous map $S^1 \times I \xrightarrow{f} \mathbb{R}^3$ such that the following conditions are satisfied:

(i) for all $t \in I$, the map $S^1 \xrightarrow{f_t} \mathbb{R}^3$ given by $x \mapsto f(x,t)$ is a homeomorphism,

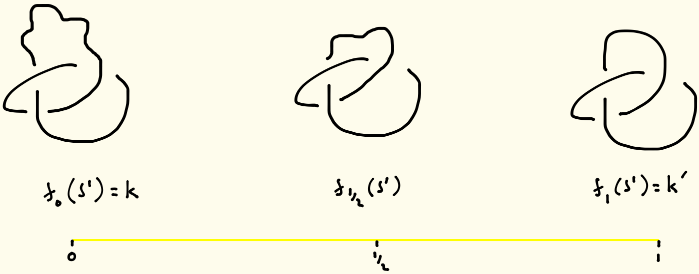
(ii) $f_0(S^1) = K$,

(iii) $f_1(S^1) = K'$.

Here as usual I denotes the unit interval, and $S^1 \times I$ is equipped with the product topology.

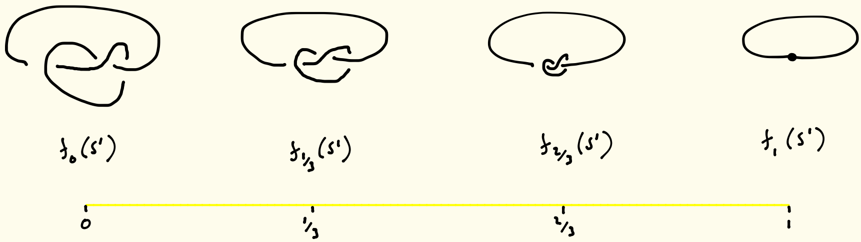
Example 17.9 We can think of an isotopy $S^1 \times I \xrightarrow{f} \mathbb{R}^3$ from a knot (K, θ_K) to a knot $(K', \theta_{K'})$ as a 'movie' of knots beginning with (K, θ_K) and ending with $(K', \theta_{K'})$.

See the next page.



We have seen these kinds of 'movies' earlier in the lectures – look back at Examples 2.13 (2) for example.

Remark 17.10 As in Remark 17.6, we must exclude certain phenomena. For example, we do not wish to allow a knot to be 'pulled tight' as indicated below.



This would mean that every knot would be isotopic to the unknot!

Just as in Remark 17.7, we can exclude this by requiring that our isotopy $S^1 \times I \xrightarrow{f} \mathbb{R}^3$ be 'smooth' or 'piecewise linear'. I will implicitly take f to be piecewise linear.

Don't worry about this! Just as in Remark 17.7, the important thing is to be aware that an additional assumption is necessary.

These remarks apply equally to the following definition.

Definition 17.10 Let $(L, \mathcal{O}_L)$ and $(L', \mathcal{O}_{L'})$ be links. Then L is *isotopic* to L' if for some integer $n \geq 0$ there is a continuous map $\left(\coprod_n S^1\right) \times I \xrightarrow{f} \mathbb{R}^3$ such that the following conditions are satisfied:

(i) for all $t \in I$, the map $\coprod_n S^1 \xrightarrow{f_t} \mathbb{R}^3$ given by $x \mapsto f(x,t)$ is a homeomorphism,

(ii) $f_0\left(\coprod_n S^1\right) = L$,

(iii) $f_1\left(\coprod_n S^1\right) = L'$.

Remark 17.11 Just as in Example 17.9, the isotopy $\left(\coprod_n S^1\right) \times I \xrightarrow{f} \mathbb{R}^3$ can be thought of as a 'movie' of links beginning with L and ending with L'.

Intuitively, two knots or links are isotopic if we can manipulate one to obtain the other in our everyday sense — a knot is isotopic to the unknot if we can untie it in our everday sense!

17.3 Link diagrams and Reidemeister moves

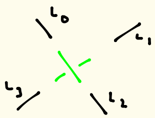
Definition 17.12 A *link diagram* consists of the following data:

(1) A set Arc, whose elements we think of as labels for 'line segments' or 'arcs'.

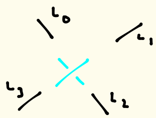
(2) A set OverCrossing, whose elements we think of as labels for 'over crossings'.

(3) A set UnderCrossing, whose elements we think of as labels for 'under crossings'.

(4) A map $\text{OverCrossing} \xrightarrow{d^{\text{Over}}} \text{Arc} \times \text{Arc} \times \text{Arc} \times \text{Arc}$, which we think of as assigning to an over crossing  four arcs (l_0, l_1, l_2, l_3) as follows.

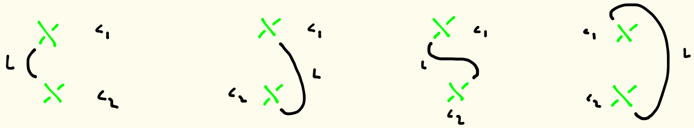


(5) A map $\text{UnderCrossing} \xrightarrow{d^{\text{Under}}} \text{Arc} \times \text{Arc} \times \text{Arc} \times \text{Arc}$, which we think of as assigning to an under crossing  four arcs (l_0, l_1, l_2, l_3) as follows.



We require that for every arc $L \in Arc$, one of the following three possibilities holds:

(i) there are exactly two over crossings $c_1, c_2 \in OverCrossing$ such that L is one of the arcs in $d^{Over}(c_1)$ and L is one of the arcs in $d^{Over}(c_2)$.

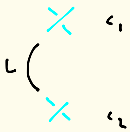


etc.

We allow that $c_1 = c_2$.

etc.

(ii) there are exactly two under crossings $c_1, c_2 \in UnderCrossing$ such that L is one of the arcs in $d^{Under}(c_1)$ and L is one of the arcs in $d^{Under}(c_2)$.

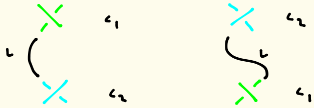


etc.

We allow that $c_1 = c_2$.

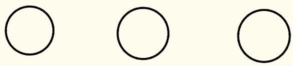
etc.

(iii) there is exactly one crossing $c_1 \in \mathrm{OverCrossing}$ and exactly one crossing $c_2 \in \mathrm{UnderCrossing}$ such that l is one of the arcs in $\partial^{\mathrm{Over}}(c_1)$ and l is one of the arcs in $\partial^{\mathrm{Under}}(c_2)$.



etc.

We allow both OverCrossings and UnderCrossings to be the empty set. In this case we think of Arc as a set of labels for a disjoint collection of circles.



Remark 17.13 Think of Definition 17.12 as like lego: we have three kinds of 'piece' (arcs, over crossings, and under crossings) which we join together in a prescribed way to build up our link diagram.

In particular, though I've drawn pictures to help our intuition, a link diagram is independent of any particular picture - a computer can understand it, for instance!

Construction 17.14 To any link (L, θ_L) we can associate a link diagram. This construction will not be asked about on the exam, but I will discuss it in a supplementary note for those who are interested.

Intuitively, the diagram associated to a link can be thought of as its projection onto a plane as shown below, together with the information as to whether a crossing is an over crossing or an under crossing.

+ remember
crossing information

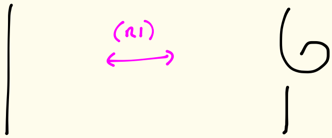
Definition 17.15 We consider three Reidemeister moves upon link diagrams.

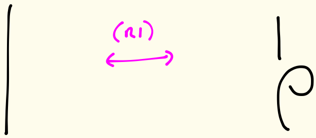
(R1) We replace an arc | by three arcs and an over crossing

or by three arcs and an under crossing

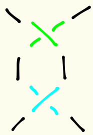
and vice versa.

We depict this as follows.

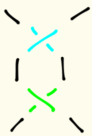




(R2) We replace a pair of arcs $||$ by six arcs, an over crossing, and an under crossing as follows

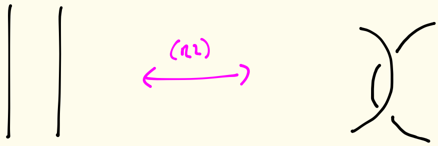


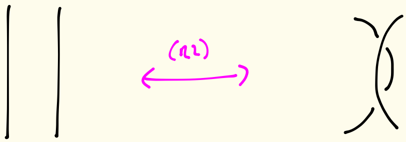
or as follows



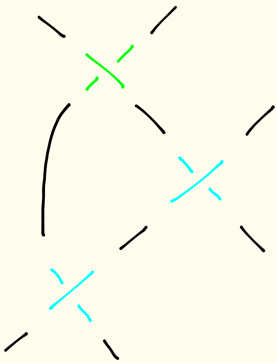
and vice versa.

We depict this as follows.

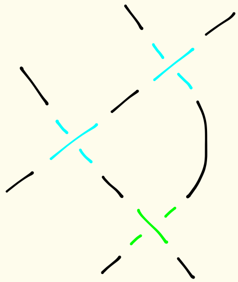




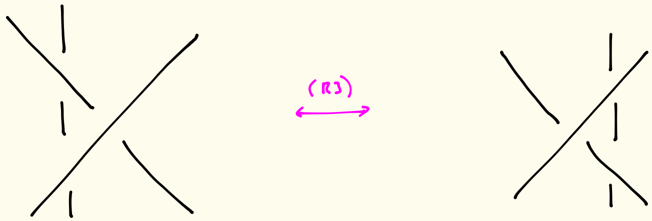
(R3) We replace a configuration of arcs and crossings as follows



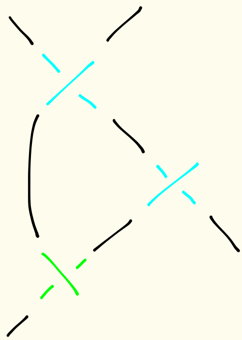
by a configuration of arcs and crossings as follows and vice versa.



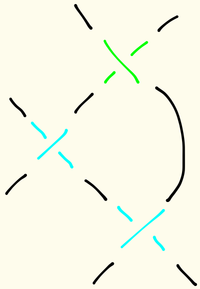
We depict this as follows.



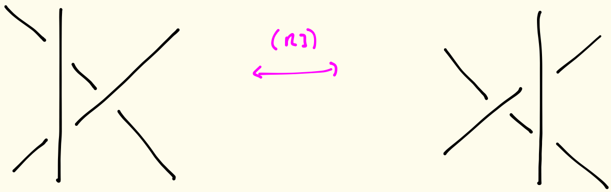
Or we replace a configuration of arcs and crossings as follows



by a configuration of arcs and crossings as follows and vice versa.



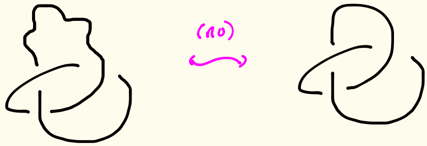
We depict this as follows.



Theorem 17.16 Let $(L, \mathcal{O}_L)$ and $(L', \mathcal{O}_{L'})$ be links, and let D_L and $D_{L'}$ be their associated link diagrams. Then L is isotopic to L' if and only if $D_{L'}$ can be obtained from D_L by a finite sequence of Reidemeister moves.

Remark 17.17 This is a true theorem! A priori all kinds of wild phenomena could occur when working in $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$. A link diagram is a much simpler gadget! The theorem will allow us to define several knot/link 'invariants'. These invariants will allow us to prove for example that various knots cannot be untied, i.e., are not isotopic to the unknot.

Remark 17.18 When drawing knots and links, let us also refer to a change in our picture corresponding to an isotopy which does not affect crossings as an (R_0)-move.



Let us refer to a change in our picture which corresponds to moving an arc under / over the entire rest of our link as an (R_∞)-move.

⚠ Neither an (R_0)-move nor an (R_∞)-move has any meaning for a link diagram! Rather they correspond to the fact that a link diagram is independent of any picture we choose to draw of the link from which it comes, as discussed in Remark 17.13.

18 Thursday 14th March

18.1 Setting the scene

Remark 18.1. Knots have been of importance throughout human history, in our everyday life and in our artistic expression.

We can think of knot theory as an art. Our mathematical understanding of knots heightens our appreciation for these mysterious gadgets which have fascinated humans from prehistoric times.

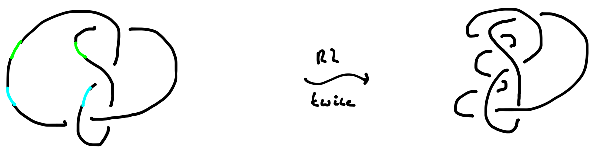
Mathematically, the theory of knots and links is of great significance in low dimensional topology, in the study of what are known as 3-manifolds and 4-manifolds. It is a bridge between representation theory, topology, and category theory which appears to be a gateway to a hitherto unexplored beautiful garden. This is of high current research interest.

18.2 Showing that two knots are isotopic through a seqeunce of Reidemeister moves

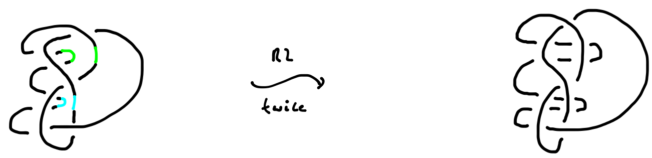
Example 18.2.

Let us explore the Reidemeister moves in practise. We shall prove that the figure of eight knot is isotopic to its mirror image.

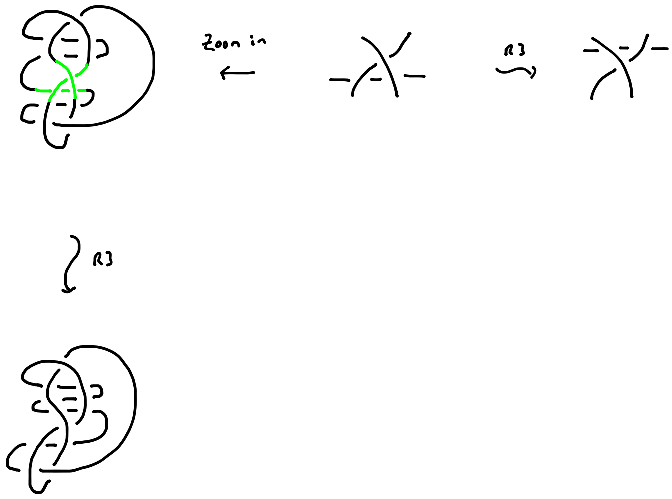
We begin with the figure of eight knot. We apply two R2 moves, one with respect to the green arcs and one with respect to the blue arcs.



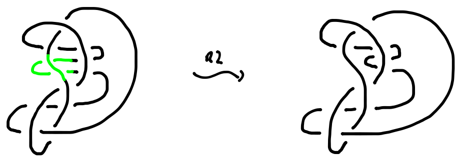
Next we once more apply two R2 moves, one with respect to the green arcs and one with respect to the blue arcs.



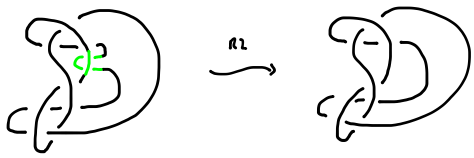
Zooming in on the green arcs below we see that we can apply an R3 move.



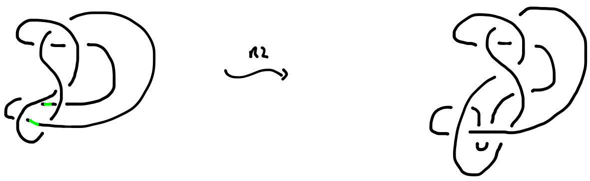
Next we apply an R2 move.



We now apply another R2 move.

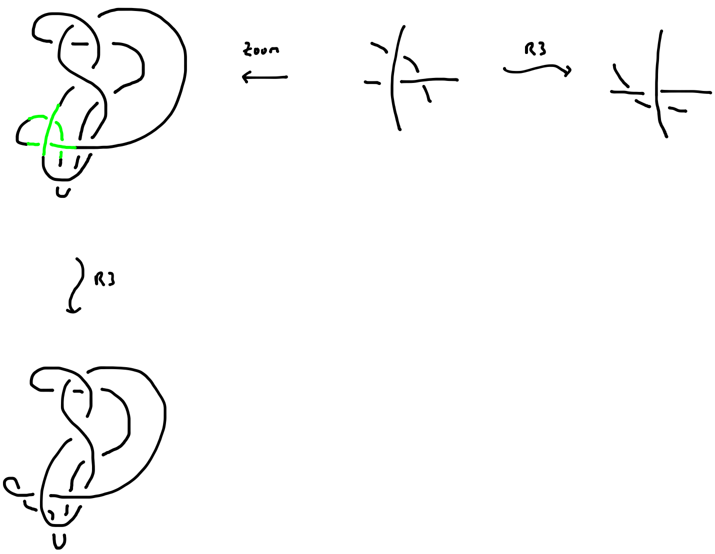


Again we apply an R2 move.

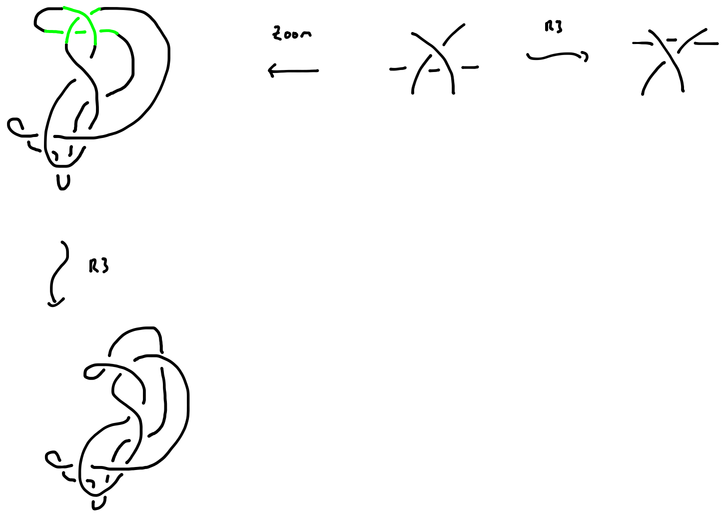


Once more we apply an R2 move.

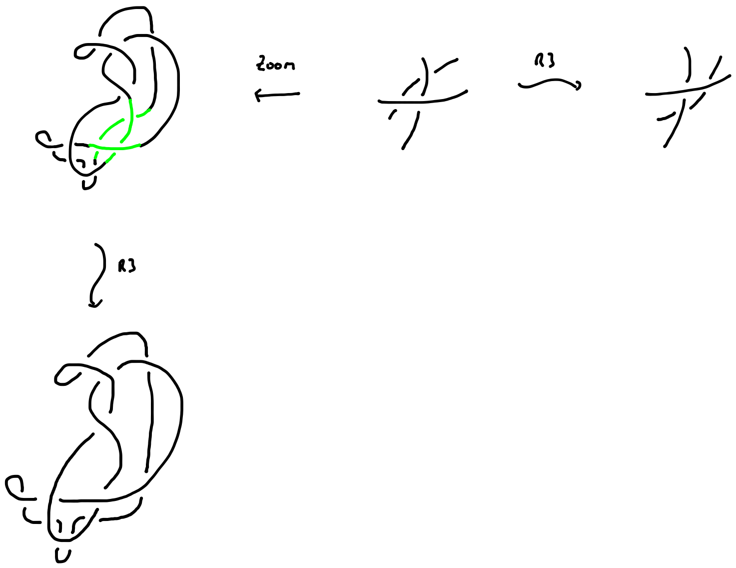
Zooming in on the green arcs below we see that we can apply an R3 move.



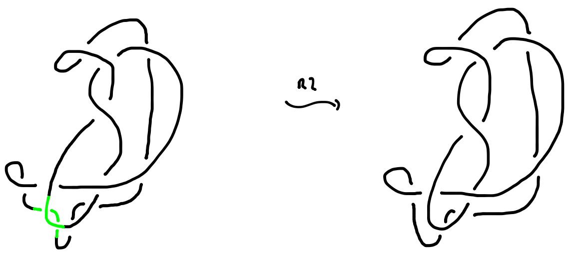
Zooming in on the green arcs below we see that we can apply another R3 move.



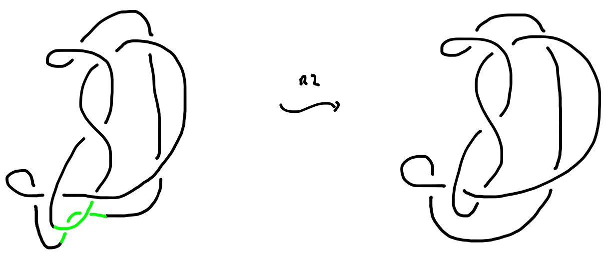
Zooming in on the green arcs below we see that we can once more apply an R3 move.



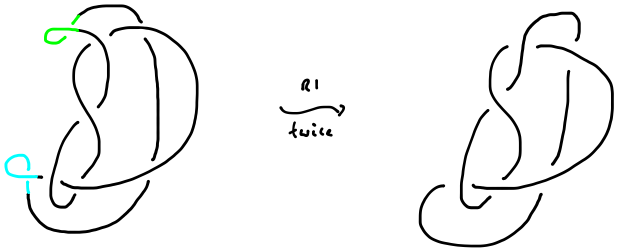
We now apply an R2 move.



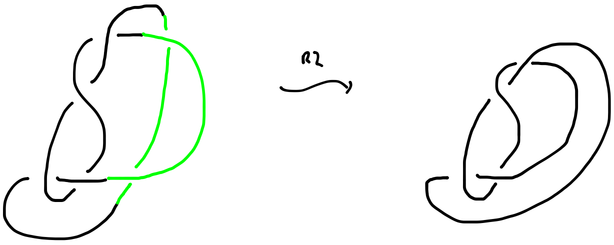
Next we apply another R2 move.



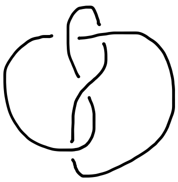
Now we apply two R1 moves, one indicated in green and one indicated in blue.



Finally we apply an R2 move.



Applying an R0 move, or in other words manipulating this knot a little, we see that we indeed have obtained the mirror image of the figure of eight knot!



Remark 18.3. This sequence of Reidemeister moves is far from the only possibility. Even for this sequence we could have carried out various moves in a different order.

Remark 18.4. A link which is isotopic to its mirror image is known as *amphichiral*. A link which is not isotopic to its mirror image is known as *chiral*.

The trefoil knot is chiral, but historically this was difficult to prove. We will later be able to prove it easily by calculating its Jones polynomial.

Remark 18.5. Note that in Example 18.2 we had to increase the number of crossings in our knot via R2 moves before we could simplify.

How do we know that we could not prove that the trefoil is isotopic to its mirror image by increasing to a very large number of crossings?

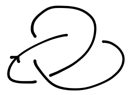
We need some tools which can tell us when one link is not isotopic to another! This is where the importance of the Reidemeister moves truly lies.

18.3 Linking number

Definition 18.6. An *oriented link* is a link $(L, \mathcal{O}_L)$ whose components have been equipped with arrows.

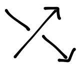
Examples 18.7.

(1) A trefoil with its two possible choices of orientation.

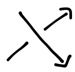


(2) A Hopf link with two choices of orientation.

Definition 18.8. Let $(L, \mathcal{O}_L)$ be an oriented link. The *sign* of a crossing



is -1, and of a crossing



is $+1$.

Remark 18.9. In Definition 18.8 the arcs of the crossings may belong to the same component of L or to distinct components of L.

Definition 18.10. Let $(L, \mathcal{O}_L)$ be an oriented link. The *linking number* of L is

$$\left|\frac{1}{2} \times \Bigg(\sum_{\substack{\text{crossings } C \text{ between} \\ \text{distinct components of } L}} \operatorname{sign}(C) \Bigg)\right|.$$

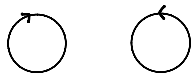
We denote it by $lk(L)$.

If there are no crossings between distinct components of L, we define $lk(L) = 0$.

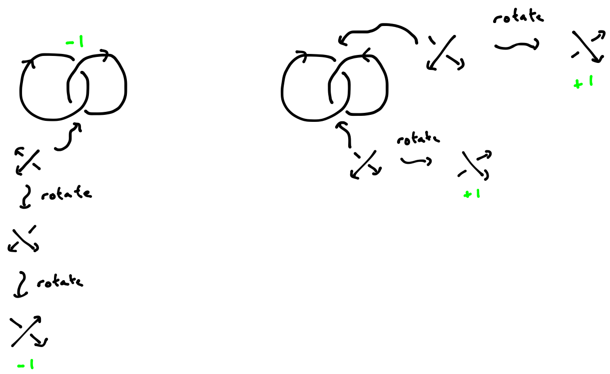
Remark 18.11. It is crucial that we only allow crossings between distinct components of L in Definition 18.10. In particular, $lk(K) = 0$ for all knots $(K, \mathcal{O}_K)$.

Examples 18.12.

(1) The linking number of the unlink with n components is 0 for any n, since there are no crossings.

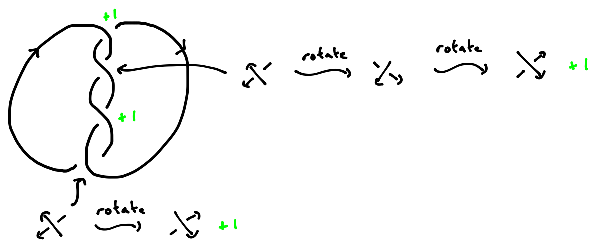


(2) The signs of the crossings for two choices of orientation on a Hopf link are as follows.



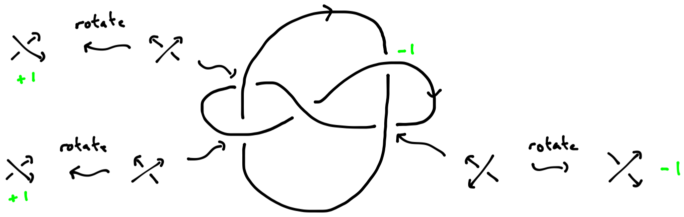
Thus the linking number of the left Hopf link is $|\frac{1}{2} \cdot (-1-1)| = 1$, and the linking number of the right Hopf link is $|\frac{1}{2} \cdot (1+1)| = 1$.

(3) Here are the signs of the crossings for another link.



Its linking number is thus $|\frac{1}{2} \cdot (1+1+1+1)| = 2$.

(4) Here are the signs of the crossings between distinct components of the Whitehead link.



Thus the linking number of the Whitehead link is $|\frac{1}{2} \cdot (1 + 1 - 1 - 1)| = 0$.

Let us emphasise that the sign of the middle crossing is omitted since this crossing does not involve distinct components.

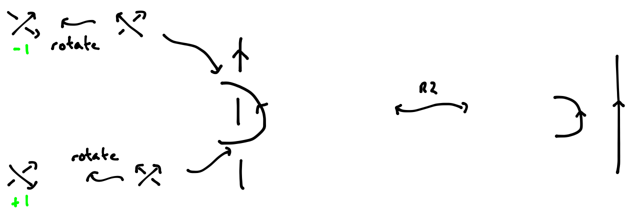
Remark 18.13. It is not a coincidence that we obtained the same linking number for the two choices of orientation on a Hopf link in Examples 18.12 (2). Though we shall not dwell on the point here, one can observe that the linking number is independent of the choice of orientation for all links.

Thus we can speak of the linking number of a link even if no orientation is specified, rather than only of the linking number of an oriented link. We just choose an orientation with which to work.

Proposition 18.14. If two links are isotopic then their linking numbers are equal.

Proof. We know by Theorem 17.16 that two links are isotopic if and only if one can be obtained from the other by a finite sequence of Reidemeister moves. Thus it suffices to prove that the linking number of a link is unchanged under the Reidemeister moves.

- R1 An R1 move does not change the linking number of a link since it only involves a crossing in which both arcs belong to the same component.
- R2 The signs of the crossings in the part of a link affected by an R2 move are indicated below.



The contribution of the crossings in the left diagram to the linking number is $-1 + 1 = 0$. Thus an R2 move does not affect the linking number of a link.

- R3 The signs of the crossings in the part of a link affected by an R3 move are indicated below.

Thus the crossings in each case make the same contribution to the linking number. Hence an R3 move does not affect the linking number of a link.

□

Remark 18.15. This proof is not complete. There is another R2 and another R3 move which must be considered. Moreover the arcs could have other combinations of orientations. However, it is the idea that is important. It adapts in a straightforward way to a proof for the other cases.

Examples 18.16.

(1) We would certainly intuitively believe that the Hopf link cannot be unlinked, or in other words that it is not isotopic to the unlink with two components!

Proposition18.14 allows us to give us a rigorous proof, since by Examples 18.12 (2) the Hopf link has linking number 1, whereas the unlink with two components has linking number 0.

(2) Linking numbers do not however allow us to prove that the Whitehead link is not isotopic to the unlink with two components.

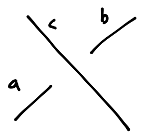
By Examples 18.12 (2) the linking number of the Whitehead link is 0, which is the same as the linking number of the unlink with two components.

19 Tuesday 19th March

19.1 Colourability

Definition 19.1. Let $(L, \mathcal{O}_L)$ be a link, and let $m \in \mathbb{N}$. Then L is *m-colourable* if we can assign an integer to every arc in L in such a way that the following hold.

(1) At every crossing the assigned integers



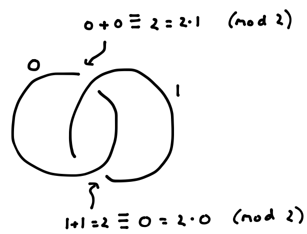
have the property that $a + b \equiv 2c \pmod{m}$.

(2) Not every arc is assigned the same integer mod m.

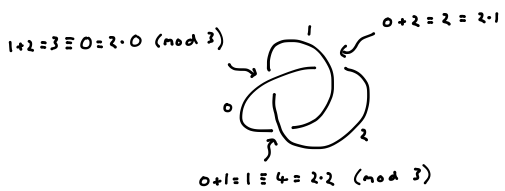
Remark 19.2. No link is 1-colourable! For every integer z we have that $z \equiv 0$ mod 1, and thus condition (2) can never be satisfied for $m = 1$.

Examples 19.3.

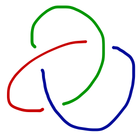
(1) The Hopf link is 2-colourable. An example of a 2-colouring along with a verification that condition (1) holds at both crossings is shown below.



(2) The trefoil knot is 3-colourable. An example of a 2-colouring along with a verification that condition (1) holds at every crossing is shown below.

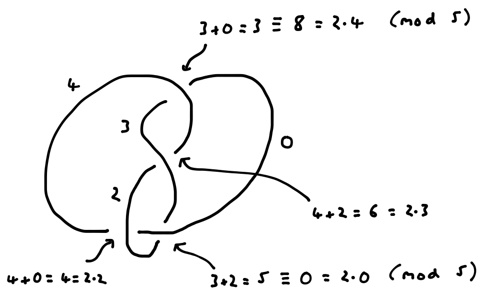


To give a 3-colouring it is equivalent to draw the arcs using three colours, such that every crossing either the three arcs have three different colours or else the three arcs have the same colour.



(3) Here is another example of a 3-colouring.

(4) The figure of eight knot is 5-colourable. An example of a 5-colouring along with the verification that condition (1) holds at every crossing is shown below.

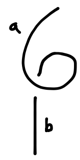


Remark 19.4. There can be many different ways to equip a link with an m-colouring.

Proposition 19.5. Let $(L, \mathcal{O}_L)$ and $(L', \mathcal{O}_{L'})$ be links. If L is isotopic to L' then L is m-colourable for an integer m if and only if L' is m-colourable.

Proof. We know by Theorem 17.16 that two links are isotopic if and only if one can be obtained from the other by a finite sequence of Reidemeister moves. Thus it suffices to prove that whether or not a link is m-colourable is unaffected by applying the Reidemeister moves.

R1 Consider the assigned integers in a part of a link which looks as follows.

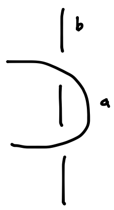


By condition (1) for an m-colouring we have that $a + b \equiv 2a \pmod{m}$. This implies that $a \equiv b \pmod{m}$.

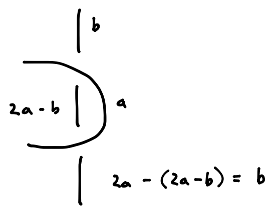
Thus we may replace this part of our link with

without affecting whether or not our link is m-colourable.

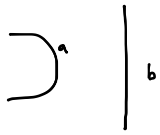
R2 Suppose that in a part of a link which looks as follows we have that two of the assigned integers a and b are as shown.



By condition (1) for an m-colouring with respect to the two crossings, the two remaining arcs must be assigned the following integers mod m.

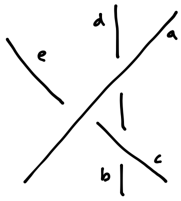


Thus we may replace this part of our link with

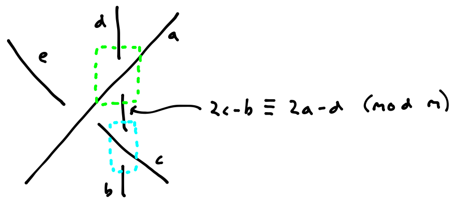


without affecting whether or not our link is m-colourable.

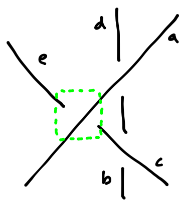
R3 Suppose that in a part of a link which looks as follows we have that five of the assigned integers a, b, c, d, and e are as shown.



By condition (1) for an m-colouring with respect to the two crossings indicated below we have that the integer assigned to the remaining arc must be equal to both $2c - b$ and $2a - d$ mod m.



Moreover by condition (1) for an m-colouring applied to the crossing indicated below we have that $c + e \equiv 2a \pmod{m}$.



Putting these two observations together we have that the following hold mod m.

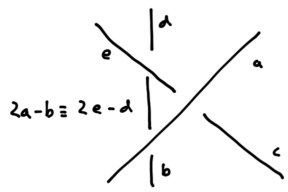
$$2c - b \equiv 2a - d$$
$$c + e \equiv 2a$$

By the second equation we deduce that $c \equiv 2a - e \pmod{m}$. By the first equation we deduce that

$$2 \cdot (2a - e) - b \equiv 2a - d \pmod{m},$$

and hence that $2a - b \equiv 2e - d \pmod{m}$.

Thus we may replace this part of our link with



without affecting whether or not our link is m-colourable.

□

Remark 19.6. This proof is not quite complete. There is another R2 and another R3 move which must be considered. However, just as in the proof of Proposition 18.14 it is the idea that is important. It adapts in a straightforward way to a proof for the other cases.

Examples 19.7.

(1) It is intuitive that the trefoil cannot be unknotted! Proposition 19.5 allows us to give a rigorous proof of this.

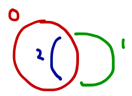
By Examples 19.3 (2) we have that the trefoil is 3-colourable. The unknot cannot be m-colourable for any m, since condition (2) clearly cannot be satisfied! In particular, it is not 3-colourable.

(2) Similarly by Examples 19.3 (4) we have that the figure of eight knot is 5-colourable. Thus it cannot be isotopic to the unknot.

(3) We have to be careful! The unlink with two 2-components is 2-colourable.

Thus we cannot deduce from Examples 19.3 (1) and Proposition 19.5 that the Hopf link is not isotopic to the unlink with two components, which we already proved using its linking number in Examples 18.16.

(4) The unlink with two components is also 3-colourable.



We will see shortly that this will allow us to prove that the Whitehead link is not isotopic to the unlink with two components, which we could not prove using linking numbers.

Lemma 19.8. Let $(L, \mathcal{O}_L)$ be an m-colourable link for $m \in \mathbb{N}$. Let $k \in \mathbb{Z}$. For any arc A of L we can find an m-colouring of L in which A is assigned the integer k.

Proof. Let a, b, c be integers. If $a+b \equiv 2c \pmod{m}$, then for any integer l we have that

$$(a+l)+(b+l) = a+b+2l \equiv 2c+2l = 2(c+l) \pmod{m}.$$

Thus given any m-colouring of L we obtain another m-colouring of by adding l to the integer assigned to every arc.

Let z be the integer assigned to A in a given m-colouring of L. Adding $k-z$ to every arc we obtain another m-colouring. In this m-colouring the integer assigned to A is $z+(k-z)=k$, as required. □

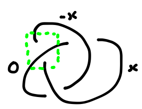
Examples 19.9.

(1) To see the full power of colourability we need to be able to determine for which integers m a given link is m-colourable.

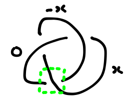
We have already seen that the trefoil is 3-colourable. By part of the argument in the proof of Lemma 19.8 it follows that the trefoil is m-colourable for any $m \in \mathbb{Z}$ with $3 \mid m$. Let us now prove that if the trefoil is m-colourable then $m \equiv 0 \pmod{3}$.

Suppose that we have an m-colouring of the trefoil. By Lemma 19.8 we can fix the integer assigned to one of the arcs to be 0. Let us denote the integer assigned to one of the other arcs by x.

By condition (1) for an m-colouring applied to the indicated crossing the integer assigned to the third arc must be equal to $-x$ mod m.



By condition (1) applied to the indicated crossing we have that $-x \equiv 2x \pmod{m}$, and hence that $3x \equiv 0 \pmod{m}$.



We thus have that $3x = km$ for some $k \in \mathbb{Z}$. Since 3 is prime we deduce that either $3 \mid k$ or $3 \mid m$. If $3 \mid k$ then $x \equiv 0 \pmod{m}$, and hence $-x \equiv 0 \pmod{m}$. Thus our colouring would be

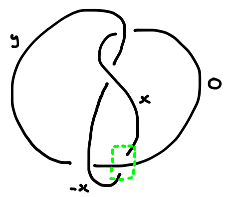
which would contradict condition (2) for an m-colouring.

We deduce that $3 \mid m$.

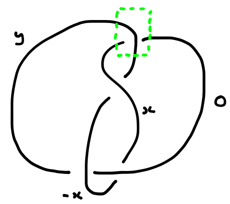
(2) Let us prove in a similar way that the figure of eight knot is m-colourable if and only if $m \equiv 0 \pmod{5}$.

Suppose that we have an m-colouring of the figure of eight knot. By Lemma 19.8 we can fix the integer assigned to one of the arcs to be 0. Let us denote the integer assigned to two of the other arcs by x and y.

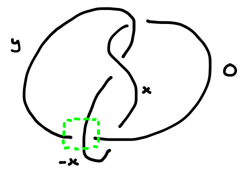
By condition (1) for an m-colouring applied to the crossing indicated below, the integer assigned to the arc shown must be equal to $-x$ mod m.



By condition (1) for an m-colouring applied to the crossing indicated below, we must have that $x \equiv 2y \pmod{m}$.

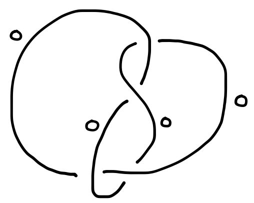


By condition (1) for an m-colouring applied to the crossing indicated below, we have that $y \equiv -2x \pmod{m}$.



Together the fact that $x \equiv 2y \pmod{m}$ and $y \equiv -2x \pmod{m}$ implies that $x \equiv -4x \pmod{m}$, and hence that $5x \equiv 0 \pmod{m}$.

We thus have that $5x = km$ for some $k \in \mathbb{Z}$. Since 5 is prime we deduce that either $5 \mid k$ or $5 \mid m$. If $5 \mid k$ then $x \equiv 0 \pmod{m}$, and hence $-x \equiv 0 \pmod{m}$ and $y \equiv 2 \cdot 0 = 0 \pmod{m}$. Thus our colouring would be



which would contradict condition (2) for an m-colouring.

We deduce that $5 \mid m$.

(3) By (1), (2), and Proposition 19.5 we can conclude that the trefoil is not isotopic to the figure of eight knot.

20 Thursday 21st March

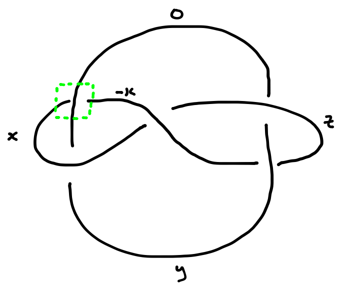
20.1 Link colourability, continued

Examples 20.1.

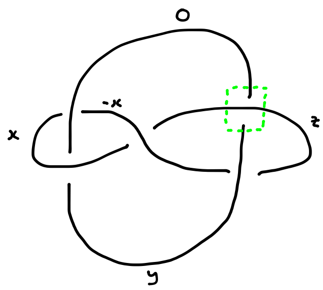
(4) Let us prove that the Whitehead link is not p-colourable for any odd prime p.

Suppose that we have a p-colouring of the Whitehead link. By Lemma 19.8 we can fix the integer assigned to one of the arcs to be 0. Let us denote the integers assigned to three of the other arcs by x, y, and z.

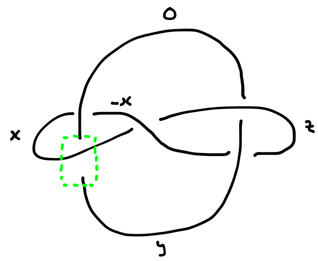
By condition (1) for an m-colouring applied to the crossing indicated below, the integer assigned to the remaining arc must be equal to $-x$ mod p.



By condition (1) for an m-colouring applied to the crossing indicated below, we must have that $y \equiv 2z \pmod{p}$.

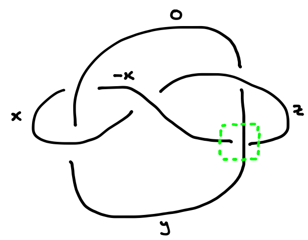


By condition (1) for an m-colouring applied to the crossing indicated below, we must have that $y \equiv 2x \pmod{p}$.



Thus we have that $2x \equiv 2z \pmod{p}$. We deduce since p is a prime that $p \mid 2(x-z)$. Since p is odd we conclude that $p \mid (x-z)$, or in other words that $x \equiv z \pmod{p}$.

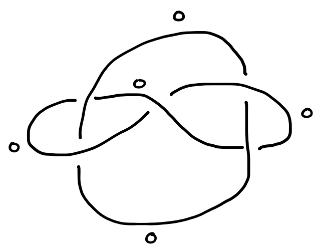
By condition (1) for an m-colouring applied to the crossing indicated below, we must have that $z - x \equiv 2y \pmod{p}$.



Since $x \equiv z \pmod{p}$ we deduce that $2y \equiv 0 \pmod{p}$. Since p is an odd prime, we deduce that $y \equiv 0 \pmod{p}$.

Then since $y \equiv 2x \pmod{p}$ we have that $2x \equiv 0 \pmod{p}$. Since p is an odd prime, we deduce that $x \equiv 0 \pmod{p}$. Since $x \equiv z \pmod{p}$ we then have that $z \equiv 0 \pmod{p}$.

Thus our colouring would be



which would contradict condition (2) for a p-colouring.

(2) In particular, the Whitehead link is not 3-colourable. By Examples 19.7 (4) the unlink with two components is 3-colourable. We conclude by Proposition 19.5 that

the Whitehead link is not isotopic to the unlink with two components, or in other words that it is genuinely linked.

Question 20.2. Is every knot m-colourable for some m?

Answer 20.3. No, but the simplest example — denoted 10_{124} — has 10 crossings!

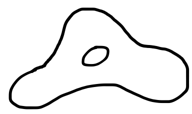
Question 20.4. Can we find an even better link invariant?

Answer 20.5. Yes! It will take us a little while to construct it.

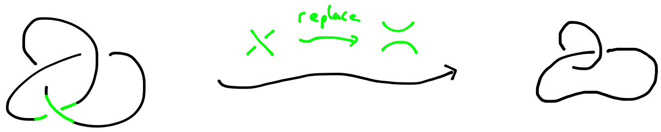
20.2 Bracket polynomial

Definition 20.6. Let $(L, \mathcal{O}_L)$ be a link, and let $\mathcal{D}$ be its corresponding link diagram. A *state* of L is a link diagram obtained by replacing every crossing of $\mathcal{D}$ with one of the following two possibilities.

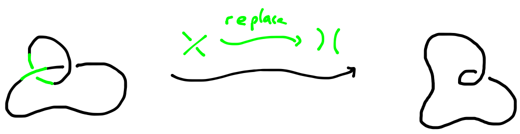
Example 20.7. The link diagram



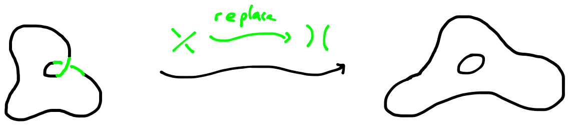
is a state of the trefoil knot. It can be obtained by first replacing a crossing as follows,



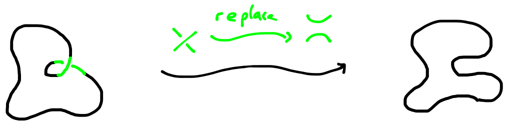
then replacing a crossing as follows,



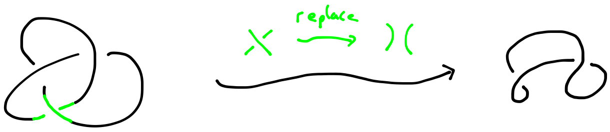
and finally replacing a crossing as follows.



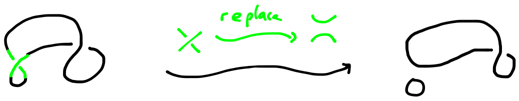
There are several other states. We shall see them all shortly. For example, at the third step we could instead proceed as follows, obtaining a different state.



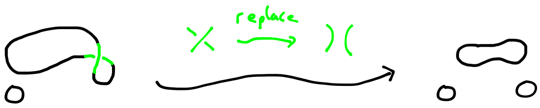
At the first step we could instead proceed as follows.



Next we could proceed as follows.



Finally we could proceed as follows, obtaining a different state to the two we have already encountered.



Definition 20.8. Let $(L, \mathcal{O}_L)$ be a link, and let $\mathcal{D}$ be its corresponding link diagram. We inductively associate to a state S of L an expression $\ll S \gg = A^i B^j$ in two variables A and B as follows.

(1) Begin by defining $\ll S \gg = 1$.

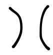
(2) If we replaced a crossing

by

when obtaining S from $\mathcal{D}$, we multiply $\ll S \gg$ by A.

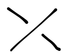
(3) If we replaced a crossing

by

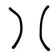


when obtaining S from $\mathcal{D}$, we multiply $\ll S \gg$ by B.

(4) If we replaced a crossing

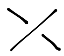


by



when obtaining S from $\mathcal{D}$, we multiply $\ll S \gg$ by A.

(5) If we replaced a crossing

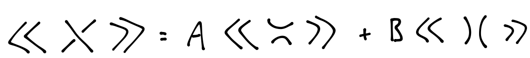


by

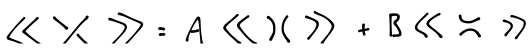
when obtaining S from $\mathcal{D}$, we multiply $\ll S \gg$ by B.

Remark 20.9. Since there are no orientations involved in Definition 20.8, rules (4) and (5) can be viewed as obtained from rules (2) and (3) by rotating, and vice versa.

Remark 20.10. To express rules (2) and (3) concisely we write the following.



To express rules (2) and (3) concisely we write the following.



Notation 20.11. Let $(L, \mathcal{O}_L)$ be a link, and let S be a state of L. We denote by $|S|$ the number of components of S.

Definition 20.12. Let $(L, \mathcal{O}_L)$ be a link. The *bracket polynomial* of L is a polynomial in three variables A, B, d given by

$$\sum_{S \text{ a state of } L} \ll S \gg d^{|S|-1}.$$

Notation 20.13. Let $(L, \mathcal{O}_L)$ be a link. We denote the bracket polynomial of L by $\ll L \gg$.

Examples 20.14.

(1) The unknot has only one state S, namely itself. Since we did not replace any crossings to obtain it, we have that $\ll S \gg = 1$. Thus the bracket polynomial of the unknot is as follows.

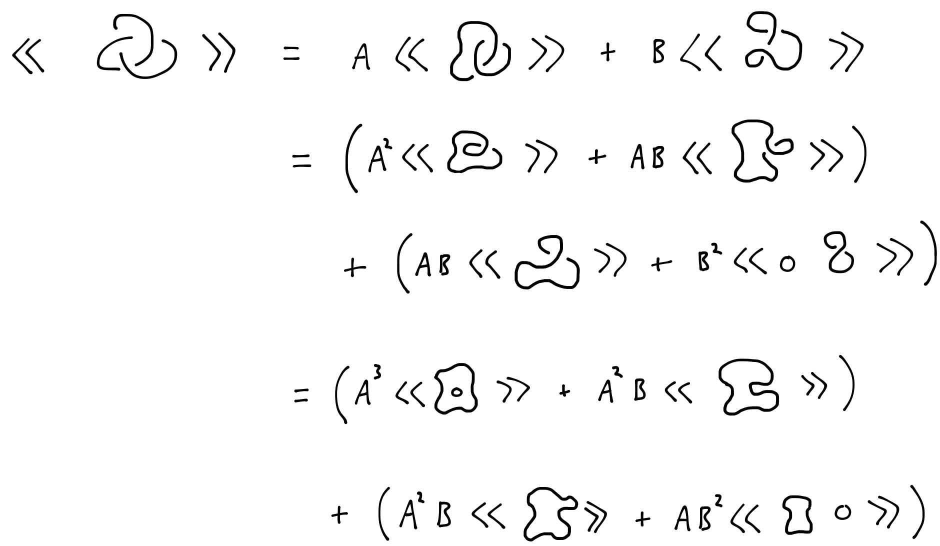
$$\ll \bigcirc \gg = 1 \cdot d^{1-1} = 1 \cdot d^0 = 1.$$

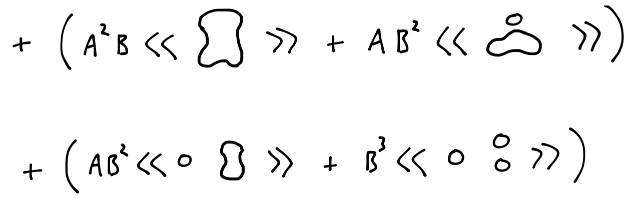
(2) The unlink with two components also has only one state S, namely itself. Again, since we did not replace any crossings to obtain it, we have that $\ll S \gg = 1$. Thus the bracket polynomial of the unlink with two components is as follows.

$$\ll \circ \ \circ \gg = 1 \cdot d^{2-1} = 1 \cdot d^1 = d$$

Similarly the bracket polynomial of the unlink with n components will be d^{n-1}.

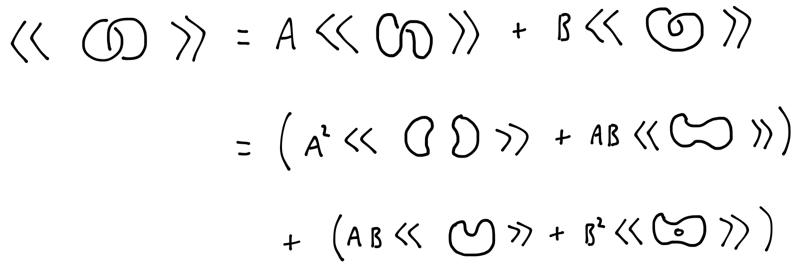
(3) Let us now calculate the bracket polynomial of the trefoil.





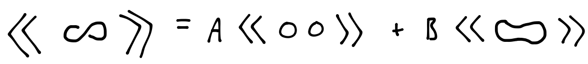
$$= A^3 d + A^2 B$$
$$+ A^2 B + AB^2 d$$
$$+ A^2 B + AB^2 d$$
$$+ AB^2 d + B^3 d^2$$
$$= A^3 d + 3A^2 B + 3AB^2 d + B^3 d^2$$

(4) Let us also calculate the bracket polynomial of the Hopf link.



$$= A^2 d + AB$$
$$+ AB + B^2 d$$
$$= A^2 d + 2AB + B^2 d$$

(5) Let us also calculate the bracket polynomial of the following knot.



$$= Ad + B$$

Definition 20.15. Let $(L, \mathcal{O}_L)$ be a link. We denote by $\langle L \rangle$ the polynomial obtained from $\ll L \gg$ by replacing d by $-A^2 - A^{-2}$ and replacing B by A^{-1}.

Remark 20.16. The reason for making this definition will become clear during the proof of Proposition 20.20.

Terminology 20.17. A polynomial such as $\langle L \rangle$ in which we have positive and negative powers of a single variable — in this case the variable A — is sometimes known as a *Laurent polynomial*.

Examples 20.18.

(1) Let O denote the unknot. By Examples 20.14 (1) we have that $\langle O \rangle = 1$.

(2) Let O_n denote the unlink with n components. By Examples 20.14 (2) we have that $\langle O_n \rangle = (-A^2 - A^{-2})^{n-1}$.

(3) Let 3_1 denote the trefoil. By Examples 20.14 (3) we have that

$$\begin{aligned}
\langle 3_1 \rangle &= A^3(-A^2 - A^{-2}) + 3A^2A^{-1} + 3AA^{-2}(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})^2 \\
&= -A^5 - A + 3A + 3A^{-1}(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})(-A^2 - A^{-2}) \\
&= -A^5 + 2A - 3A - 3A^{-3} + A^{-3}(A^4 + 2 + A^{-4}) \\
&= -A^5 - A - 3A^{-3} + A + 2A^{-3} + A^{-7} \\
&= -A^5 - A^{-3} + A^{-7}.
\end{aligned}$$

(4) Let 2_1^2 denote the Hopf link. By Examples 20.14 (4) we have that

$$\begin{aligned}
\langle 2_1^2 \rangle &= A^2(-A^2 - A^{-2}) + 2AA^{-1} + A^{-2}(-A^2 - A^{-2}) \\
&= -A^4 - 1 + 2 - 1 - A^{-4} \\
&= -A^4 - A^{-4}.
\end{aligned}$$

(5) Let K denote the knot of Examples 20.14 (5). We have that

$$\begin{aligned}
\langle K \rangle &= A(-A^2 - A^{-2}) + A^{-1} \\
&= -A^3 - A^{-1} + A^{-1} \\
&= -A^3.
\end{aligned}$$

Notation 20.19. Let $(L, \mathcal{O}_L)$ be a link. When working with $\ll L \gg$ or $\langle L \rangle$ we frequently depict only part of L inside the brackets. For example, in the proof of Proposition 20.20 we write the following.

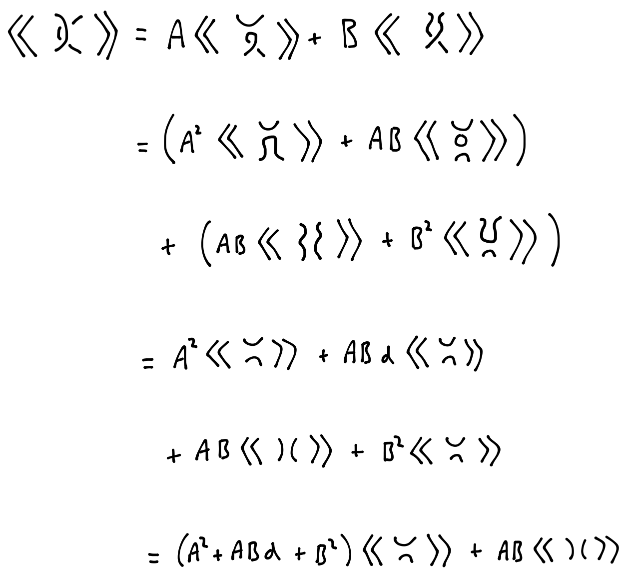
We do not typically mean to refer to the link

itself, but rather to a larger link, one part of which looks like this.

Proposition 20.20. Let $(L, \mathcal{O}_L)$ be a link. Then $\langle L \rangle$ is unchanged under the Reidemeister moves R2 and R3.

Proof. The proof for R3 relies upon the proof for R2.

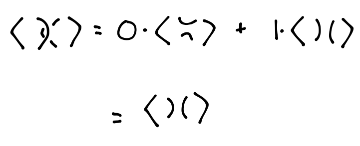
R2 We make the following calculation.



Letting $d = -A^2 - A^{-2}$ and $B = A^{-1}$ we have that

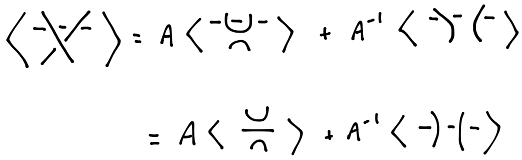
$$\begin{aligned} A^2 + ABd + B^2 &= A^2 + AA^{-1}(-A^2 - A^{-2}) + A^{-2} \\ &= A^2 - A^2 - A^{-2} + A^{-2} \\ &= 0. \end{aligned}$$

Moreover we have that $AB = AA^{-1} = 1$. We conclude that



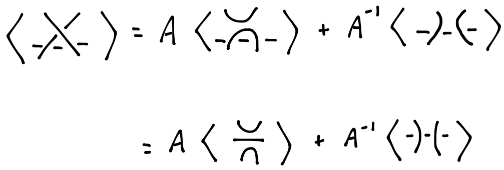
as required.

R3 We make the following calculation.



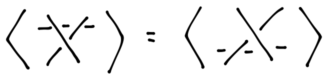
For the second equality we appeal to the fact that $\langle L\rangle$ is unchanged by an $\mathsf{R2}$ move.

We also make the following calculation.



Again for the second equality we appeal to the fact that $\langle L\rangle$ is unchanged by an $\mathsf{R2}$ move.

We conclude that



as required.

$\square$

Remark 20.21. This proof is not quite complete. There is another $\mathsf{R2}$ and another $\mathsf{R3}$ move which must be considered. However, just as in the proof of Proposition 18.14 and the proof of Proposition 19.5 it is the idea that is important. It adapts in a straightforward way to a proof for the other cases.

Remark 20.22. Let $(L, \mathcal{O}_L)$ be a link. By Examples 20.18 (1) and (5) we see that $\langle L\rangle$ is not necessarily unchanged under an $\mathsf{R1}$ move, since the knot in Examples 20.18 (5) is isotopic to the unknot.

In the next lecture we will see that we can repair this. We shall introduce a tool — the writhe of L — which roughly speaking counts twists

in L, and modify the definition of $\langle L\rangle$ to take it into account.

21 Thursday 4th April

21.1 Writhe

Definition 21.1. Let $(L, \mathcal{O}_L)$ be an oriented link. The *writhe* of L is

$$\sum_{\text{crossings } C \text{ of } L} \text{sign}(C).$$

We denote it by $w(L)$.

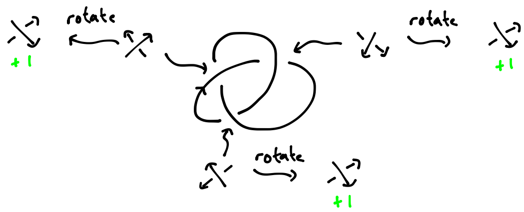
If there are no crossings, we adopt the convention that $w(L) = 0$.

Remark 21.2. Let $(L, \mathcal{O}_L)$ be an oriented link. The most important difference between the writhe of L and the linking number of L, which was introduced in Definition 18.10, is that in the definition of a linking number we consider only crossings between distinct components. Here we consider all crossings.

Examples 21.3.

(1) The writhe of the unknot is 0, since the unknot has no crossings. Similarly the writhe of the unlink with n components is 0.

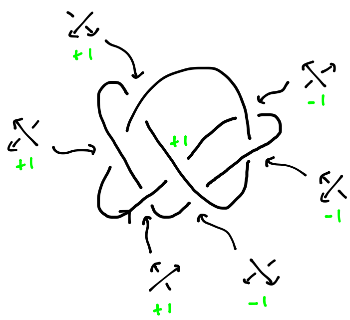
(2) The signs of the crossings of the oriented trefoil below are as shown.



Thus its writhe is $1 + 1 + 1 = 3$.

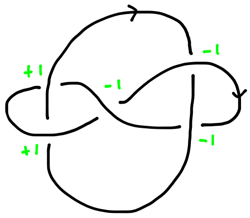
Reversing the orientation changes the sign of all three crossings, giving writhe of -3.

(3) The signs of the crossings of the oriented knot below are as shown.



Thus its writhe is $4 - 3 = 1$.

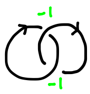
(4) The signs of the crossings of the oriented Whitehead link below are shown below.



Thus its writhe is $2 - 3 = -1$.

Compare this with Examples 18.12 (4). This time we do count the sign of the middle crossing!

(5) The signs of the crossings of the oriented Hopf link below are as shown.



Thus its writhe is -2.

21.2 Jones polynomial

Remark 21.4. The Jones polynomial was discovered in the 1980s — it is probably the most recent mathematics that you will come across in your undergraduate studies! It is of deep significance.

Definition 21.5. Let $(L, \mathcal{O}_L)$ be an oriented link. The *Jones polynomial* of L is

$$(-A)^{-3w(L)}\langle L\rangle.$$

We denote it by $V_L(A)$.

Examples 21.6.

(1) Let O be the unknot. By Examples 20.18 (1) we have that $\langle O\rangle = 1$. By Examples 21.3 (1) we have that $w(O) = 0$. Thus we have that $V_O(A) = (-A)^0 = 1$.

(2) Let O_n be the unlink with n components.. By Examples 20.18 (2) we have that $\langle O_n\rangle = (-A^{-2} - A^2)^{n-1}$. By Examples 21.3 (1) we have that $w(O_n) = 0$.

Thus we have that

$$\begin{aligned} V_{O_n}(A) &= (-A)^0(-A^{-2} - A^2)^{n-1} \\ &= (-A^{-2} - A^2)^{n-1}. \end{aligned}$$

(3) Let 3_1 be a trefoil with the orientation indicated below.

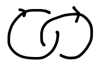
By Examples 20.18 (3) we have that $\langle 3_1 \rangle = A^{-7} - A^{-3} - A^5$. By Examples 21.3 (2) we have that the writhe of 3_1 with this orientation is 3.

Thus we have that

$$\begin{aligned} V_{3_1}(A) &= (-A)^{-9}(A^{-7} - A^{-3} - A^5) \\ &= -A^{-16} + A^{-12} + A^{-4}. \end{aligned}$$

We will later prove that the Jones polynomial of a knot does not depend on the choice of orientation.

(4) Let 2_1^2 be a Hopf link with the orientation indicated below.



By Examples 20.18 (4) we have that $\langle 2_1^2 \rangle = -A^{-4} - A^4$. By Examples 21.3 we have that the writhe of 2_1^2 with this orientation is -2.

Thus we have that

$$V_{2_1^2}(A) = (-A)^6(-A^{-4} - A^4) = -A^2 - A^{10}.$$

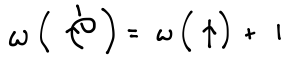
Proposition 21.7. Let $(L, \mathcal{O}_L)$ and $(L', \mathcal{O}_{L'})$ be oriented links. If L is isotopic to L' then $V_L(A) = V_{L'}(A)$.

Proof. We know by Theorem 17.16 that two links are isotopic if and only if one can be obtained from the other by a finite sequence of Reidemeister moves. Thus it suffices to prove that the Jones polynomial of an oriented link $(L, \mathcal{O}_L)$ is not changed by applying the Reidemeister moves.

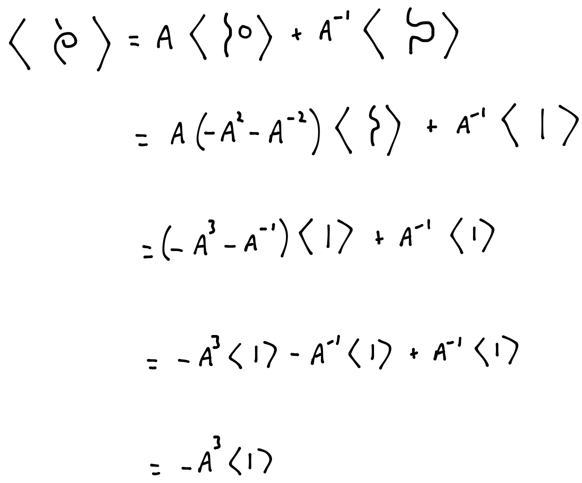
By Proposition 20.20 we have that $\langle L \rangle$ is not changed by an R2 move or an R3 move. Moreover the same argument as was given in the proof of Proposition 18.14 demonstrates that $w(L)$ is not changed by an R2 move or an R3 move. We deduce that $V_L(A)$ is not changed by an R2 move or an R3 move.

It remains to prove that $V_L(A)$ is not changed by an R1 move. We will adopt Notation 20.19 when working with the writhe of L in the following proof as well as when working with $\langle L \rangle$.

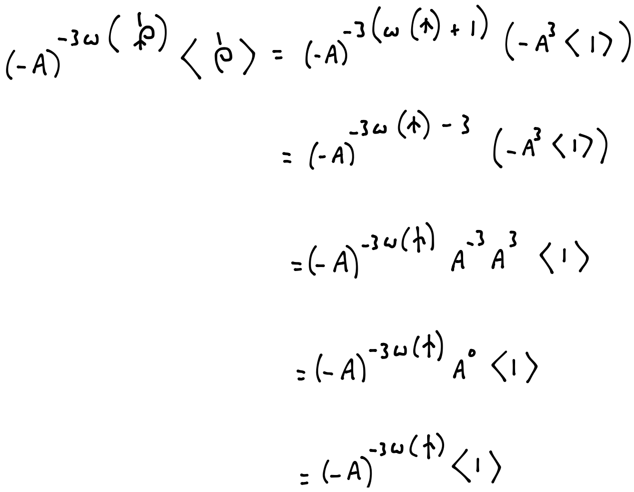
We begin by making the following observation.



Next we make the following calculation.



We deduce that the following holds, as required.



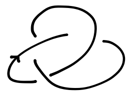
□

Examples 21.8.

(1) By Examples 21.6 (2) and (4) we have that the Jones polynomial of the oriented Hopf link

is different to the Jones polynomial of the unlink with two components. By Proposition 21.7 we conclude that the Hopf link is not isotopic to the unlink with two components, which we observed via linking numbers in Examples 18.16.

(2) By Examples 21.6 (1) and (3) we have that the Jones polynomial of the trefoil



is different to the Jones polynomial of the unknot. By Proposition 21.7 we conclude that the trefoil is not isotopic to the unknot, which we observed via 3-colourability in Examples 19.9.

Definition 21.9. Let $(L, \mathcal{O}_L)$ be an oriented link. We denote by $V_L(t)$ the polynomial obtained from $V_L(A)$ by replacing A by $t^{-\frac{1}{4}}$. We refer to $V_L(t)$ also as the Jones polynomial of L.

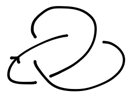
Remark 21.10. Passing from $V_L(A)$ to $V_L(t)$ has no deep meaning! The Jones polynomial was originally constructed in the form $V_L(t)$ by different methods.

Examples 21.11.

(1) Let O be the unknot. By Examples 21.6 (1) we have that $V_O(A) = 1$. Thus we have that $V_O(t) = 1$.

(2) Let O_n be the unlink with n components. By Examples 21.6 (2) we have that $V_{O_n}(A) = (-A^{-2} - A^2)^{n-1}$. Thus we have that $V_{O_n}(t) = (-A^{-\frac{1}{2}} - A^{\frac{1}{2}})^{n-1}$.

(3) Let 3_1 be a trefoil with the orientation indicated below.



By Examples 21.6 (1) we have that $V_{3_1}(A) = -A^{-16} + A^{-12} + A^{-4}$. Thus we have that $V_{3_1}(t) = t + t^3 - t^4$.

(4) Let 2_1^2 be a Hopf link with the orientation indicated below.

By Examples 21.6 (2) we have that $V_{2_1^2}(A) = -A^2 - A^{10}$. Thus we have that $V_{2_1^2}(t) = -t^{-\frac{5}{2}} - t^{-\frac{1}{2}}$.

21.3 Skein relations

Definition 21.12. The *skein relations* for the Jones polynomial are as follows.

1) $V_{O}(t) = 1$

unknot

2) $t^{-1} V(t) - t V(t) = (t^{1/2} - t^{-1/2}) V(t)$

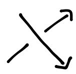
Remark 21.13. By Examples 21.11 we have that the Jones polynomial satisfies 1). We will prove later that it also satisfies 2). Before we do so we shall explore the meaning of 2) by using the skein relations to calculate inductively the Jones polynomials of a few links.

Examples 21.14.

(1) Let us calculate the Jones polynomial of the unlink with two components via the skein relations.

We begin with the following oriented knot.

We choose a crossing. Here there is only one possibility! The way in which the skein relations work is that since the crossing is of the form



we take

$$V(t)$$

to be the Jones polynomial of the knot with which we began.

$$V(t) = V(t)$$

$$= V(t)$$

$$= V_{O}(t)$$

$$= 1$$

We also make the following calculation.

$$V(t) = V(t)$$

$$= V(t)$$

$$= V_{O}(t)$$

Thus by the second skein relation we have the following.

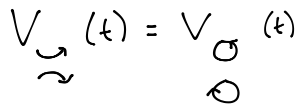
$$t^{-1} \cdot 1 - t \cdot 1 = (t^{1/2} - t^{-1/2}) \, V(t)$$

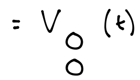
$$\Rightarrow V(t) = \frac{t^{-1} - t}{t^{1/2} - t^{-1/2}}$$

$$= \frac{(t^{1/2} - t^{-1/2})(-t^{1/2} - t^{-1/2})}{t^{1/2} - t^{-1/2}}$$

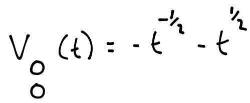
$$= -t^{1/2} - t^{-1/2}$$

In addition we have the following.





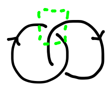
Putting everything together we have the following.



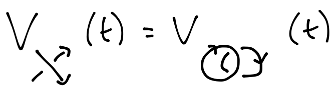
This agrees with our calculation in Examples 21.11 (2).

(2) Let us now use the skein relations to calculate the Jones polynomial of an oriented Hopf link.

Let us work at the indicated crossing.



We have the following, appealing to our calculation in (1).



$$= -t^{-1/2} - t^{1/2}$$

We also have the following.

V  $(t) = V$  (t)

$$= V_{O}(t)$$

$$= 1$$

Thus by the second skein relation we have the following.

$t^{-1}\left(-t^{-1/2} - t^{1/2}\right) - t\,V$  $(t) = \left(t^{1/2} - t^{-1/2}\right) \cdot 1$

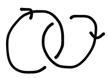
$\Rightarrow\ -t^{-3/2} - t^{-1/2} - t\,V$  $(t) = t^{1/2} - t^{-1/2}$

$\Rightarrow\ -t\,V$  $(t) = t^{-3/2} + t^{1/2}$

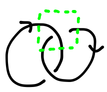
$\Rightarrow\ V$  $(t) = -t^{-5/2} - t^{-1/2}$

This agrees with our calculation in Examples 21.11 (4).

(3) Let us also use the skein relations to calculate the Jones polynomial of the mirror image of the oriented Hopf link of (2).



Let us work at the indicated crossing.



We have the following, appealing to our calculation in (1).

$$V(t) = V(t)$$
$$= V_{O\,O}(t)$$
$$= -t^{-1/2} - t^{1/2}$$

We also have the following.

$$V(t) = V(t)$$
$$= V_{O}(t)$$
$$= 1$$

Thus by the second skein relation we have the following.

$$t^{-1}V(t) - t\left(-t^{-1/2} - t^{1/2}\right) = \left(t^{1/2} - t^{-1/2}\right)\cdot 1$$

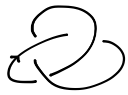
$$\Rightarrow \quad t^{-1}V(t) + t^{1/2} + t^{3/2} = t^{1/2} - t^{-1/2}$$

$$\Rightarrow \quad t^{-1}V(t) = -t^{3/2} - t^{-1/2}$$

$$\Rightarrow \quad V(t) = -t^{5/2} - t^{1/2}$$

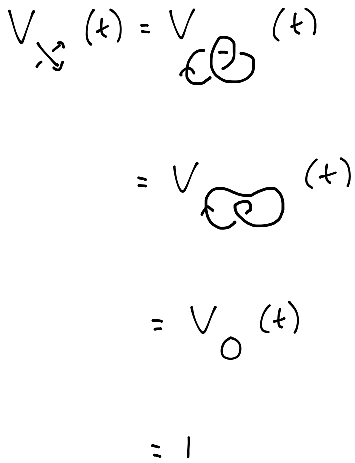
In particular this polynomial is not equal to the Jones polynomial we calculated in (2). We conclude by Proposition 21.7 that the oriented Hopf link of (2) is not isotopic to its mirror image.

(4) Let us now use the skein relations to calculate the Jones polynomial of an oriented trefoil.

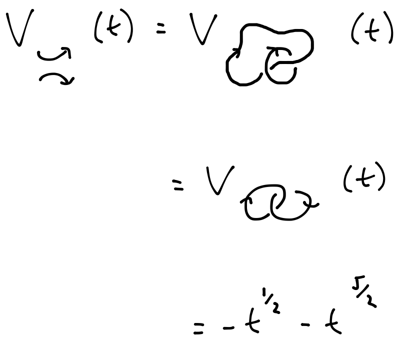


Let us work at the indicated crossing.

We have the following.



We also have the following, appealing to our calculation in (3).



Thus by the second skein relation we have the following.

$$t^{-1} V(t) - t \cdot 1 = (t^{1/2} - t^{-1/2})(-t^{1/2} - t^{5/2})$$

$$\Rightarrow t^{-1} V(t) - t = -t + 1 - t^3 + t^2$$

$$\Rightarrow t^{-1} V(t) = 1 + t^2 - t^3$$

$$\Rightarrow V(t) = t + t^3 - t^4$$

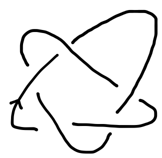
This agrees with our calculation in Examples 21.11 (3).

22 Tuesday 9th April

22.1 Skein relations, continued

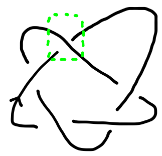
Examples 22.1.

(4) Let us use the skein relations to calculate the Jones polynomial of the knot 5_1, known as the cinquefoil.

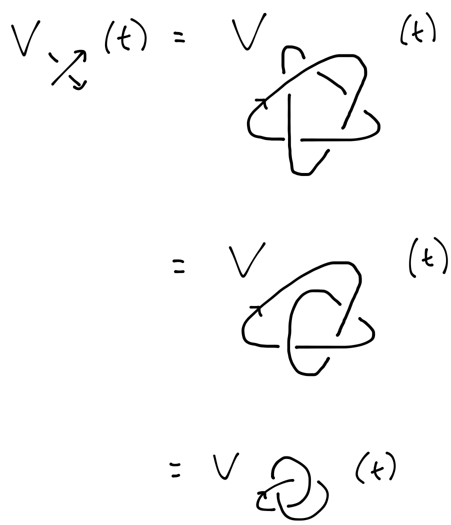


The argument illustrates the inductive nature of a calculation using the skein relations: we apply the skein relations until we encounter an oriented link whose Jones polynomial we already know.

Let us work at the indicated crossing.

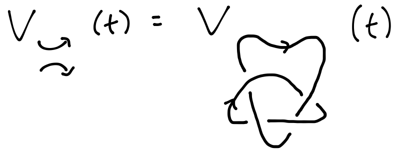


We have the following, appealing to Examples 21.14 (4) for the last equality.

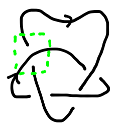


$$= t + t^3 - t^4$$

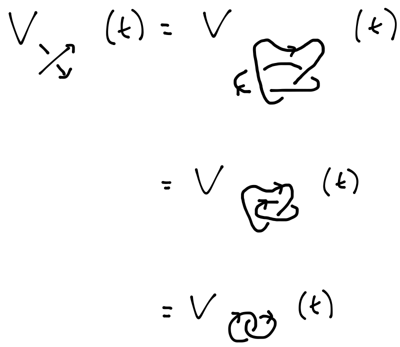
We also have the following.



We have not yet calculated this Jones polynomial. Thus we must interrupt our main calculation in order to use the skein relations to work it out. Let us work at the indicated crossing.

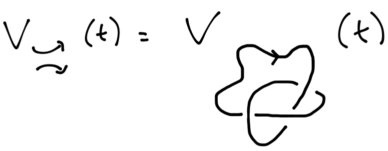


We have the following, appealing to Examples 21.14 (3) for the last equality.



$$= -t^{5/2} - t^{1/2}$$

We also have the following, appealing to Examples 21.14 (4) for the last equality.



$=$ V  (t)

$$= t + t^{3} - t^{4}$$

Thus by the second skein relation we have the following.

$t^{-1} V$  $(t) - t\left(-t^{5/2} - t^{1/2}\right) = \left(t^{1/2} - t^{-1/2}\right)\left(t + t^{3} - t^{4}\right)$

$\Rightarrow t^{-1} V$  $(t) + t^{7/2} + t^{3/2} = t^{3/2} + t^{7/2} - t^{9/2} - t^{1/2} - t^{5/2} + t^{7/2}$

$\Rightarrow t^{-1} V$  $(t) = -t^{1/2} - t^{5/2} + t^{7/2} - t^{9/2}$

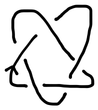
$\Rightarrow V$  $(t) = -t^{3/2} - t^{7/2} + t^{9/2} - t^{11/2}$

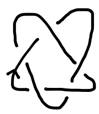
We can now complete our main calculation. We have the following.

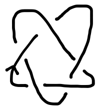
V  $(t) = V$  (t)

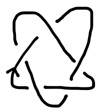
$$= -t^{3/2} - t^{7/2} + t^{9/2} - t^{11/2}$$

Thus by the second skein relation we have the following.

$t^{-1} V$  $(t) - t\,(t + t^3 - t^4) = (t^{1/2} - t^{-1/2})\,(-t^{3/2} - t^{7/2} + t^{9/2} - t^{11/2})$

$\Rightarrow t^{-1} V$  $(t) - t^2 - t^4 + t^5 = -t^2 - t^4 + t^5 - t^6 + t + t^3 - t^4 + t^5$

$\Rightarrow t^{-1} V$  $(t) = t + t^3 - t^4 + t^5 - t^6$

$\Rightarrow V$  $(t) = t^2 + t^4 - t^5 + t^6 - t^7$

Proposition 22.2. The Jones polynomial for oriented links satisfies the second skein relation of Definition 21.12.

Proof. Throughout this proof we will adopt Notation 20.19. Let us first make several observations.

(1) By definition of $\langle L \rangle$ for a link $(L, \mathcal{O}_L)$ we have the following.

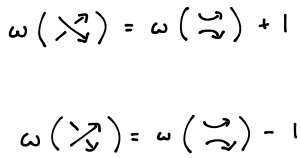
$$A \langle \times \rangle - A^{-1} \langle \times \rangle = A\left(A \langle \asymp \rangle + A^{-1} \langle)(\rangle\right)$$

$$- A^{-1}\left(A \langle)(\rangle + A^{-1} \langle \asymp \rangle\right)$$

$$= A^2 \langle \asymp \rangle + \langle)(\rangle - \langle)(\rangle - A^{-2} \langle \asymp \rangle$$

$$= (A^2 - A^{-2}) \langle \asymp \rangle$$

(2) We have the following.



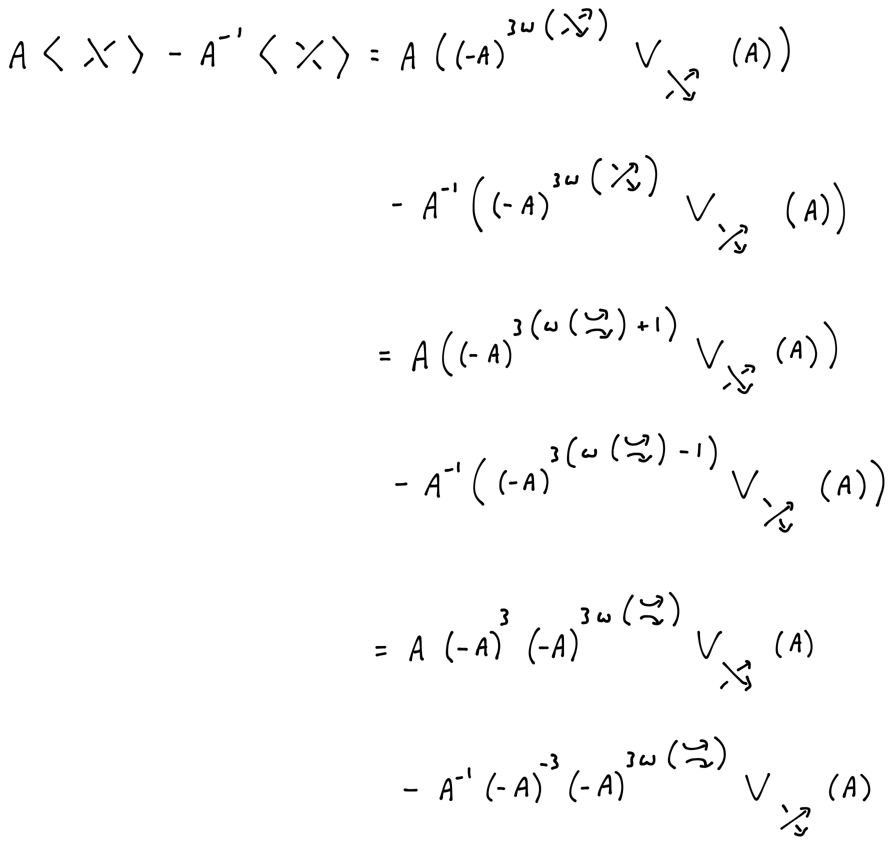
(3) By definition of $V_L(A)$ for an oriented link $(L, \mathcal{O}_L)$ we have that

$$V_L(A) = (-A)^{-3w(L)} \langle L \rangle.$$

Thus we have that

$$\langle L \rangle = (-A)^{3w(L)} V_L(A).$$

We now make the following calculation, appealing to (3) for the first equality and to (2) for the second equality.



$$= -A^{4}\,(-A)^{3w(\;)}\;V(A)$$

$$+\,A^{-4}\,(-A)^{3w(\;)}\;V(A)$$

$$= (-A)^{3w(\;)}\left(-A^{4}V(A) + A^{-4}V(A)\right)$$

We deduce that the following holds, appealing to (1) for the first equality and to (3) for the second equality.

$$(-A)^{3w(\;)}\left(-A^{4}V(A) + A^{-4}V(A)\right) = (A^{2}-A^{-2})\langle\;\rangle$$

$$= (A^{2}-A^{-2})\left((-A)^{3w(\;)}V(A)\right)$$

Thus we have the following.

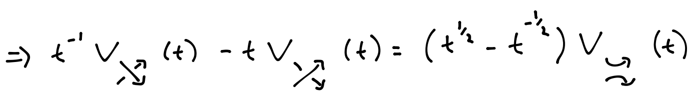
$$(-A)^{3w(\;)}\left(-A^{4}V(A) + A^{-4}V(A)\right) = (-A)^{3w(\;)}(A^{2}-A^{-2})\,V(A)$$

$$\Rightarrow\; -A^{4}V(A) + A^{-4}V(A) = (A^{2}-A^{-2})\,V(A)$$

Thus we have the following, as required.

$$-\left(t^{-1/4}\right)^{4}V(t) + \left(t^{-1/4}\right)^{-4}V(t) = \left(\left(t^{-1/4}\right)^{2} - \left(t^{-1/4}\right)^{-2}\right)V(t)$$

$$\Rightarrow\; -t^{-1}V(t) + t\,V(t) = \left(t^{-1/2} - t^{1/2}\right)V(t)$$



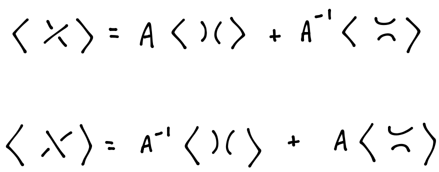
$\square$

22.2 Jones polynomial of a mirror image

Proposition 22.3. Let $(L, \mathcal{O}_L)$ be a link, and let $(L_m, \mathcal{O}_{L_m})$ be its mirror image. Then

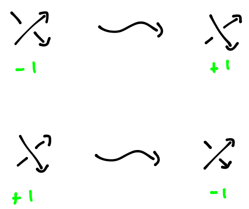
$$V_L(t) = V_{L_m}(t^{-1}).$$

Proof. By definition of $\langle L' \rangle$ we have the following for any link $(L', \mathcal{O}_{L'})$.



Thus $\langle L_m \rangle$ is obtained from $\langle L \rangle$ by replacing A by A^{-1}.

Let us now equip L with a choice of orientation. We have that $w(L) = -w(L_m)$, since passing from L to L_m has the following effect on the sign of a crossing.



Thus we have that

$$(-A)^{-3w(L)} \langle L \rangle = (-A)^{3w(L_m)} = \big(-(A^{-1})\big)^{-3w(L_m)}.$$

Together with the fact that $\langle L_m \rangle$ is obtained from $\langle L \rangle$ by replacing A by A^{-1}, this implies that $V_L(A) = V_{L_m}(A^{-1})$. We conclude that $V_L(t) = V_{L_m}(t^{-1})$.

$\square$

Terminology 22.4. A Laurent polynomial is *palindromic* if it is not changed by replacing t by t^{-1}.

Corollary 22.5. Let $(L, \mathcal{O}_L)$ be a link, and let $(L_m, \mathcal{O}_{L_m})$ be its mirror image. If L is isotopic to L' then $V_L(t)$ is palindromic.

Proof. Follows immediately from Proposition 22.3. $\square$

Examples 22.6.

(1) By Examples 21.14 (2) and Corollary 22.5 the Hopf link is not isotopic to its mirror image, since its Jones polynomial with respect to the orientation we chose is $-t^{-\frac{5}{2}} - t^{-\frac{1}{2}}$. This is not palindromic!

We saw by hand in Examples 21.14 (3) that the Jones polynomial of the mirror image of the Hopf link in Examples 21.14 (2) is $-t^{\frac{1}{2}} - t^{\frac{5}{2}}$, as we now know must be true by Proposition 22.3.

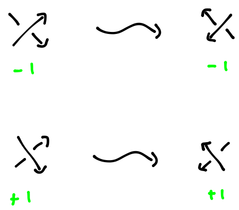
(2) By Examples 21.14 (4) and Corollary 22.5 the trefoil is not isotopic to its mirror image, since its Jones polynomial is $t + t^3 - t^4$. This is not palindromic!

(3) We know by Example 18.2 that the figure of eight knot is isotopic to its mirror image. Its Jones polynomial can be calculated to be $t^{-2} - t^{-1} + 1 - t + t^2$. This is palindromic, as we know by Corollary 22.5 that it must be!

Proposition 22.7. Let $(K, \mathcal{O}_K)$ be a knot. Then $V_K(t)$ does not depend on the choice of orientation of K.

Proof. The definition of $\langle K \rangle$ does not involve an orientation of K, and therefore does not depend on any choice of orientation.

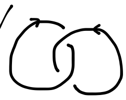
Since a knot has only one component, the only way to change orientation is to reverse it. This has the following effect on the sign of a crossing.



Thus the writhe of K is not changed by reversing its orientation. We deduce that $V_K(A)$ is not changed by reversing the orientation of K. We conclude that $V_K(t)$ is not changed by reversing the orientation of K. $\square$

Remark 22.8. Thus we can speak of the Jones polynomial of a knot, rather than only of an oriented knot. In other words, when we calculate the Jones polynomial we are to choose any orientation we wish.

Remark 22.9. Proposition 22.7 does not necessarily hold for a link with more than one component. For example one can prove the following.

V  $(t) = -t^{5/2} - t^{1/2}$

This is not the same as the Jones polynomial which we calculated in Examples 21.14 (2), where we were working with a different choice of orientation.

23 Lectures 23–27

23.1 Δ-complexes

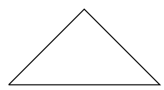
Remark 23.1. In the remaining lectures we'll introduce ideas around the classification of surfaces. We'll focus on the essence of this beautiful story, and not be completely precise. Rest assured that everything can be made entirely rigorous!

Terminology 23.2. A 0-*simplex* is a point in $\mathbb{R}^2$. A 1-*simplex* is a closed line segment in $\mathbb{R}^2$. A 2-*simplex* is a closed filled in triangle in $\mathbb{R}^2$.

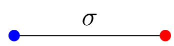
By a *simplex* we shall mean a 0-simplex, a 1-simplex, or a 2-simplex.

Remark 23.3. We will often regard a simplex as equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$, but will omit to mention this from now on.

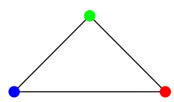
Remark 23.4. From now on all triangles in our pictures are to be regarded as filled in, or in other words as 2-simplices. For example, the following is to be regarded as a picture of a 2-simplex.



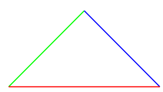
Terminology 23.5. A *vertex* of a 1-simplex is one of the following two 0-simplices.



A *vertex* of a 2-simplex is one of the following three 0-simplices.

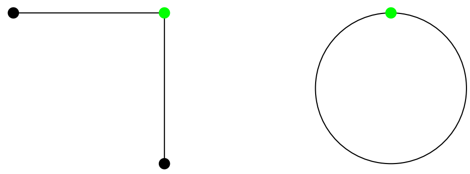


An *edge* of a 2-simplex is one of the following three 1-simplices.

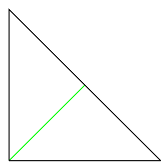


Definition 23.6. A Δ-*complex structure* on a topological space $(X, \mathcal{O}_X)$ is a recipe for constructing $(X, \mathcal{O}_X)$ up to homeomorphism by glueing together simplices in the following ways.

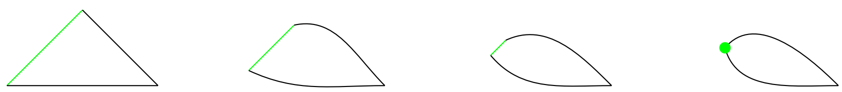
(1) A vertex of a 1-simplex may be glued to a vertex of a 1-simplex. These two 1-simplices may be the same or different.



(2) An edge of a 2-simplex may be glued to an edge of a 2-simplex. These two 2-simplices may be the same or different. We may glue with or without a twist.

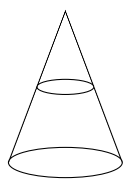


(3) An edge of a 2-simplex may be glued to a 0-simplex. This means that we identify all points on an edge of a 2-simplex together, and can be thought of as shrinking the edge to a point.

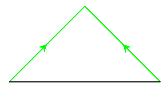


Examples 23.7.

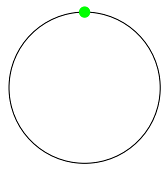
(1) A Δ-complex structure on a hollow cone



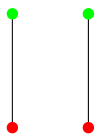
is given by glueing two edges of a single 2-simplex together.



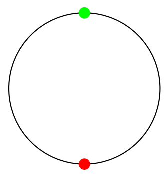
(2) A Δ-complex structure on a circle is given by glueing the two vertices of a single 1-simplex together.



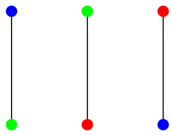
There are many other ways to equip a circle with a Δ-complex structure. For example we can glue two 1-simplices



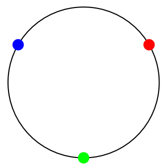
together by identifying the green vertices and identifying the red vertices.



We could glue three 1-simplices

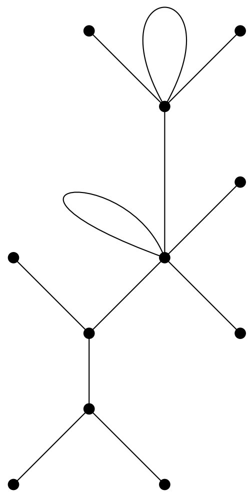


together, identifying each pair of vertices with the same colour.

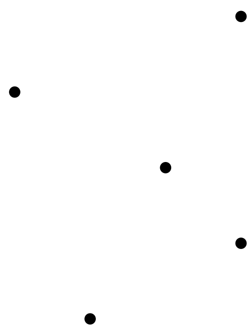


And so on!

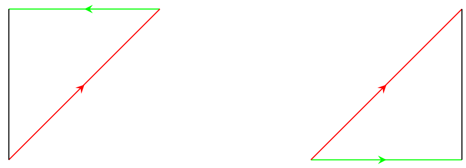
(3) Glueing together vertices of lots of 1-simplices we can equip a tree — possibly with loops — with the structure of a Δ-complex.



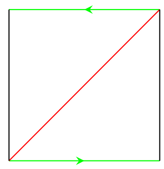
(4) A collection of points has the structure of a Δ-complex.



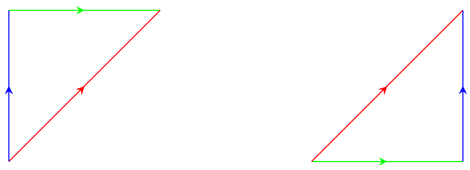
(6) A Δ-complex structure on the Möbius band $(M^2, \mathcal{O}_{M^2})$ is given by glueing together the green edges and glueing together the red edges of two 2-simplices as follows.



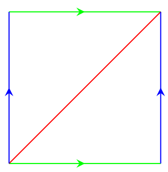
We often depict this in the following manner.



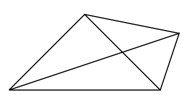
(7) A Δ-complex structure on the torus $(T^2, \mathcal{O}_{T^2})$ is given by glueing together the green edges, glueing together the blue edges, and glueing together the red edges of two 2-simplices as follows.



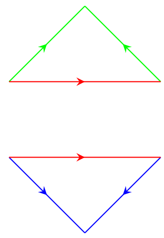
We often depict this in the following manner.



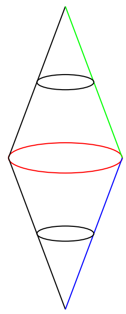
(8) A Δ-complex structure on the 2-sphere $(S^2, \mathcal{O}_{S^2})$ is given by glueing together edges of four 2-simplices as follows.



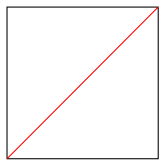
There are many other ways to equip $(S^2, \mathcal{O}_{S^2})$ with a Δ-complex structure. For example, we can glue together edges of two 2-simplices as follows.



This can be thought of as glueing the hollow cone from (1) to an upside hollow cone.

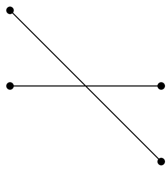


A third Δ-complex structure on $(S^2, \mathcal{O}_{S^2})$ is given by glueing two 2-simplices together to obtain a square



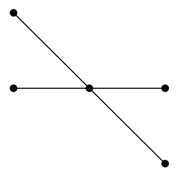
and moreover glueing all four of the remaining edges to a 0-simplex. This is the same idea as in the construction of $(S^2, \mathcal{O}_{S^2})$ in Examples 3.9 (6).

(9) Glueing two 1-simplices as follows does not define a Δ-complex structure.

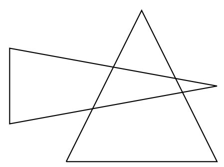


We are only permitted to glue in the three ways prescribed in Definition 23.6. Here we have glued the two 1-simplices in the middle, rather than glueing a vertex to a vertex.

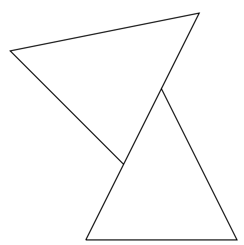
Nevertheless we can certainly equip this topological space with the structure of a Δ-complex by glueing together more 1-simplices.



(9) Glueing two 2-simplices as follows does not define a Δ-complex structure.



Nor does glueing two 2-simplices as follows.



We are only allowed to glue edges to edges.

23.2 Surfaces

Terminology 23.8. Let $(X, \mathcal{O}_X)$ be a topological space. Then $(X, \mathcal{O}_X)$ is *locally homeomorphic to an open disc* if for every $x \in X$ there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ such that U equipped with its subspace topology with respect to $(X, \mathcal{O}_X)$ is homeomorphic to an open disc.

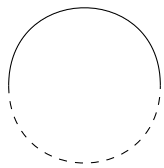
Definition 23.9. A topological space $(X, \mathcal{O}_X)$ is a *surface* if it is compact, connected, Hausdorff and is locally homeomorphic to an open disc.

Remark 23.10. A surface in the sense of Definition 23.9 is also known as a *closed surface.*

Terminology 23.11. We refer to the set

$$\{(x,y) \in \mathbb{R}^2 \mid 0 \leq \|(x,y)\| \leq 1 \text{ and } y \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 \mid 0 \leq \|(x,y)\| < 1 \text{ and } y < 0\}$$

equipped with its subspace topology with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ as a *half open disc.*

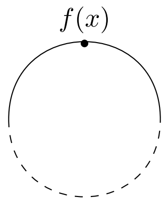


We denote it by $(D^2_{\text{half}}, \mathcal{O}_{D^2_{\text{half}}})$.

Remark 23.12. When deciding whether or not a given topological space $(X, \mathcal{O}_X)$ is a surface, we frequently encounter the situation that for some point x in X there is a neighbourhood U of x in $(X, \mathcal{O}_X)$ such that, letting U be equipped with its subspace topology with respect to $(X, \mathcal{O}_X)$, there is a homeomorphism

$$U \xrightarrow{f} D^2_{\text{half}}$$

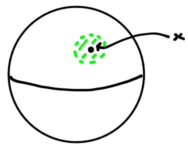
with the property that $f(x)$ belongs to the boundary of D^2_{half} in $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.



This can be shown to imply that there does not exist a neighbourhood of x which is homeomorphic to an open disc. One needs techniques a little more sophisticated than those we have studied to prove this, which you will meet if you take Algebraic Topology I in the autumn. We shall take it on faith.

Examples 23.13.

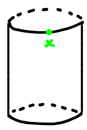
(1) $(S^2, \mathcal{O}_{S^2})$ is a surface. A point x on S^2 and a neighbourhood of x which equipped with its subspace topology is homeomorphic to an open disc is depicted below.



(2) $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$ is not a surface. It is connected, Hausdorff, and locally homeomorphic to an open disc, but is not compact.

(3) The cylinder $(S^1 \times I, \mathcal{O}_{S^1 \times I})$ is not a surface. It is compact, connected, and Hausdorff, but is not locally homeomorphic to an open disc.

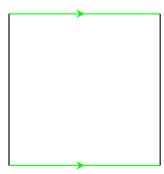
To see this, let x be a point on one of the boundary circles.



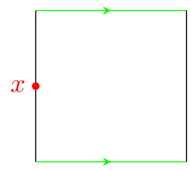
Then x admits a neighbourhood with the property dicussed in Remark 23.12.

We conclude that x does not admit a neighbourhood which is homeomorphic to an open disc.

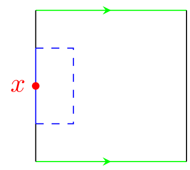
Let us carry out this argument if we instead view the cylinder as the quotient of I^2 by the equivalence relation indicated below.



We let x be a point on one of the black boundary edges.



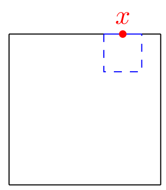
Then x admits a neighbourhood with the property dicussed in Remark 23.12.



We conclude that x does not admit a neighbourhood which is homeomorphic to an open disc.

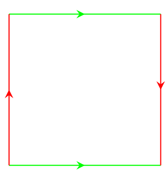
(3) $(I^2, \mathcal{O}_{I^2})$ is not a surface. It is compact, connected, and Hausdorff, but is not locally homeomorphic to an open disc.

Every point on its boundary admits a neighbourhood with the property dicussed in Remark 23.12. Thus it cannot admit a neighbourhood which is homeomorphic to an open disc.

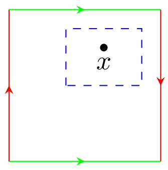


(5) $(T^2, \mathcal{O}_{T^2})$ and $(K^2, \mathcal{O}_{K^2})$ are surfaces. Let us explain why $(K^2, \mathcal{O}_{K^2})$ is locally homeomorphic to an open disc.

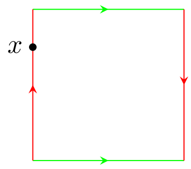
We view $(K^2, \mathcal{O}_{K^2})$ as the quotient of I^2 by the equivalence relation indicated below.



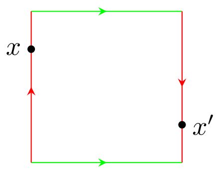
We can clearly find a neighbourhood homeomorphic to an open disc of any $[x] \in K^2$ such that x does not belong to $\partial_{\mathbb{R}^2} I^2$.



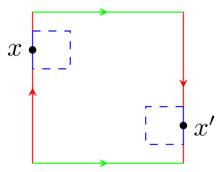
This was also true for the cylinder in (3). The difference with the cylinder is that we can also find a neighbourhood homeomorphic to an open disc of $[x] \in K^2$ for any $x \in \partial_{\mathbb{R}^2} I^2$.



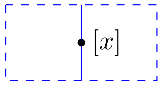
Let us explore this. For such an x there is a point x' on the opposite edge such that $[x'] = [x]$.



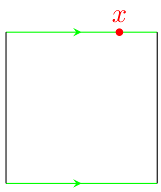
We can take a neighbourhood of each point in I^2 as indicated below.



Each neighbourhood is homeomorphic to a half open disc in $(I^2, \mathcal{O}_{I^2})$, but in $(K^2, \mathcal{O}_{K^2})$ they become glued together to give a neighbourhood of $[x] = [x']$ which is homeomorphic to an open disc.



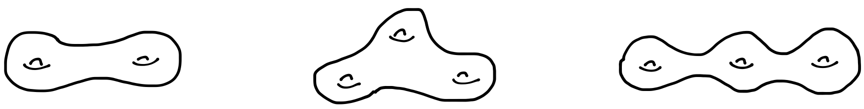
A similar argument proves that $(T^2, \mathcal{O}_{T^2})$ is locally homeomorphic to an open disc. Moreover, let us view the cylinder as a quotient $(I^2/\sim, \mathcal{O}_{I^2/\sim})$ of I^2 as in (3). A similar argument proves that $[x] \in I^2/\sim$ has a neighbourhood which is homeomorphic to an open disc for every point x belonging to a green edge.



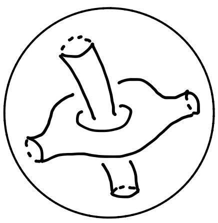
It is exactly the points on the black edges that do not admit a neighbourhood which is homeomorphic to an open disc.

(6) Here are a few, more exotic, examples of surfaces!

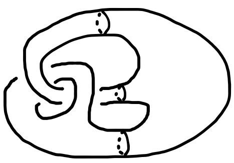
Gadgets similar to $(T^2, \mathcal{O}_{T^2})$ except with two or more holes.



A sphere with two intertwining tunnels.



A kind of knotted torus-like gadget.



23.3 Euler characteristic

Definition 23.14. Let $(X, \mathcal{O}_X)$ be a topological space equipped with a Δ-complex structure. For $0 \leq i \leq 2$, let m_i denote the number of i-simplices involved in this Δ-complex structure, counting only once any simplices which are to be glued together.

The *Euler characteristic* of X with respect to this Δ-complex structure is

$$\sum_{0 \leq i \leq 2} (-1)^i m_i.$$

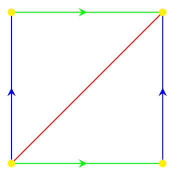
Remark 23.15. Miraculously, the Euler characteristic of $(X, \mathcal{O}_X)$ does not depend on the choice of Δ-complex structure — this is one of my favourite observations in mathematics!

It is of profound mathematical significance — the quest for a rigorous proof mirrored the historical evolution of algebraic topology — and yet the miracle of it can be appreciated by a child. I'll discuss this a little in the examples below.

Notation 23.16. Let $(X, \mathcal{O}_X)$ be a topological space equipped with a Δ-complex structure. We denote the Euler characteristic of X by $\chi(X)$.

Examples 23.17.

(1) Let us consider the Δ-complex structure on $(T^2, \mathcal{O}_{T^2})$ of Examples 23.7 (7). For clarity the 0-simplices are also indicated, in yellow.

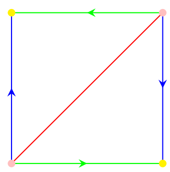


Before considering glueing we have four 0-simplices, five 1-simplices, and two 2-simplices. After considering glueing we have one 0-simplex, three 1-simplices, and two 2-simplices.

Thus we have that

$$\chi(T^2) = 1 - 3 + 2 = 0.$$

(2) The real projective plane $(\mathbb{P}^2(\mathbb{R}), \mathcal{O}_{\mathbb{P}^2(\mathbb{R})})$ can be defined to be the quotient of I^2 by the equivalence relation indicated below. It is explored in Question 12 of Exercise Sheet 4.

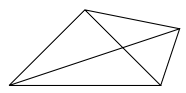


Before considering glueing we have four 0-simplices, five 1-simplices, and two 2-simplices. After considering glueing we have two 0-simplices, three 1-simplices, and two 2-simplices.

Thus we have that

$$\chi\big(\mathbb{P}^2(\mathbb{R})\big) = 2 - 3 + 2 = 1.$$

(3) Let us calculate $\chi(S^2)$ via the Δ-complex structure

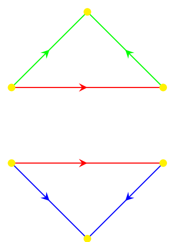


of Examples 23.7 (8). After considering glueing we have four 0-simplices, six 1-simplices, and four 2-simplices.

Thus we have that

$$\chi(S^2) = 4 - 6 + 4 = 2.$$

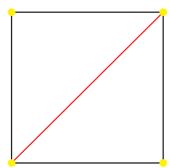
Let us instead calculate $\chi(S^2)$ using the Δ-complex structure of Examples 23.7 (8) below.



After considering glueing we have three 0-simplices, three 1-simplices, and two 2-simplices. Thus we have that

$$\chi(S^2) = 3 - 3 + 2 = 2.$$

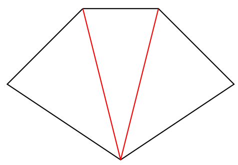
Let us now calculate $\chi(S^2)$ using the Δ-complex structure of Examples 23.7 (8) below in which we glue all four edges to a 0-simplex.



After considering glueing we have one 0-simplex, one 1-simplex and two 2-simplices. The 1-simplex is that drawn in red in the above figure. Thus we have that

$$\chi(S^2) = 1 - 1 + 2 = 2.$$

All the five platonic solids can also be regarded as equipping S^2 with a Δ-complex structure. For example the dodecahedron can be obtained by glueing together twelve pentagons. Each pentagon can be obtained by glueing together three 2-simplices as follows.



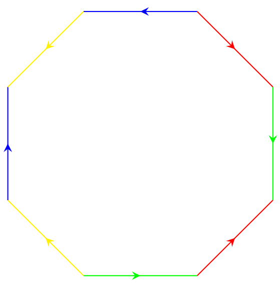
In this way we obtain a Δ-complex structure with twenty 0-simplices, fifty four 1-simplices, and thirty six 2-simplices. Thus we have that

$$\chi(S^2) = 20 - 54 + 36 = 2.$$

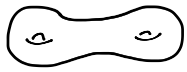
Without there being any pattern, we always arrive at the answer $\chi(S^2) = 2$! This is a wonderful way I feel for a child, or indeed anybody, to experience a sense of the beauty of mathematics.

That the calculation of $\chi(S^2)$ is independent of the choice of the Δ-complex structure was probably known to Archimedes around 200 BC. Post-renaissance the story goes back to Euler, around 1750.

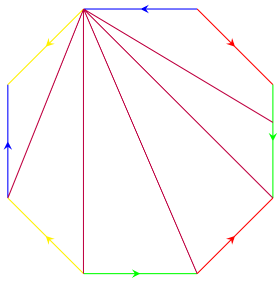
(4) It can be proven that by glueing the sides of an octagon as follows



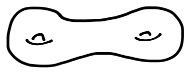
we obtain a topological space which is homeomorphic to the following surface. This is explored in Question 17 of Exercise Sheet 4.



Thus glueing together six 2-simplices in the following manner



equips the surface

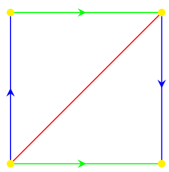


with a Δ-complex structure.

Hence its Euler characteristic is

$$1 - 9 + 6 = -2.$$

(5) Let us equip $(K^2, \mathcal{O}_{K^2})$ with the following Δ-complex structure.



We find that

$$\chi(K^2) = 1 - 3 + 2 = 0.$$

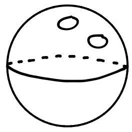
In particular we have that $\chi(T^2) = \chi(K^2) = 0$, whereas it can be proven using more sophiscated techniques — that you will meet if you take Algebraic Topology I in the autumn — that $(T^2, \mathcal{O}_{T^2})$ is not homeomorphic to $(K^2, \mathcal{O}_{K^2})$.

Thus the Euler characteristic does not necessarily detect whether or not two given surfaces are homeomorphic. Nevertheless it does a very good job! We will see that $(T^2, \mathcal{O}_{T^2})$ and $(K^2, \mathcal{O}_{K^2})$ are the only two surfaces whose Euler characteristic is 0.

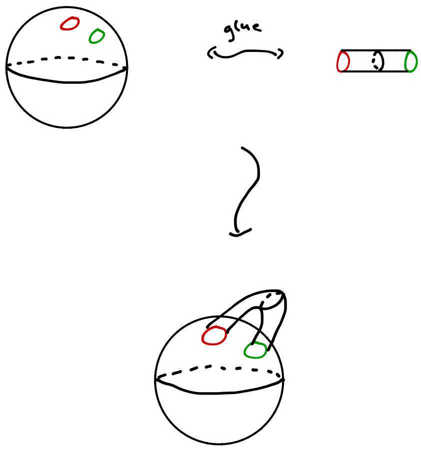
23.4 Statement of the classification of surfaces

Definition 23.18. Let $(X, \mathcal{O}_X)$ be a surface. We refer to the following procedure as *glueing a handle* onto $(X, \mathcal{O}_X)$.

(1) Cut out the interiors of two dijsoint discs in $(X, \mathcal{O}_X)$.

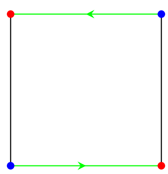


(2) Glue the boundary circles of a cylinder to the boundary circles of the discs whose interiors we cut out.

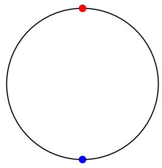


Example 23.19. By glueing a handle onto $(S^2, \mathcal{O}_{S^2})$ we obtain a topological space which is homeomorphic to $(T^2, \mathcal{O}_{T^2})$.

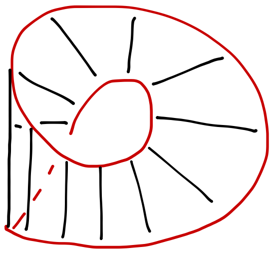
Observation 23.20. Let us view the Möbius band $(M^2, \mathcal{O}_{M^2})$ as the quotient of I^2 by the equivalence relation indicated below.



In $(M^2, \mathcal{O}_{M^2})$ the two black edges glue together to give a circle.



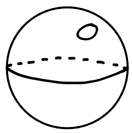
We refer to it as the *boundary circle* of $(M^2, \mathcal{O}_{M^2})$. It is depicted in red in the following picture.



If you find it hard to visualise this, it is a very good idea to colour the edges of a piece of paper and make yourself a Möbius band!

Definition 23.21. Let $(X, \mathcal{O}_X)$ be a surface. We refer to following procedure as *glueing a Möbius band* onto $(X, \mathcal{O}_X)$.

(1) Cut out the interior of a disc in $(X, \mathcal{O}_X)$.

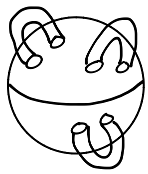


(2) Glue the boundary circle of a Möbius band to the boundary circle of the disc whose interior we cut out.

Remark 23.22. By glueing a Möbius band onto $(S^2, \mathcal{O}_{S^2})$ we obtain a topological space which is homeomorphic to the projective plane $(\mathbb{P}^2(\mathbb{R}), \mathcal{O}_{\mathbb{P}^2(\mathbb{R})})$. This cannot be truly visualised in $\mathbb{R}^3$. Nevertheless we can understand it geometrically! I omit the argument for now. Hopefully I will have time to make an update this evening.

If we glue two Möbius bands onto $(S^2, \mathcal{O}_{S^2})$ we obtain a topological space which is homeomorphic to $(K^2, \mathcal{O}_{K^2})$.

Definition 23.23. Let $n \geq 0$. An *n-handlebody* is a topological space which can be constructed up to homeomorphism by glueing n handles onto $(S^2, \mathcal{O}_{S^2})$.



Definition 23.24. Let $n \geq 1$. An *n-crosscap* is a topological space which can be constructed up to homeomorphism by glueing n Möbius bands onto $(S^2, \mathcal{O}_{S^2})$.

Theorem 23.25. Let $(X, \mathcal{O}_X)$ be a surface. There is an $n \geq 0$ such that $(X, \mathcal{O}_X)$ is homeomorphic to either an n-handlebody or an n-crosscap.

Remark 23.26. Theorem 23.25 is known as the *classification of surfaces*. It is a truly deep result. We must not lose sight of how remarkable it is — we can cook up all kinds of weird and wonderful surfaces which when we draw them in $\mathbb{R}^3$ do not posssibly look as though they could be homeomorphic to an n-handlebody. Yet we can prove that they are!

The proof I will sketch relies on one deep tool and one deep technique. The tool is the Euler characteristic of a surface, which we have already met. The technique is known as surgery, which we will now explore.