MA3002 Generell Topologi — Vår 2014

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May 19, 2014

9 Monday 3rd February

9.1 A local characterisation of closed sets

Proposition 9.1.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let V be a subset of X. Then V is closed with respect to $\mathcal{O}_X$ if and only if $\mathsf{cl}_{(X,\mathcal{O}_X)}(V)$ is V.

Proof. Suppose first that V is closed with respect to $\mathcal{O}_X$. Suppose that x does not belong to V. We make the following observations.

(1) By definition of $X \setminus V$, we have that x belongs to $X \setminus V$. Moreover, since V is closed with respect to $\mathcal{O}_X$, we have that $X \setminus V$ belongs to $\mathcal{O}_X$. In other words, $X \setminus V$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$.

(2) By definition of $X \setminus V$ once more, we have that $V \cap (X \setminus V)$ is empty.

Together (1) and (2) establish that x is not a limit point of V in X with respect to $\mathcal{O}_X$, for any x which does not belong to V. We conclude that $\mathsf{cl}_{(X,\mathcal{O}_X)}(V)$ is V.

Suppose now that $\mathsf{cl}_{(X,\mathcal{O}_X)}(V)$ is V. Suppose that $x \in X$ does not belong to V. By definition of $\mathsf{cl}_{(X,\mathcal{O}_X)}(V)$, we have that x is not a limit point of V in X with respect to $\mathcal{O}_X$. By definition of a limit point, we deduce that there is a neighbourhood U_x of x such that $V \cap U_x$ is empty. We make the following observations.

(1) We have that

$$X \setminus V = \bigcup_{x \in X \setminus V} \{x\}.$$

We also have that x belongs to U_x for every $x \in X \setminus V$, or, in other words, that $\{x\}$ is a subset of U_x for every $x \in X \setminus V$. Thus we have that $\bigcup_{x \in X \setminus V} \{x\}$ is a subset of $\bigcup_{x \in X \setminus V} U_x$. We deduce that $X \setminus V$ is a subset of $\bigcup_{x \in X \setminus V} U_x$.

(2) We have that

$$V \cap \left(\bigcup_{x \in X \setminus V} U_x \right) = \bigcup_{x \in X \setminus V} (V \cap U_x).$$

Since $V \cap U_x$ is empty for every $x \in X \setminus V$, we have that $\bigcup_{x \in X \setminus V} (V \cap U_x)$ is empty. We deduce that $V \cap \left(\bigcup_{x \in X \setminus V} U_x \right)$ is empty. In other words, $\bigcup_{x \in X \setminus V} U_x$ is a subset of $X \setminus V$.

(3) Since U_x belongs to $\mathcal{O}_X$, for every $x \in X \setminus V$, and since $\mathcal{O}_X$ is a topology on X, we have that $\bigcup_{x \in X \setminus V} U_x$ belongs to $\mathcal{O}_X$.

By (1) and (2) together, we have that $\bigcup_{x \in X \setminus V} U_x = X \setminus V$. By (3), we deduce that $X \setminus V$ belongs to $\mathcal{O}_X$. Thus V is closed with respect to $\mathcal{O}_X$. $\square$

Remark 9.1.2. We now, by Proposition 9.1.1, have two ways to understand closed sets. The first is 'global' in nature: that $X \setminus V$ belongs to $\mathcal{O}_X$. The second is 'local' in nature: that every limit point of V belongs to V.

For certain purposes in mathematics it can be appropriate to work 'locally', whilst for others it can be appropriate to work 'globally'. To know that 'local' and 'global' variants of a particular mathematical concept coincide allows us to move backwards and forwards between these points of view. This is often a very powerful technique.

9.2 Boundary

Definition 9.2.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. The *boundary* of A in X with respect to $\mathcal{O}_X$ is the set of $x \in X$ such that, for every neighbourhood U of x in X with respect to $\mathcal{O}_X$, there is an $a \in U$ which belongs to A, and there is a $y \in U$ which belongs to $X \setminus A$.

Notation 9.2.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. We shall denote the boundary of A in X with respect to $\mathcal{O}_X$ by $\partial_{(X,\mathcal{O}_X)} A$.

Remark 9.2.3. Suppose that $x \in X$ belongs to $\partial_{(X,\mathcal{O}_X)} A$. Then x is a limit point of A in X with respect to $\mathcal{O}_X$.

Remark 9.2.4. Let x be a limit point of A which does not belong to A. Then x belongs to $\partial_{(X,\mathcal{O}_X)} A$.

However, as we shall see in Example 9.3.1, it is not necessarily the case that if a belongs to A, then a belongs to $\partial_{(X,\mathcal{O}_X)} A$. In particular, not every limit point of A belongs to $\partial_{(X,\mathcal{O}_X)} A$.

9.3 Boundary in a finite example

Example 9.3.1. , Let $X = \{a, b, c, d, e\}$ be a set with five elements. Let $\mathcal{O}_X$ be the topology on X given by

$$\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,e\}, \{c,d\}, \{a,b,e\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}, \{b,c,d,e\}, X\}.$$

Let $A = \{b, d\}$. The neighbourhoods in X with respect to $\mathcal{O}_X$ of each of the elements of A are listed in a table in Example 8.4.2. To determine $\partial_{(X,\mathcal{O}_X)} A$, we check, for each element of A, whether each of its neighbourhoods both contain either a, c, or e, and contain either b or d. We determined the limit points of A in X with respect to $\mathcal{O}_X$ in Example 8.4.2, which saves us a little work.

Element	Belongs to $\partial_{(X,\mathcal{O}_X)} A$?	Reason
a	✗	Not a limit point.
b	✗	The neighbourhood $\{b\}$ does not contain any element of $X \setminus A$.
c	✓	Limit point which does not belong to A.
d	✓	Every neighbourhood of d contains both c and d. We have that d belongs to A, and that c belongs to $X \setminus A$.
e	✓	Limit point which does not belong to A.

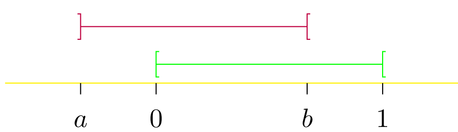
Thus $\partial_{(X,\mathcal{O}_X)} A = \{c, d, e\}$.

9.4 Geometric examples of boundary

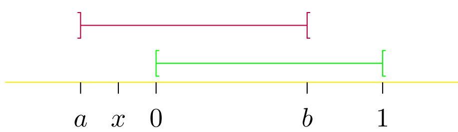
Example 9.4.1. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Let $A = [0, 1[$.

By Example 8.4.4, we have that 1 is a limit point of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ which does not belong to $[0, 1[$. Thus 1 belongs to $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})} [0, 1[$. By Example **??**, we have that all other limit points of $[0, 1[$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$ belong to $[0, 1[$. To determine $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})} [0, 1[$, it therefore remains to check which elements of $[0, 1[$ have the property that each of their neighbourhoods contains at least one element of $\mathbb{R} \setminus [0, 1[$.

Let U be a neighbourhood of 0 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$. By definition of $\mathcal{O}_{\mathbb{R}}$, there is an open interval $]a, b[$ such that $a < 0 < b$, and which is a subset of U.

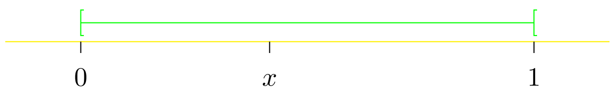


There is an $x \in \mathbb{R}$ such that $a < x < 0$. In particular, x belongs to $\mathbb{R} \setminus [0, 1[$.

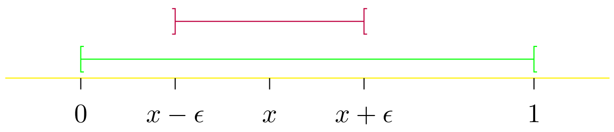


Since $]a, 0[$ is a subset of $]a, b[$, and since $]a, b[$ is a subset of U, we also have that x belongs to U. This proves that if U is a neighbourhood of 0 in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, then $(\mathbb{R} \setminus [0,1[) \cap U$ is not empty. Thus 0 belongs to $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})} [0,1[$.

Suppose now that $0 < x < 1$.

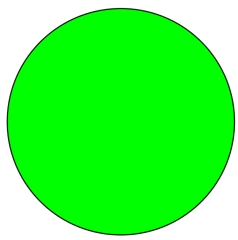


Let $0 < \epsilon \leq \min\{x, 1-x\}$. Then $]x-\epsilon, x+\epsilon[$ is a neighbourhood of x in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$, and $(\mathbb{R} \setminus [0,1[) \cap \,]x-\epsilon, x+\epsilon[$ is empty.

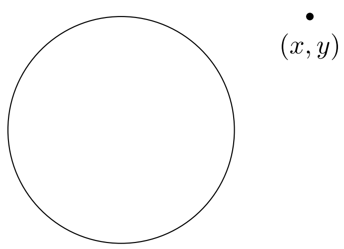


Thus x does not belong to $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})} [0,1[$. We conclude that $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})} [0,1[$ is $\{0,1\}$.

Example 9.4.2. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $A = D^2$.



Suppose that (x, y) belongs to $\mathbb{R}^2 \setminus D^2$.



Let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon < \|(x,y)\| - 1.$$

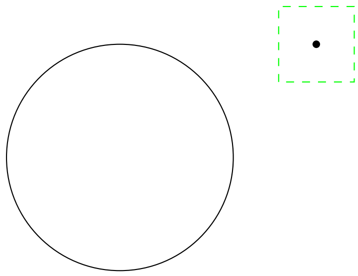
Let U_x be the open interval given by

$$\left]x - \tfrac{\epsilon\sqrt{2}}{\epsilon}, x + \tfrac{\epsilon\sqrt{2}}{\epsilon}\right[.$$

Let U_y be the open interval given by

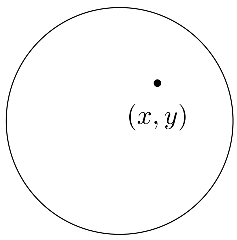
$$\left]y - \tfrac{\epsilon\sqrt{2}}{\epsilon}, y + \tfrac{\epsilon\sqrt{2}}{\epsilon}\right[.$$

Then $U_x \times U_y$ is a neighbourhood of (x, y) in $\mathbb{R}^2$ whose intersection with D^2 is empty.



This can be proven by the same argument as is needed to carry out Task E8.2.4. Thus (x, y) is not a limit point of D^2 in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. In particular, (x, y) does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$.

Suppose now that $(x, y) \in \mathbb{R}^2$ has the property that $\|(x, y)\| < 1$.



Let $\epsilon \in \mathbb{R}$ be such that

$$0 < \epsilon < 1 - \|(x, y)\|.$$

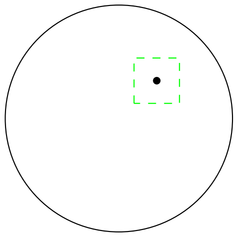
Let U_x be the open interval given by

$$\left]x - \tfrac{\epsilon\sqrt{2}}{\epsilon}, x + \tfrac{\epsilon\sqrt{2}}{\epsilon}\right[.$$

Let U_y be the open interval given by

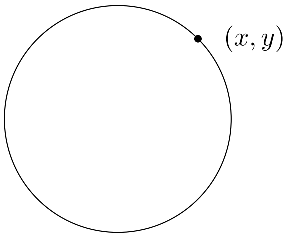
$$\left]y - \tfrac{\epsilon\sqrt{2}}{\epsilon}, y + \tfrac{\epsilon\sqrt{2}}{\epsilon}\right[.$$

Then $U_x \times U_y$ is a neighbourhood of (x, y) in $\mathbb{R}^2$ whose intersection with D^2 is empty. To check this is Task E9.2.2.

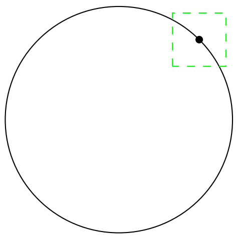


In other words, $\left(\mathbb{R}^2 \setminus D^2\right) \cap (U_x \times U_y)$ is empty. Thus (x, y) does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$.

Suppose now that $\|(x, y)\| = 1$. In other words, we have that (x, y) belongs to S^1.

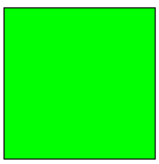


Every neighbourhood of (x, y) in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ contains an 'open rectangle' U to which (x, y) belongs. Both $D^2 \cap U$ and $\left(\mathbb{R}^2 \setminus D^2\right) \cap U$ are not empty.

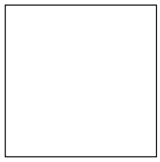


Thus (x, y) belongs to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$. To fill in the details of this argument is the topic of Task E9.2.4. We conclude that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$ is S^1.

Example 9.4.3. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let $A = I^2$.

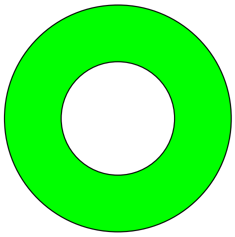


Then $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} I^2$ is the 'border around I^2'.



In other words, $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} I^2$ is $(\{0,1\} \times I) \cup (I \times \{0,1\})$. To prove this is the topic of Task E9.2.3.

Example 9.4.4. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let A be an annulus A_k, for some $k \in \mathbb{R}$ with $0 < k < 1$, as in Notation 4.1.17.

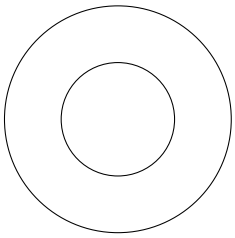


Then $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} A_k$ is the union of the outer and the inner circle of A_k. In other words, the union of the set

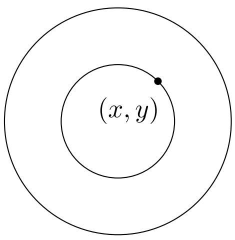
$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1\right\}$$

and the set

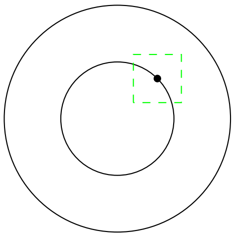
$$\left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = k\right\}.$$



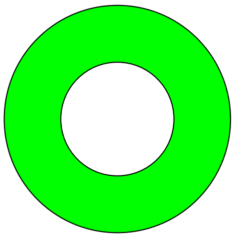
To prove this is the topic of Task E9.2.5. Suppose, for instance, that $(x, y) \in \mathbb{R}^2$ belongs to the inner circle of A_k.



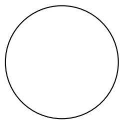
Every neighourhood of (x, y) contains an 'open rectangle' around (x, y) which overlaps both A_k and the open disc which we can think of as having been cut out from D^2 to obtain A_k.



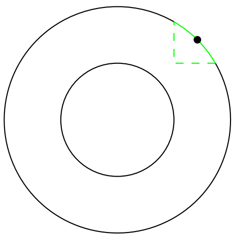
Example 9.4.5. Let $(X, \mathcal{O}_X)$ be $(D^2, \mathcal{O}_{D^2})$. Let A be an annulus A_k as in Example 9.4.4.



Then $\partial_{(D^2, \mathcal{O}_{D^2})} A_k$ is the inner circle of A_k.



To prove this is the topic of Task E9.2.6. In particular if $(x, y) \in S^1$ then, unlike in Example 9.4.4, (x, y) does not belong to $\partial_{(D^2, \mathcal{O}_{D^2})} A_k$. For there is a neighbourhood of (x, y) in D^2 with respect to $\mathcal{O}_{D^2}$ which does not overlap $D^2 \setminus A_k$, the open disc which we can think of as having cut out of D^2 to obtain A_k.



Example 9.4.4 and Example 9.4.5 demonstrate that given a set A, and a topological space $(X, \mathcal{O}_X)$ such that A is a subset of X, the boundary of A in X with respect to $\mathcal{O}_X$ depends upon $(X, \mathcal{O}_X)$. The next examples illustrate this further.

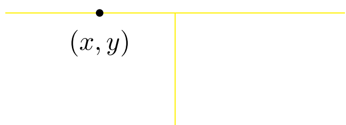
Example 9.4.6. Let $(X, \mathcal{O}_X)$ be $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let T denote the subset of $\mathbb{R}^2$ given by the union of

$$\{(0, y) \mid 0 \leq y \leq 1\}$$

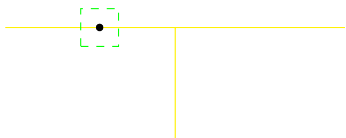
and

$$\{(x, 1) \mid -1 \leq x \leq 1\}.$$

Then $\partial_X \mathsf{T}$ is T. We have that T^2 is closed in $\mathbb{R}^2$. To prove this is the topic of Task E9.2.7. Thus every limit point of T^2 belongs to T. Suppose that (x, y) belongs to T.



Then the intersection with $\mathbb{R}^2 \setminus \mathsf{T}$ of every neighbourhood of (x, y) in $\mathbb{R}^2$ is not empty. To prove this is the topic of Task E9.2.8.



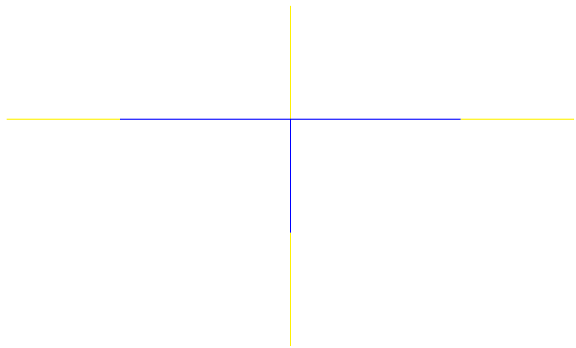
Example 9.4.7. Let X be the subset of $\mathbb{R}^2$ given by the union of

$$\{(0, y) \mid -1 \leq y \leq 2\}$$

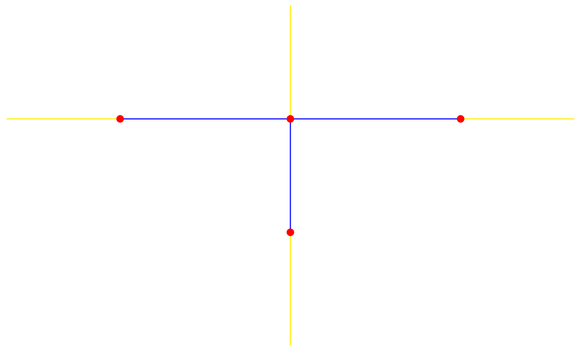
and

$$\{(x, 1) \mid -2 \leq x \leq 2\}.$$

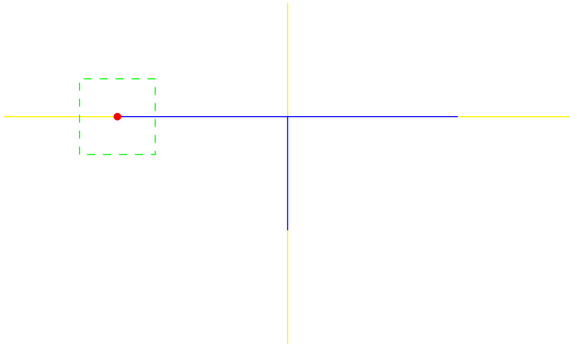
Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let T be as in Example 9.4.6.



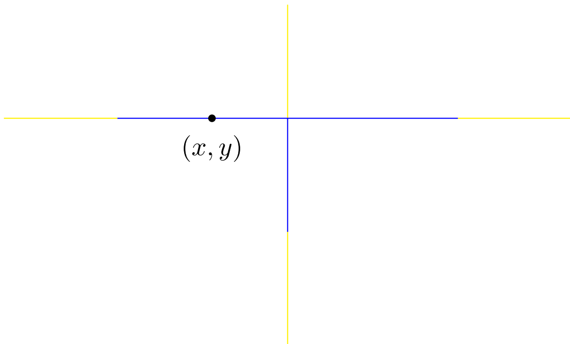
Then $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$ is $\{(-1,1),(0,1),(1,1),(0,0)\}$.



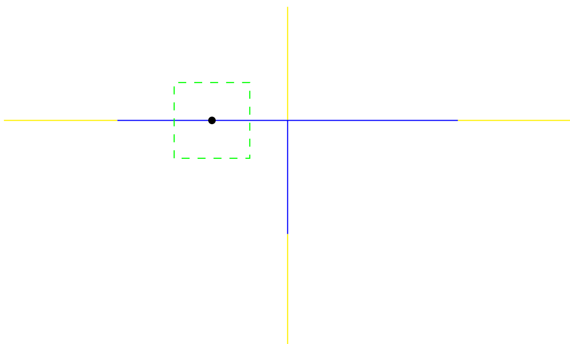
Every neighbourhood of each of these four points contains both a segment of $X \setminus \mathsf{T}$ and a segment of T. A typical neighbourhood of $(-1,1)$, for instance, is the intersection of an 'open rectangle' around $(-1,1)$ in $\mathbb{R}^2$ with T as depicted below.



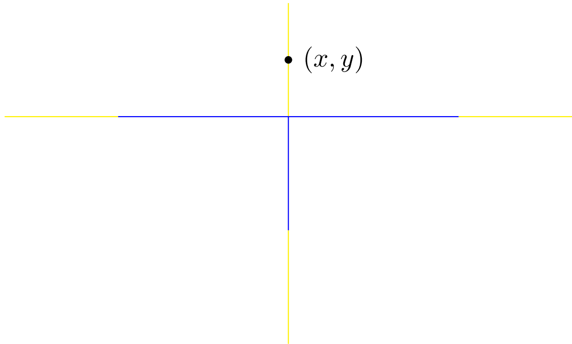
Let (x, y) be a point of T which is not one of these four.



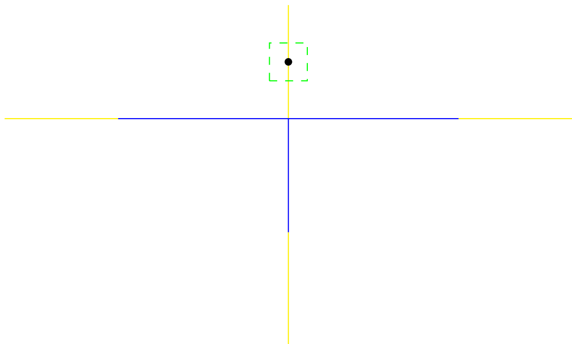
Then we can find a neighbourhood of (x, y) whose intersection with $X \setminus \mathsf{T}$ is empty. For instance, an intersection of a sufficiently small 'open rectangle' around (x, y) in $\mathbb{R}^2$ with X.



Suppose that $(x, y) \in X$ does not belong to T.



Then we can find a neighbourhood of (x, y) whose intersection with $X \setminus \mathsf{T}$ is empty. For instance, an intersection of a sufficiently small 'open rectangle' around (x, y) in $\mathbb{R}^2$ with X.



To fill in the details of this argument is the topic of Task E9.2.9.

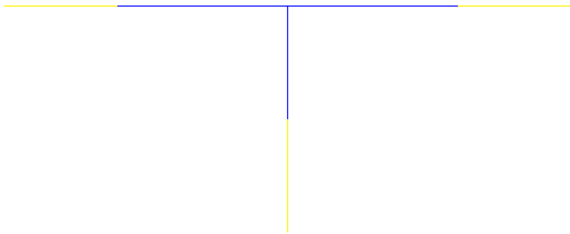
Example 9.4.8. Let X be the subset of $\mathbb{R}^2$ given by the union of

$$\{(0, y) \mid -2 \leq y \leq 1\}$$

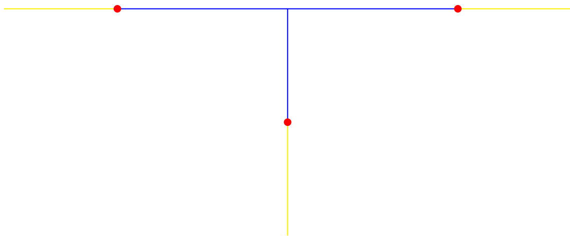
and

$$\{(x, 1) \mid -2 \leq x \leq 2\}.$$

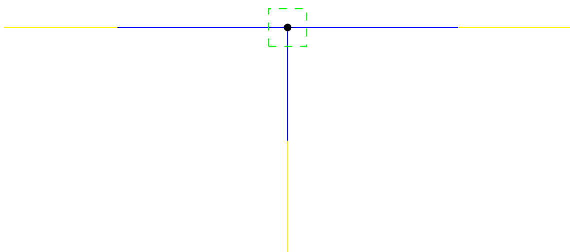
Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Let T be as in Example 9.4.6.



Then $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$ is $\{(-1,1),(1,1),(0,0)\}$.



In particular $(0,1)$ does not belong to $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$, unlike in Example 9.4.7. We can find a neighbourhood of $(0,1)$ whose intersection with $X \setminus \mathsf{T}$ is empty, such as the intersection of a sufficiently small 'open rectangle' around $(0,1)$ in $\mathbb{R}^2$ with X.



To give the details of the calculation of $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$ is the topic of Task E9.2.10.

9.5 Connected topological spaces

Terminology 9.5.1. Let X be a set. Let X_0 and X_1 be subsets of X. The union $X_0 \cup X_1$ of X_0 and X_1 is *disjoint* if $X_0 \cap X_1$ is the empty set.

Notation 9.5.2. Let X be a set. Let X_0 and X_1 be subsets of X. If $X = X_0 \cup X_1$, and this union is disjoint, we write $X = X_0 \sqcup X_1$.

Definition 9.5.3. A topological space $(X, \mathcal{O}_X)$ is *connected* if there do not exist subsets X_0 and X_1 of X such that the following hold.

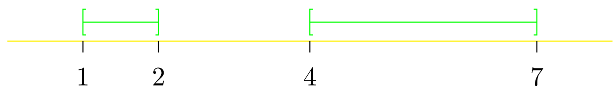
(1) Neither X_0 nor X_1 is empty, and both belong to $\mathcal{O}_X$.

(2 We have that $X = X_0 \sqcup X_1$.

9.6 An example of a topological space which is not connected

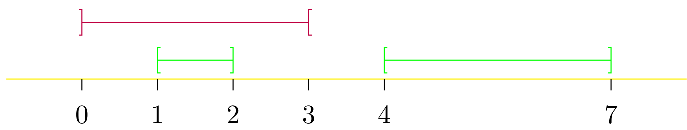
Remark 9.6.1. We shall have to work quite hard to prove that any of our geometric examples of topological spaces are connected. Instead, we shall begin with some examples of topological spaces which are not connected.

Example 9.6.2. Let $X = [1,2] \cup [4,7]$.



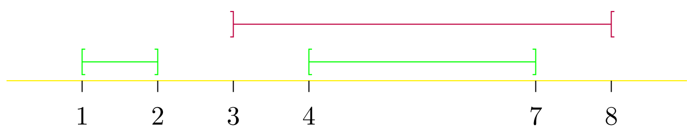
Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. The following hold.

(1) By Example 1.6.3, we have that $]0,3[$ belongs to $\mathcal{O}_{\mathbb{R}}$. We have that $[1,2] = X \cap]0,3[$.



By definition of $\mathcal{O}_X$, we deduce that $[1,2]$ belongs to $\mathcal{O}_X$.

(2) By Example 1.6.3, we have that $]3,8[$ belongs to $\mathcal{O}_{\mathbb{R}}$. We have that $[4,7] = X \cap]3,8[$.



By definition of $\mathcal{O}_X$, we conclude that $[4,7]$ belongs to $\mathcal{O}_X$.

(3) We have that $X = [1,2] \sqcup [4,7]$, since $[1,2] \cap [4,7]$ is empty.

We conclude that $(X, \mathcal{O}_X)$ is not connected.

Remark 9.6.3. In (1), we could have chosen instead of $]0,3[$ any subset of $\mathbb{R}$ which belongs to $\mathcal{O}_{\mathbb{R}}$, which does not intersect $[4,7]$, and of which $[1,2]$ is a subset. In (2), we could have chosen instead of $]3,8[$ any subset of $\mathbb{R}$ which belongs to $\mathcal{O}_{\mathbb{R}}$, which does not intersect $[1,2]$, and of which $[4,7]$ is a subset.

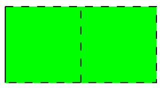
E9 Exercises for Lecture 9

E9.1 Exam questions

Task E9.1.1. We saw in Example 9.4.1 that $\partial_{(\mathbb{R},\mathcal{O}_{\mathbb{R}})}\, [0,1[$ is $\{0,1\}$.

Prove that $\{0,1\}$ is also the boundary of each of $]0,1[$, $]0,1]$, and $[0,1]$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$.

Task E9.1.2. Let A be the subset of $\mathbb{R}^2$ given by the union of $]0,1[\times]0,1[$ and $[-1,0[\times$ $]0,1[$.



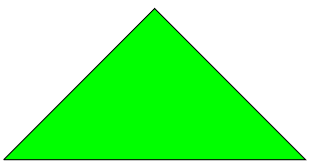
(1) What is the boundary of A in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$?

(2) What is the boundary of A in $\mathbb{R}\times]0,\infty[$, where $\mathbb{R}\times]0,\infty[$ is equipped with the subspace topology with respect to $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$?

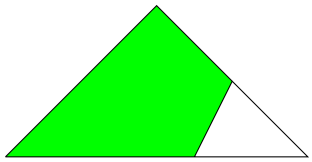
(3) Let X be the union of $]-\infty,0[\times\mathbb{R}$ and $]0,\infty[\times\mathbb{R}$. Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$. What is the boundary of A in X with respect to $\mathcal{O}_X$?

Task E9.1.3. Let $(X,\mathcal{O}_X)$ be as in Task E8.1.2. What is the boundary of $\{a,c\}$ in X with respect to $\mathcal{O}_X$? What is the boundary of $\{b,c\}$ in X with respect to $\mathcal{O}_X$? What is the boundary of $\{d\}$ in X with respect to $\mathcal{O}_X$?

Task E9.1.4. Let X be the subset of $\mathbb{R}^2$ which is a 'solid triangle'.

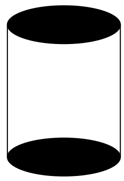


Let A be the subset of X depicted below. All of the lines, and the entire shaded area, belong to A.



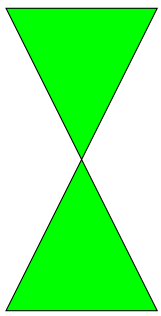
In other words, A is obtained from X by cutting out the inside of smaller 'solid triangle' inside it. What is $\partial_{(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})}A$? What is $\partial_{(X,\mathcal{O}_X)}A$, where $\mathcal{O}_X$ is the subspace topology on X with respect to $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$?

Task E9.1.5. What is the boundary of $D^2 \times I$ in $\mathbb{R}^3$ with respect to $\mathcal{O}_{\mathbb{R}^3}$?

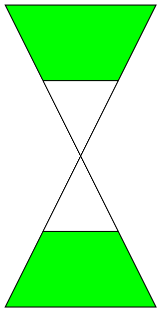


Give a proof by appealing to Example 9.4.2, Task E9.1.1, and Task E9.3.11. What is the boundary of $D^2 \times I$ in $\mathbb{R}^2 \times I$ with respect to $\mathcal{O}_{\mathbb{R}^2 \times I}$? What is the boundary of $D^2 \times I$ in $D^2 \times \mathbb{R}$ with respect to $\mathcal{O}_{D^2 \times \mathbb{R}}$?

Task E9.1.6 (Continuation exam, August 2013)**.** Let X be a subset of $\mathbb{R}^2$ as depicted below. In other words, we have two triangles which 'meet at their tips'.



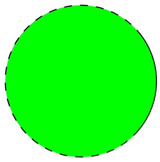
Let A be the subset of X obtained by removing the inside of a smaller copy of this shape, as depicted below. All of the lines, and the entirety of both shaded areas, belong to A.



What is $\partial_{(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})}A$? What is $\partial_{(X,\mathcal{O}_X)}A$?

Task E9.1.7. What is the boundary of $\mathbb{Q}$ in $\mathbb{R}$ with respect to $\mathcal{O}_{\mathbb{R}}$?

Task E9.1.8. Let A be the subset of D^2 of Task E8.1.4.

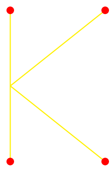


What is the boundary of A in D^2 with respect to $\mathcal{O}_{D^2}$?

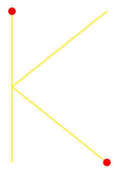
Task E9.1.9. View the letter K as a subset of $\mathbb{R}^2$.

For each of the following, find a subset X of $\mathbb{R}^2$ such that K is a subset of X, and such that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ is as described, where $\mathcal{O}_X$ denotes the subspace topology on X with respect to $(\mathbb{R}^2,\mathcal{O}_{\mathbb{R}^2})$.

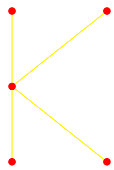
(1) We have that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ consists of the four points depicted below.



(2) We have that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ consists of the two points depicted below.



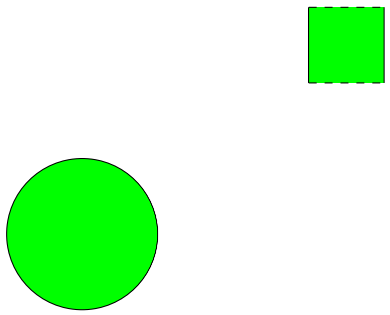
(3) We have that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ consists of the five points depicted below.



(4) We have that $\partial_{(X,\mathcal{O}_X)}\mathsf{K}$ consists of the union of the two lines depicted below.

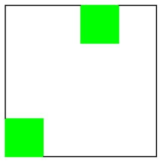
Task E9.1.10. Let $(X, \mathcal{O}_X)$ be a topological space. Explain why $\partial_{(X,\mathcal{O}_X)}X$ is the empty set.

Task E9.1.11. Let X be the union of D^2 and $[3,4] \times]2,3[$.



Let $\mathcal{O}_X$ denote the subspace topology on X with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$. Prove that $(X, \mathcal{O}_X)$ is not connected.

Task E9.1.12. Let X be the subset of I^2 given by the union of $\left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{4}\right]$ and $\left[\frac{1}{2}, \frac{3}{4}\right] \times \left[\frac{3}{4}, 1\right]$.



Let

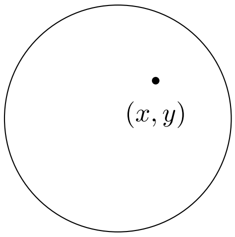
$$I^2 \xrightarrow{\pi} T^2$$

be the quotient map. Let $\mathcal{O}_{\pi(X)}$ denote the subspace topology on $\pi(X)$ with respect to $(T^2, \mathcal{O}_{T^2})$. Prove that $\left(\pi(X), \mathcal{O}_{\pi(X)}\right)$ is not connected.

E9.2 In the lecture notes

Task E9.2.1. Do the same as in Task E2.2.2 for the proof of Proposition 9.1.1.

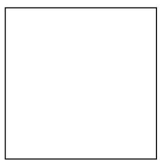
Task E9.2.2. Let $(x, y) \in \mathbb{R}^2$ be such that $\|(x, y)\| < 1$.



Prove that (x, y) does not belong to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$, following the argument outlined in Example 9.4.2. You may find it helpful to look back at Example 3.2.3.

Task E9.2.3. It was asserted in Example 9.4.3 that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} I^2$ is

$$(\{0, 1\} \times I) \cup (I \times \{0, 1\}) \,.$$



Prove this first as follows, along the lines of Example 9.4.2.

(1) Demonstrate that if $(x, y) \in \mathbb{R}^2$ does not belong to I^2, then (x, y) is not a limit point of I^2 in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

(2) Demonstrate that if $0 < x < 1$ and $0 < y < 1$, then there is a neighbourhood U of (x, y) in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$ such that $\left(\mathbb{R}^2 \setminus I^2\right) \cap U$ is empty.

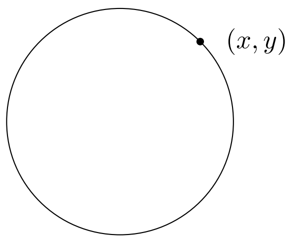
(3) Demonstrate that if (x, y) belongs to

$$(\{0,1\} \times I) \cup (I \times \{0,1\}),$$

then every neighbourhood U of (x, y) in $\mathbb{R}^2$ has the property that both $I^2 \cap U$ and $\left(\mathbb{R}^2 \setminus I^2\right) \cap U$ are not empty.

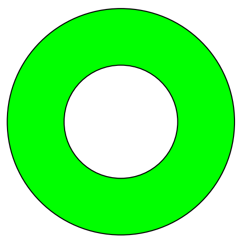
Give a second proof by appealing to Task E9.3.11. Give a third proof by appealing to Task E7.2.9 and Task E9.3.12.

Task E9.2.4. Let $(X, \mathcal{O}_X)$ and A be as in Example 8.4.8. Suppose that $(x, y) \in \mathbb{R}^2$ belongs to S^1.



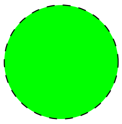
Prove that (x, y) belongs to $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} D^2$.

Task E9.2.5. Let A be an annulus A_k, for some $k \in \mathbb{R}$ with $0 < k < 1$, as in Notation 4.1.17.



Prove that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} A_k$ is the union of the outer and the inner circle of the annulus, as claimed in Example 9.4.4. You may wish to proceed as follows.

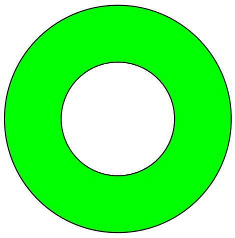
(1) Let B be the 'open disc' of radius k centred at $(0, 0)$.



Demonstrate that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})} B$ is the circle of radius k centred at $(0, 0)$.

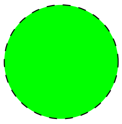
(2) Observing that A_k is $D^2 \setminus B$, appeal to Example 9.4.2 and Task E9.3.15.

Task E9.2.6. Let A be an annulus A_k, for some $k \in \mathbb{R}$ with $0 < k < 1$, as in Notation 4.1.17.



Prove that $\partial_{(D^2, \mathcal{O}_{D^2})} A_k$ is the inner circle of the annulus, as claimed in Example 9.4.5. You may wish to proceed as follows.

(1) Let B be the 'open disc' of radius k centred at $(0, 0)$.



Appealing to (1) of Task E9.2.5and Task E9.3.13, observe that $\partial_{(D^2, \mathcal{O}_{D^2})} B$ is the circle of radius k centred at $(0, 0)$.

(2) Appeal to Task E9.1.10 and Task E9.3.15.

Task E9.2.7. Let T be the subset of $\mathbb{R}^2$ of Example 9.4.6.

Prove that T is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$. You may wish to proceed as follows.

(1) Observe that T is the union of $\{0\} \times [0, 1]$ and $[0, 1] \times \{1\}$.

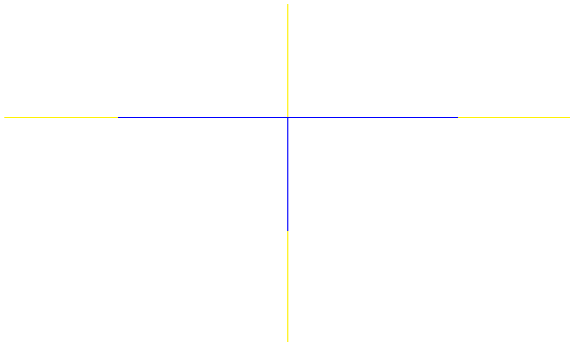
(2) Appealing to Task E3.3.1, observe that $\{0\} \times [0,1]$ and $[0,1] \times \{1\}$ are both closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

(3) Appealing to Task E9.3.5, conclude from (1) and (2) that T is closed in $\mathbb{R}^2$ with respect to $\mathcal{O}_{\mathbb{R}^2}$.

Task E9.2.8. Let T be the subset of $\mathbb{R}^2$ of Example 9.4.6.

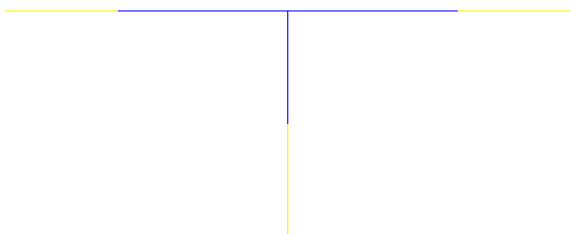
Prove that $\partial_{(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})}\mathsf{T}$ is T. You may wish to follow the argument outlined in Example 9.4.6.

Task E9.2.9. Let $(X, \mathcal{O}_X)$ and T be as in Example 9.4.7.



Prove that $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$ is $\{(-1,1),(0,1),(1,1),(0,0)\}$. You may wish to follow the argument outlined in Example 9.4.7.

Task E9.2.10. Let $(X, \mathcal{O}_X)$ and T be as in Example 9.4.8.



Prove that $\partial_{(X,\mathcal{O}_X)}\mathsf{T}$ is $\{(-1,1),(1,1),(0,0)\}$.

E9.3 For a deeper understanding

Task E9.3.1. Let $(X, \mathcal{O}_X)$ be a topological space. Let V be a subset of X which is closed with respect to $\mathcal{O}_X$. Let A be a subset of V. Prove that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is a subset of V. You may wish to appeal to Proposition 9.1.1.

Task E9.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Prove that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is equal to the intersection of all subsets V of X with the following properties.

(1) V is closed with respect to $\mathcal{O}_X$.

(2) A is a subset of V.

You may wish to appeal to Task E9.3.1.

Corollary E9.3.3. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Then $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is closed.

Proof. Follows immediately from Task E9.3.2 and the fact, observed as part of Remark E1.3.2, that an intersection of (possibly infinitely many) subsets of X which are closed with respect to $\mathcal{O}_X$ is closed with respect to $\mathcal{O}_X$. □

Remark E9.3.4. In other words, $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ is the smallest subset of X which contains A, and which is closed with respect to $\mathcal{O}_X$.

Task E9.3.5. Let $(X, \mathcal{O}_X)$ be a topological space. Let A and B be subsets of X. Prove that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A \cup B)$ is $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cup \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$. You may wish to proceed as follows.

(1) By Corollary E9.3.3, we have $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ and $\mathsf{cl}_{(X,\mathcal{O}_X)}(B)$ are closed with respect to $\mathcal{O}_X$. By Remark E1.3.2, we thus have that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cup \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$ is closed with respect to $\mathcal{O}_X$. Deduce by Task E9.3.1 that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A \cup B)$ is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cup \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$.

(2) Observe that if $x \in X$ is a limit point of A or B in X with respect to $\mathcal{O}_X$, then x is a limit point of $A \cup B$ in X with respect to $\mathcal{O}_X$.

Task E9.3.6. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\{A_i\}_{i \in I}$ be an infinite set of subsets of X. Give an example to demonstrate that $\mathsf{cl}_{(X,\mathcal{O}_X)}(\cup_{i \in I} A_i)$ is not necessarily $\cup_{i \in I} \mathsf{cl}_{(X,\mathcal{O}_X)}(A_i)$.

Task E9.3.7. Let $(X, \mathcal{O}_X)$ be a topological space. Let A and B be subsets of X. Prove that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A \cap B)$ is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$.

Task E9.3.8. Let $(X, \mathcal{O}_X)$ be a topological space. Let A and B be subsets of X. Give an example to demonstrate that $\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$ is not necessarily a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A \cap B)$. In particular, these sets are not necessarily equal.

Task E9.3.9. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a map. Prove that f is continuous if and only if for every subset A of X, we have that $f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}(A)\right)$ is a subset of $\mathsf{cl}_{(Y,\mathcal{O}_Y)}(f(A))$. You may wish to proceed as follows.

(1) Suppose that the condition holds. Let V be a subset of Y which is closed with respect to $\mathcal{O}_Y$. By one of the relations of Table A.2, observe that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)$ is a subset of

$$f^{-1}\left(f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)\right)\right).$$

(2) By hypothesis, we have that $f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)\right)$ is a subset of

$$\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right).$$

By one of the relations of Table A.2, deduce that

$$f^{-1}\left(f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)\right)\right)$$

is a subset of

$$f^{-1}\left(\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)\right).$$

(3) By (1) and (2), deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)$ is a subset of

$$f^{-1}\left(\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)\right).$$

(4) By one of the relations of Table A.2, observe that $f\left(f^{-1}(V)\right)$ is a subset of V. By Task E8.3.12, deduce that $\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)$ is a subset of $\mathsf{cl}_{(Y,\mathcal{O}_Y)}(V)$.

(5) Since V is closed in Y with respect to $\mathcal{O}_Y$, we have by Proposition 9.1.1 that $V = \mathsf{cl}_{(Y,\mathcal{O}_Y)}(V)$. By (4), deduce that $\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)$ is a subset of V.

(6) By (5) and one of the relations of Table A.2, deduce that $f^{-1}\left(\mathsf{cl}_{(Y,\mathcal{O}_Y)}\left(f\left(f^{-1}(V)\right)\right)\right)$ is a subset of $f^{-1}(V)$.

(7) By (3) and (6), deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)$ is a subset of $f^{-1}(V)$.

(8) By Remark 8.5.4, we have that $f^{-1}(V)$ is a subset of $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right)$. By (7), deduce that $\mathsf{cl}_{(X,\mathcal{O}_X)}\left(f^{-1}(V)\right) = V$.

(9) By Proposition 9.1.1, deduce that $f^{-1}(V)$ is closed in X with respect to $\mathcal{O}_X$. By Task **??**, conclude that f is continuous.

(10) Conversely, suppose that f is continuous. Suppose that x is a limit point of A in X with respect to $\mathcal{O}_X$. Let $U_{f(x)}$ be a neighbourhood of $f(x)$ in Y with respect to $\mathcal{O}_Y$. Since f is continuous, observe that, by Task E8.3.3, there is a neighbourhood U_x of x in X with respect to $\mathcal{O}_X$ such that $f(U_x)$ is a subset of $U_{f(x)}$.

(11) Since x is a limit point of A in X with respect to $\mathcal{O}_X$, we have that $U_x \cap A$ is not empty. Thus $f(U_x \cap A)$ is not empty. Since $f(U_x \cap A)$ is a subset of $f(U_x) \cap f(A)$, deduce that $f(U_x) \cap f(A)$ is not empty.

(12) Since $f(U_x)$ is a subset of $U_{f(x)}$, deduce that $U_{f(x)} \cap f(A)$ is not empty.

(13) Conclude that $f(x)$ is a limit point of $f(A)$ in Y with respect to $\mathcal{O}_Y$. Thus $f\left(\mathsf{cl}_{(X,\mathcal{O}_X)}(A)\right)$ is a subset of $\mathsf{cl}_{(Y,\mathcal{O}_Y)}(f(A))$.

Task E9.3.10. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Prove that $\partial_{(X,\mathcal{O}_X)}A$ is the intersection of $\mathsf{cl}_{(X,\mathcal{O}_X)}(A)$ and $\mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus A)$.

Task E9.3.11. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let A be a subset of X, and let B be a subset of Y. Prove that $\partial_{(X\times Y,\mathcal{O}_{X\times Y})}A \times B$ is the union of

$$\left(\partial_{(X,\mathcal{O}_X)}A\right) \times \mathsf{cl}_{(Y,\mathcal{O}_Y)}(B)$$

and

$$\mathsf{cl}_{(X,\mathcal{O}_X)}(A) \times \left(\partial_{(Y,\mathcal{O}_Y)}B\right).$$

For proving that $\partial_{(X\times Y,\mathcal{O}_{X\times Y})}A \times B$ is a subset of this union, you may wish to make use of one of the set theoretic equalities listed in Remark A.1.1.

Task E9.3.12. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Let

$$X \xrightarrow{f} Y$$

be a homeomorphism. Let A be a subset of X. Prove that $\partial_{(Y,\mathcal{O}_Y)}f(A)$ is $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$. You may wish to proceed as follows.

(1) Suppose that y belongs to $\partial_{(Y,\mathcal{O}_Y)}f(A)$. Let U be a neighbourhood of $f^{-1}(y)$ in X with respect to $\mathcal{O}_X$. Observe that since f is a homeomophism, $f(U)$ is a neighbourhood of y in Y with respect to $\mathcal{O}_Y$.

(2) We have that

$$f^{-1}(f(A) \cap f(U)) = f^{-1}(f(A)) \cap f^{-1}(f(U)).$$

Since f is a bijection, we have that $f^{-1}(f(A)) = A$, and that $f^{-1}(f(U)) = U$. Deduce that

$$f^{-1}(f(A) \cap f(U)) = A \cap U.$$

(3) Observe that by (1) and the fact that y belongs to $\partial_{(Y,\mathcal{O}_Y)}f(A)$, we have that $f(A) \cap f(U)$ is not empty. Conclude by means of (2) that $A \cap U$ is not empty.

(4) We have that

$$\begin{aligned} f^{-1}((Y \setminus f(A)) \cap f(U)) &= f^{-1}(Y \setminus f(A)) \cap f^{-1}(f(U)) \\ &= \left(X \setminus f^{-1}(f(A))\right) \cap f^{-1}(f(U)). \end{aligned}$$

In a similar manner as in (2) and (3), deduce that $(X \setminus A) \cap U$ is not empty.

(5) Observe that (2) and (3) demonstrate that $f^{-1}(y)$ belongs to $\partial_{(X,\mathcal{O}_X)}A$. Conclude that y belongs to $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$. Thus we have proven that $\partial_{(Y,\mathcal{O}_Y)}f(A)$ is a subset of $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$.

(6) Suppose now that x belongs to $\partial_{(X,\mathcal{O}_X)}A$. Let U be a neighbourhood of $f(x)$ in Y with respect to $\mathcal{O}_Y$. Observe that then $f^{-1}(U)$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$.

(7) Since x belongs to $\partial_{(X,\mathcal{O}_X)}A$, we have that $A \cap f^{-1}U$ is not empty. Thus

$$f\left(A \cap f^{-1}(U)\right)$$

is not empty. We have that $f\left(A \cap f^{-1}(U)\right)$ is a subset of

$$f(A) \cap f\left(f^{-1}(U)\right).$$

Since f is a surjection, we also have that $f\left(f^{-1}(U)\right) = U$. Deduce that $f(A) \cap U$ is not empty.

(8) Since x belongs to $\partial_{(X,\mathcal{O}_X)}A$, we have that $(X \setminus A) \cap f^{-1}U$ is not empty. Observe that since f is a bijection, we have that $f(X \setminus A) = Y \setminus f(A)$. In a similar manner as in (6), deduce that $(Y \setminus f(A)) \cap U$ is not empty.

(9) Observe that (7) and (8) demonstrate that $f(x)$ belongs to $\partial_{(Y,\mathcal{O}_Y)}f(A)$. Thus we have proven that $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$ is a subset of $\partial_{(Y,\mathcal{O}_Y)}f(A)$.

(10) Conclude from (5) and (9) that $f\left(\partial_{(X,\mathcal{O}_X)}A\right)$ is $\partial_{(Y,\mathcal{O}_Y)}f(A)$.

Task E9.3.13. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X. Let $\mathcal{O}_A$ denote the subspace topology on A with respect to $(X, \mathcal{O}_X)$. Let B be a subset of A which belongs to $\mathcal{O}_X$. Prove that $\partial_{(A,\mathcal{O}_A)}B$ is $A \cap \partial_{(X,\mathcal{O}_X)}B$. You may wish to proceed as follows.

(1) By Task E9.3.10, we have that $\partial_{(A,\mathcal{O}_A)}B$ is the intersection of $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$ and $\mathsf{cl}_{(A,\mathcal{O}_A)}(A \setminus B)$.

(2) Observe that since B belongs to $\mathcal{O}_X$, and since B is a subset of A, we have that B belongs to $\mathcal{O}_A$. Thus $A \setminus B$ is closed in A with respect to $\mathcal{O}_A$. Deduce by Proposition 9.1.1 that $\mathsf{cl}_{(A,\mathcal{O}_A)}(A \setminus B)$ is $A \setminus B$.

(3) Since B belongs to $\mathcal{O}_X$, we have that $X \setminus B$ is closed in X with respect to $\mathcal{O}_X$. Deduce by Proposition 9.1.1 that $\mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus B)$ is $X \setminus B$. Thus $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus B)$ is $A \setminus B$.

(4) Observe that by (2) and (3), we have that $\mathsf{cl}_{(A,\mathcal{O}_A)}(A \setminus B)$ is $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus B)$.

(5) Observe that by Task E8.3.13, we have that $\mathsf{cl}_{(A,\mathcal{O}_A)}(B)$ is $A \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)$.

(6) By (1), (4), and (5), conclude that $\partial_{(A,\mathcal{O}_A)}B$ is $A \cap \left(\mathsf{cl}_{(X,\mathcal{O}_X)}(X \setminus B) \cap \mathsf{cl}_{(X,\mathcal{O}_X)}(B)\right)$.

(7) Conclude by Task E9.3.10 that $\partial_{(A,\mathcal{O}_A)}B$ is $A \cap \partial_{(X,\mathcal{O}_X)}B$.

Task E9.3.14. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X, and let B be a subset of A. Prove that $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$ is a subset of the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$. You may wish to proceed as follows.

(1) Suppose that x belongs to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$. Suppose first that every neighbourhood U of x in X with respect to $\mathcal{O}_X$ has the property that $(X \setminus A) \cap U$ is not empty. Since x belongs $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$, we also have that $(A \setminus B) \cap U$ is not empty. In particular, $A \cap U$ is not empty. Deduce that x belongs to $\partial_{(X,\mathcal{O}_X)}A$.

(2) Suppose instead that there is a neighbourhood U of x in X with respect to $\mathcal{O}_X$ such that $(X \setminus A) \cap U$ is empty. We have that $X \setminus (A \setminus B)$ is the union of $X \setminus A$ and B. Since x belongs $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$, we have that $(X \setminus (A \setminus B)) \cap U$ is not empty. Deduce that $B \cap U$ is not empty.

(3) Let U' be any neighbourhood of x in X with respect to $\mathcal{O}_X$. Suppose that $B \cap U'$ is empty. We have that $U \cap U'$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$. Moreover, observe that $(X \setminus A) \cap (U \cap U')$ is empty, and that $B \cap (U \cap U')$ is empty. Conclude that $B \cap U'$ is not empty.

(4) Since x belongs to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$, we have that $(A \setminus B) \cap U'$ is not empty. In particular, we have that $(X \setminus B) \cap U'$ is not empty.

(5) Observe that, by (2) – (4), if there is a neighbourhood U of x in X with respect to $\mathcal{O}_X$ such that $(X \setminus A) \cap U$ is empty, then x belongs to $\partial_{(X,\mathcal{O}_X)}B$.

(6) Observe that, by (1) and (5), we have that $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$ is a subset of the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$.

Task E9.3.15. Let $(X, \mathcal{O}_X)$ be a topological space. Let A be a subset of X which is closed with respect to $\mathcal{O}_X$. Let B be a subset of X which belongs to $\mathcal{O}_X$. Prove that $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$ is the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$. You may wish to proceed as follows.

(1) Observe that, by Task E9.3.14, we have that $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$ is a subset of the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$.

(2) Since B is a subset of A, we have that $X \setminus A$ is a subset of $X \setminus B$. Deduce that $(X \setminus A) \cap B$ is empty.

(3) Suppose that $x \in X$ belongs to B. Then B is a neighbourhood of x in X with respect to $\mathcal{O}_X$. Deduce by (2) that x does not belong to $\partial_{(X,\mathcal{O}_X)}A$.

(4) Suppose that $x \in X$ belongs to $X \setminus A$. Since A is closed in X with respect to $\mathcal{O}_X$, we then have that $X \setminus A$ is a neighbourhood of x in X with respect to $\mathcal{O}_X$. Deduce that x does not belong to $\partial_{(X,\mathcal{O}_X)}A$.

(5) Suppose that x belongs to $\partial_{(X,\mathcal{O}_X)}A$. By (3) and (4), we have that x belongs to $A \setminus B$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_X$. Observe that since x belongs to $A \setminus B$, we have that $(A \setminus B) \cap U$ is not empty.

(6) Since x belongs to $\partial_{(X,\mathcal{O}_X)}A$, we also have that $(X \setminus A) \cap U$ is not empty. We have that $X \setminus (A \setminus B)$ is the union of $X \setminus A$ and B. Deduce that $(X \setminus (A \setminus B)) \cap U$ is not empty.

(7) Conclude from (5) and (6) that if x belongs to $\partial_{(X,\mathcal{O}_X)}A$, then x belongs to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$.

(8) Arguing in a similar way, prove that if x belongs to $\partial_{(X,\mathcal{O}_X)}B$, then x belongs to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$.

(9) By (7) and (8), we have that the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$ is a subset of $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$. Conclude by (1) that the union of $\partial_{(X,\mathcal{O}_X)}A$ and $\partial_{(X,\mathcal{O}_X)}B$ is equal to $\partial_{(X,\mathcal{O}_X)}(A \setminus B)$.

E9.4 Exploration — limit points in a metric space

Task E9.4.1. Let (X, d) be a metric space. Let $\mathcal{O}_d$ be the topology on X corresponding to d of Task E3.4.9. Let A be a subset of X. Suppose that x belongs to X. Prove that x is a limit point of A in X with respect to $\mathcal{O}_d$ if and only if, for every $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, there is an a which belongs to A such that $d(x, a) < \epsilon$. You may wish to proceed as follows.

(1) Suppose that x is a limit point of A in X with respect to $\mathcal{O}_d$. By Task E4.3.2, we have that $B_\epsilon(x)$ is a neighourhood of x in X with respect to $\mathcal{O}_d$. Deduce that $A \cap B_\epsilon(x)$ is not empty, and thus that there is an a which belongs to A such that $d(x, a) < \epsilon$.

(2) Suppose instead that, for every $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, there is an a which belongs to A such that $d(x, a) < \epsilon$. Let U be a neighbourhood of x in X with respect to $\mathcal{O}_d$. By definition of $\mathcal{O}_d$, there is a $\zeta \in \mathbb{R}$ with $\epsilon > 0$ such that $B_\zeta(x)$ is a subset of U. By assumption, there is an a in A such that a belongs to $B_\zeta(x)$. Deduce that $A \cap U$ is not empty. Conclude that that x is a limit point of A in X with respect to $\mathcal{O}_d$.

Task E9.4.2. Let (X, d) be a metric space. Let A be a subset of X. Suppose that x belongs to X. Let X be equipped with the topology $\mathcal{O}_d$ corresponding to d of Task E3.4.9. Prove that if A is closed in X with respect to $\mathcal{O}_d$, then $d(x, A) > 0$ for every x which does not belong to A. You may wish to proceed as follows.

(1) Since A is closed in X with respect to $\mathcal{O}_d$, we have, by Proposition 9.1.1, that x is not a limit point of A in X with respect to $\mathcal{O}_X$. By Task E9.4.1, deduce that there is an $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ such that $d(x, a) \geq \epsilon$ for all a which belong to A.

(2) Deduce that $d(x, A) \geq \epsilon$, and thus that $d(x, A) > 0$.