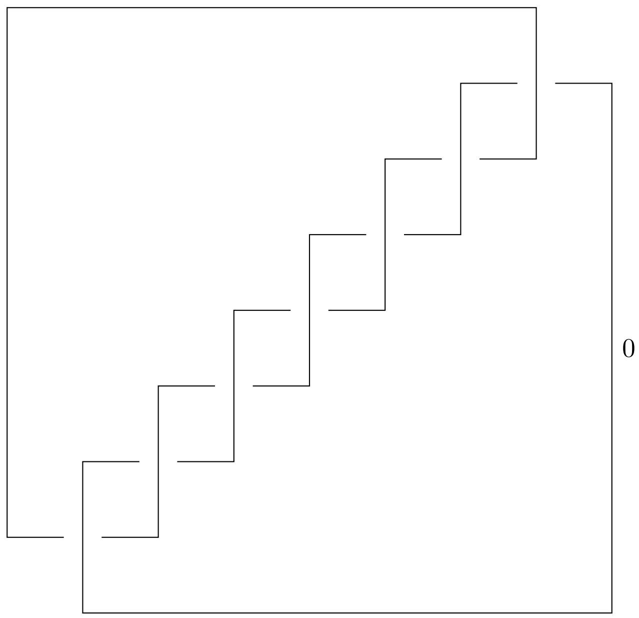
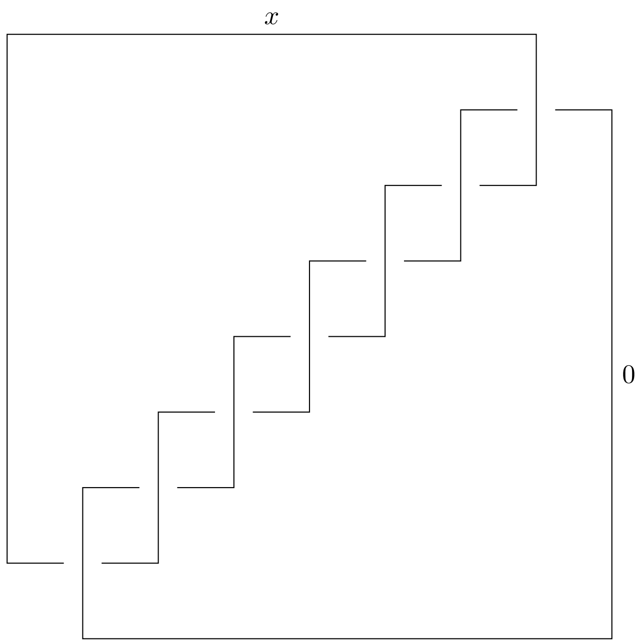
Solutions 3

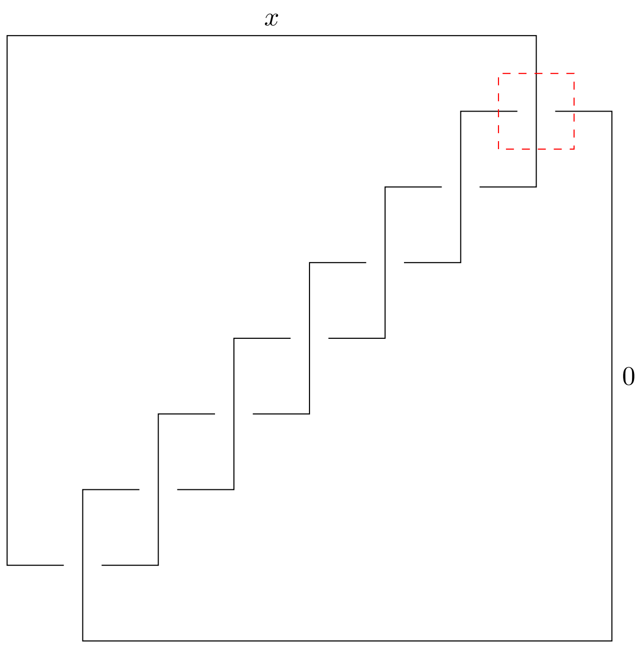
a) Suppose that 7_1 is m-colourable. By a result from the course, we can assume that the integer assigned to one of the arcs is 0, for instance the arc indicated below.



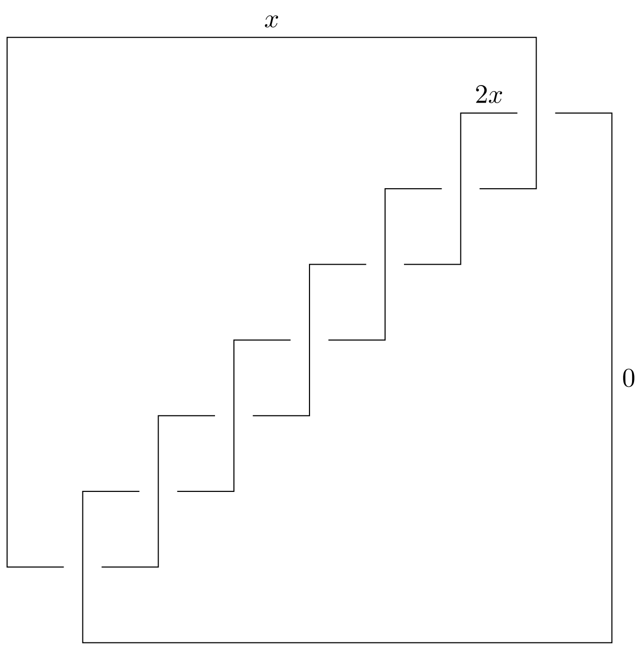
Suppose that the integer, mod m, assigned to the arc indicated below is x.



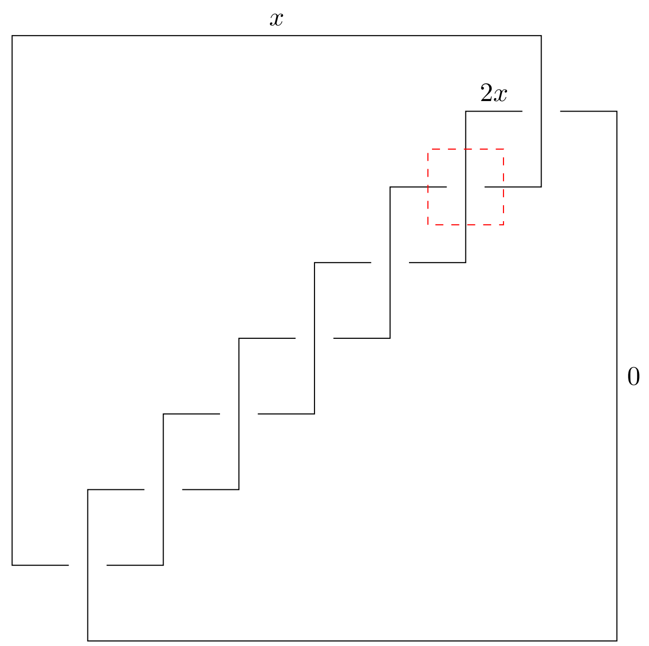
Consider the following crossing.



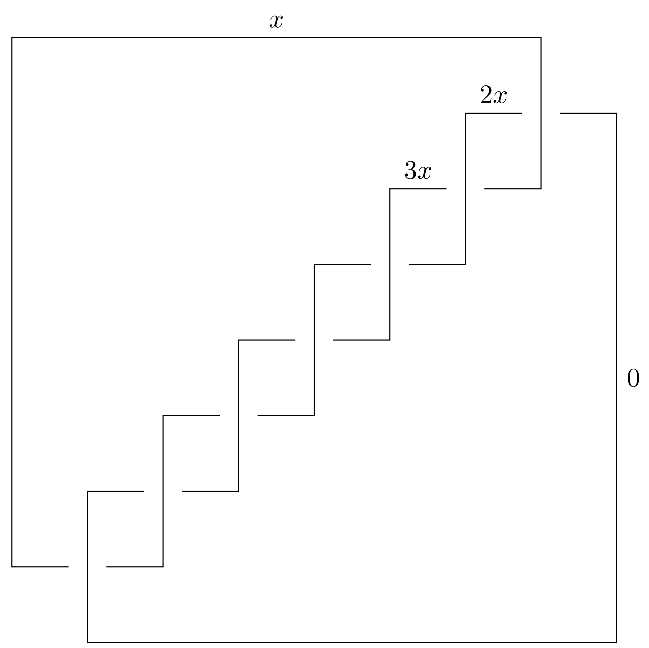
By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $2x \pmod{m}$.



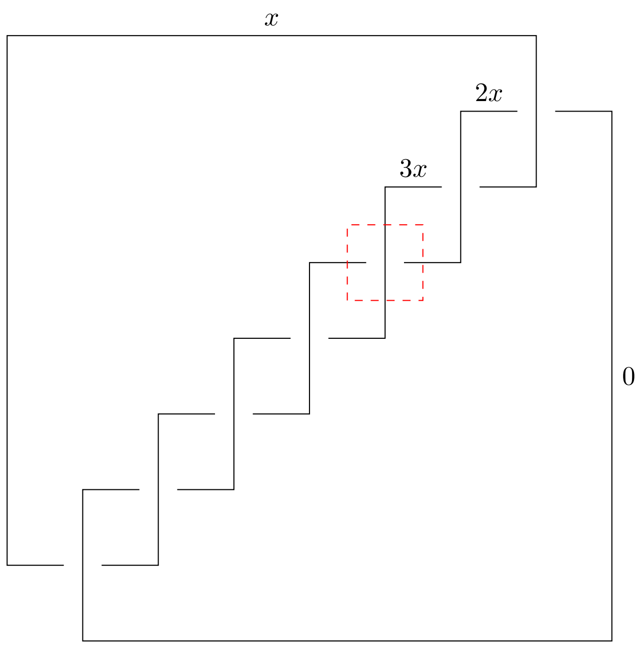
Consider the following crossing.



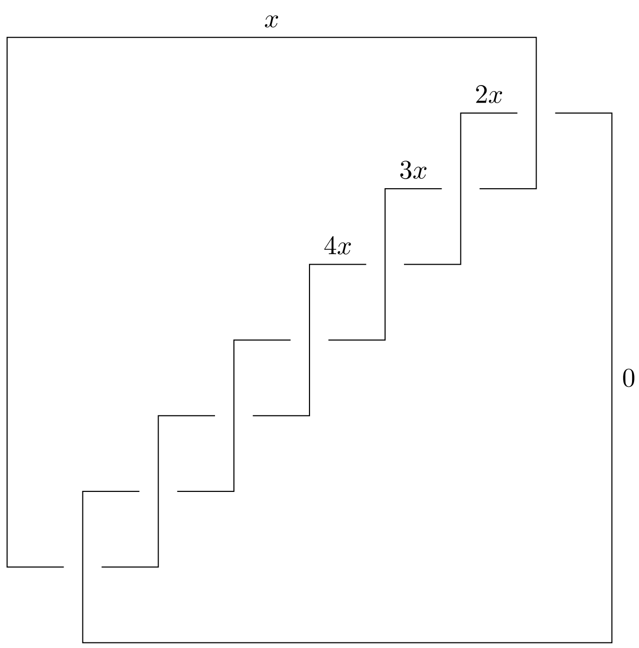
By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $4x - x \pmod{m}$, namely $3x \pmod{m}$.



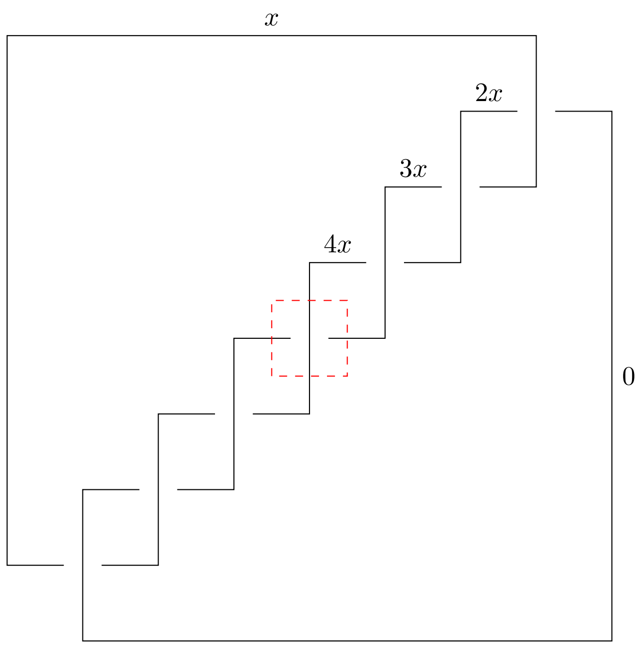
Consider the following crossing.



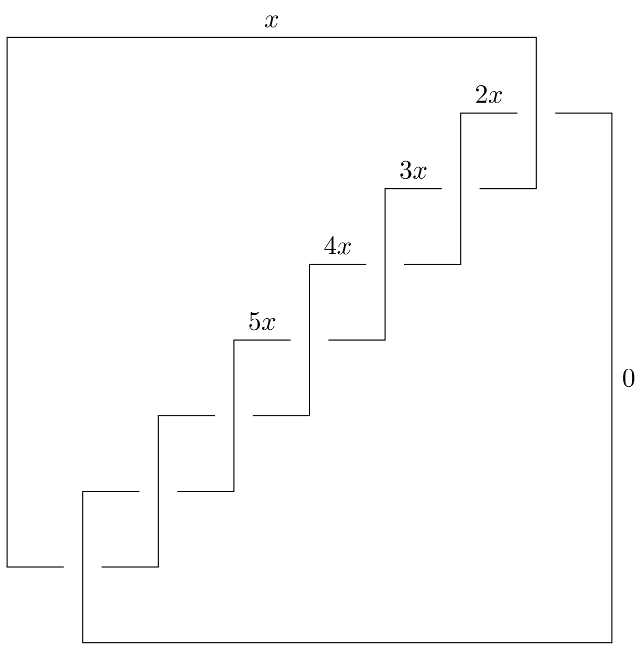
By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $6x - 2x \pmod{m}$, namely $4x \pmod{m}$.



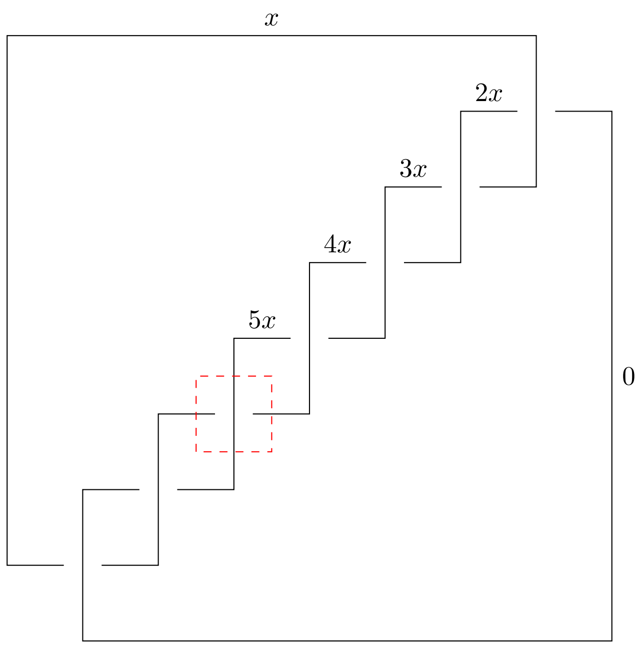
Consider the following crossing.



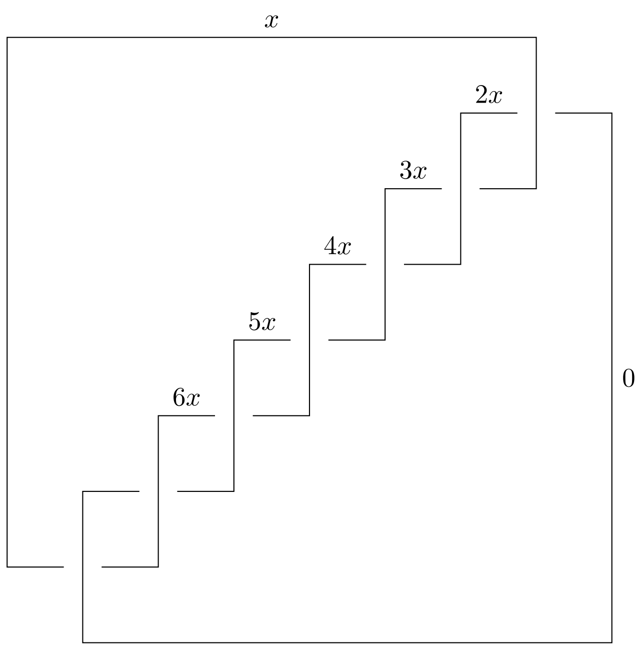
By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $8x - 3x \pmod{m}$, namely $5x \pmod{m}$.



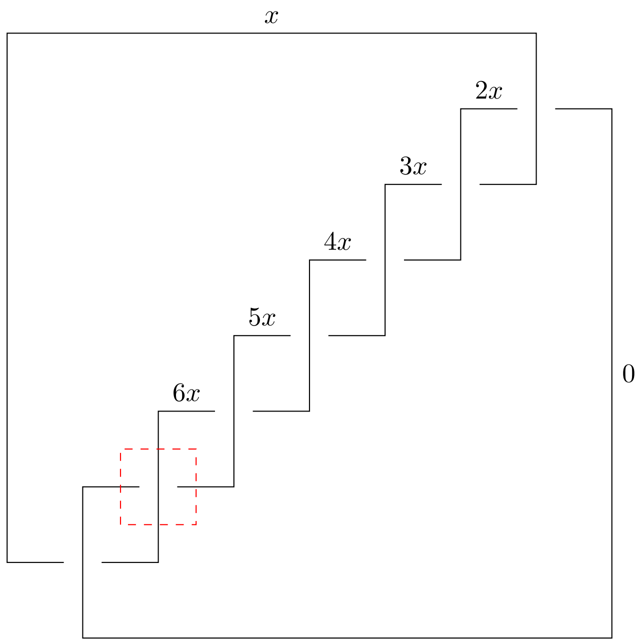
Consider the following crossing.



By the condition for a knot colouring which must hold at this crossing, we have that the integer assigned to the indicated arc is $10x - 4x \pmod{m}$, namely $6x \pmod{m}$.



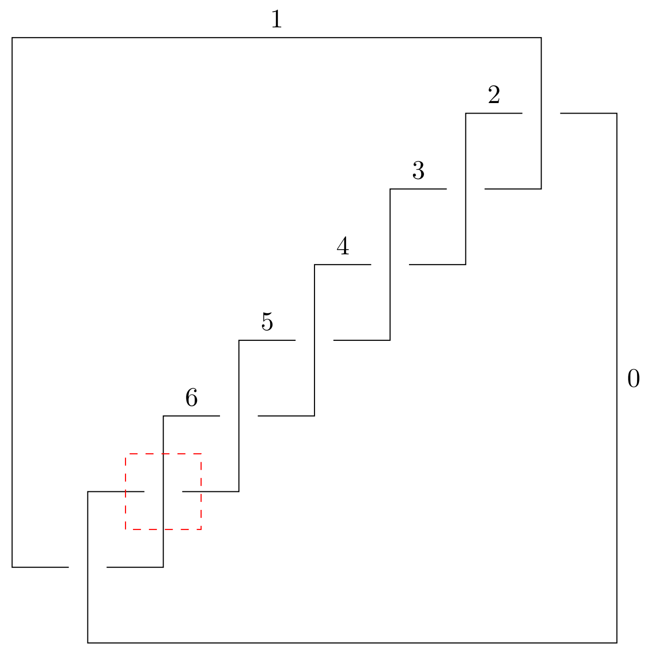
Consider the following crossing.



We have already assigned integers to all the arcs involved in this crossing. By the condition for a knot colouring which must hold at this crossing, we have that $12x \equiv 5x + 0 \pmod{m}$. We deduce that $7x \equiv 0 \pmod{m}$.

By definition of equivalence mod m, we thus have that there is an integer k such that $7x = km$. Since 7 is a prime, we have, by the the fundamental theorem of arithmetic, that either $7 \mid k$ or $7 \mid m$. Suppose that $7 \mid k$. Then we have that $x = \frac{k}{7} \cdot m$, and $\frac{k}{7}$ is an integer. Thus we have that $x \equiv 0 \pmod{m}$. Then 0 is assigned to every arc mod m, which contradicts one of the conditions for a colouring.

We conclude that $7 \mid m$. Conversely, a 7-colouring of 7_1 is obtained by, for instance, taking x to be 1 in the previous figure.



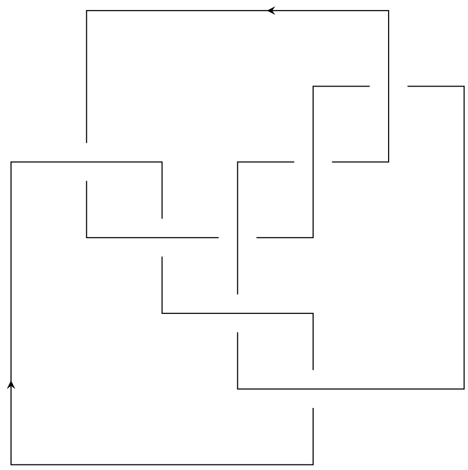
A $7k$-colouring for any integer k is obtained by multiplying the integer assigned to each arc in the previous figure by k.

b) Omitted.

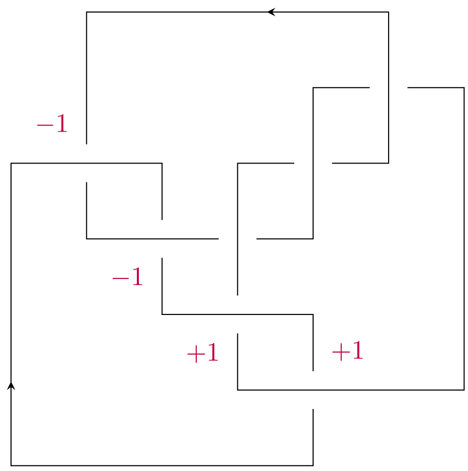
c) No. By a result from the course, if 6_1 is isotopic to 7_1, then 6_1 is m-colourable for an integer m if and only if 7_1 is m-colourable. By part b), we have that 6_1 is 15-colourable. Hence, if 6_1 is isotopic to 7_1, then 7_1 is 15-colourable. Since 15 is not divisible by 7, this contradicts part a).

d) An argument using the Jones polynomial of each knot is needed. Omitted.

e) The linking number of 7_5^2 is 0. Suppose, for instance, that we choose the following orientation.



The signs of the crossings between distinct components are as indicated.



We then calculate: $\frac{1}{2} \cdot |-1-1+1+1| = 0$.

f) We calculated in the lectures that the linking number of 2_1^2 is 1. By a result from the course, if 7_5^2 were isotopic to 2_1^2, then their linking numbers would be the equal. Since the linking number of 7_5^2 is 0 by part e), we conclude that 7_5^2 is not isotopic to 2_1^2.

g) No. The unlink with two components also has linking number 0.

Discussion

Knot colouring

The solution to part a) is in the style that I used in the lectures this year. It is much more efficient than the approach taken in last year's lecture notes.

It is very important that you have a clear understanding of the conclusion of the proof that $7 \mid m$ in the solution to part a), using the fundamental theorem of arithmetic and so on, and that you feel confident that you can carry out this kind of argument.

Linking number

Don't forget: only count the signs between *distinct* crossings! Remember also that one needs an orientation to calculate the linking number of a link, but any orientation can be chosen. This is by contrast with the Jones polynomial of a link, in which the answer depends upon the orientation, unless the link is a knot.