Revision Question 10 — 09/05/14

Let X be the set $[0,1] \times]0,1[\times]0,1[$. Let $\mathcal{O}_X$ be the subspace topology on X with respect to $(\mathbb{R}^3, \mathcal{O}_{\mathbb{R}^3})$.

a) Find an open covering of X which does not admit a finite subcovering. [3 marks]

b) Let A be the subset of X given by $\left]\frac{1}{2}, 1\right] \times \left]\frac{1}{4}, \frac{3}{4}\right[\times \left]\frac{1}{4}, \frac{3}{4}\right[$. Decide whether the following are true or false, and justify your answers.

i) The set A belongs to $\mathcal{O}_X$.

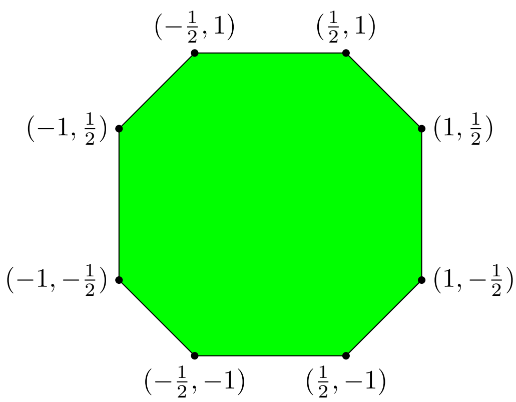
ii) The set A is closed in X with respect to $\mathcal{O}_X$.

[6 marks]

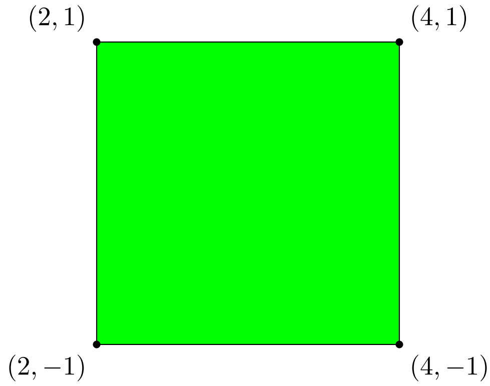
c) Let $\mathcal{O}_{[0,1]}$ be the subspace topology on $[0,1]$ with respect to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}})$. Give a counterexample to the following statement: for every subset W of X such that W belongs to $\mathcal{O}_X$, and such that $\left\{\frac{1}{2}\right\} \times]0,1[\times]0,1[$ is a subset of W, there is a neighbourhood U of $\frac{1}{2}$ in $[0,1]$ with respect to $\mathcal{O}_{[0,1]}$ such that $U \times]0,1[\times]0,1[$ is a subset of W. [4 marks]

d) Is it possible to find a counterexample if we replace X by $[0,1] \times [0,1] \times [0,1]$? Justify your answer. You may quote without proof any results from the course. [5 marks]

Let O be the octagon in $\mathbb{R}^2$ depicted below.



Let S be the square in $\mathbb{R}^2$ depicted below.

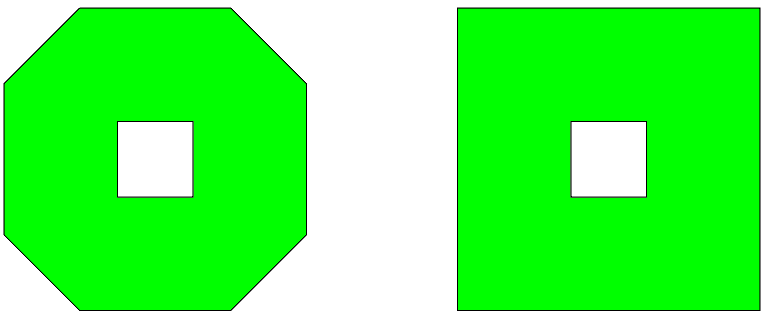


Let Y be the subset of $\mathbb{R}^2$ given by the union of

$$O \setminus \left(\left]-\tfrac{1}{4}, \tfrac{1}{4}\right[\times \left]-\tfrac{1}{4}, \tfrac{1}{4}\right[\right)$$

and

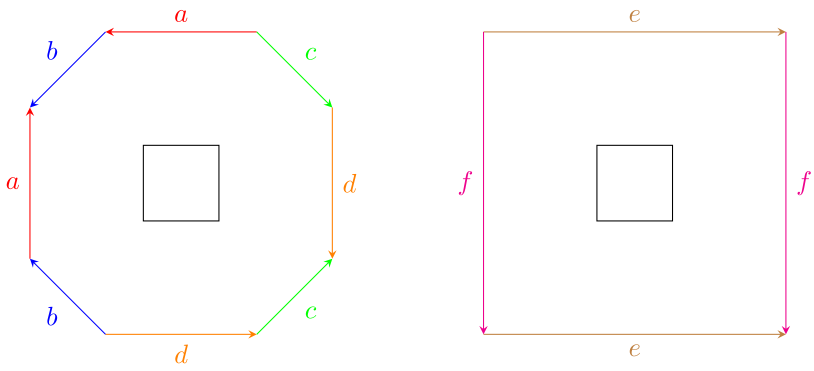
$$S \setminus \left(\left]\tfrac{11}{4}, \tfrac{13}{4}\right[\times \left]-\tfrac{1}{4}, \tfrac{1}{4}\right[\right).$$



Let $\mathcal{O}_Y$ be the subspace topology on Y with respect to $(\mathbb{R}^2, \mathcal{O}_{\mathbb{R}^2})$.

e) Prove that $(Y, \mathcal{O}_Y)$ is not connected. [5 marks]

f) Let $\sim$ be the equivalence relation on Y which identifies, without a twist, the edges with the same letter in the figure below.



Define an equivalence relation $\approx$ on Y such that the following hold.

i) Suppose that x_0 and x_1 belong to $\mathbb{R}^2$, and that $x_0 \sim x_1$. Then $x_0 \approx x_1$.

ii) We have that $(Y/\approx, \mathcal{O}_{Y/\approx})$ is a surface.

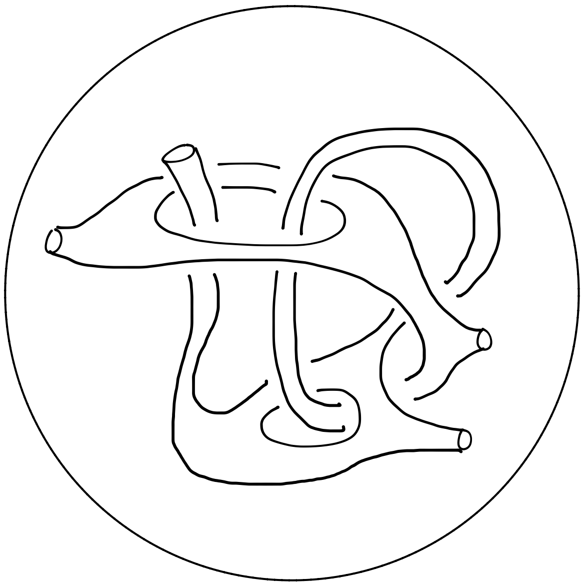
You do not need to prove that i) and ii) hold. [6 marks]

g) Equip $(Y/\approx, \mathcal{O}_{Y/\approx})$ with the structure of a Δ-complex. [4 marks]

h) Calculate the Euler characteristic of $(Y/\approx, \mathcal{O}_{Y/\approx})$. [3 marks]

i) Assume that $(Y/\approx, \mathcal{O}_{Y/\approx})$ is an n-handlebody for some n. Using your answer to part h) or otherwise, determine n. You may quote without proof any results from the course. [3 marks]

Let $(X, \mathcal{O}_X)$ be the surface given by the 'sphere with tunnels' depicted below.



j) Which of the surfaces in the classification is $(X, \mathcal{O}_X)$ homeomorphic to? Justify your answer by a surgery argument. [6 marks]

k) Hence or otherwise, calculate the Euler characteristic of $(X, \mathcal{O}_X)$.